

ZERO PATTERNS AND UNITARY SIMILARITY: THE CELL AN AND ĐOKOVIĆ COULD NOT COMPUTE, AND THEIR CONJECTURE 3.6 THROUGH $n = 12$

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ABSTRACT. For a zero pattern I inside $\mathbb{Z}_n \times \mathbb{Z}_n$ put $\chi_I = \prod_{(i,j) \in I} (x_i - x_j)$, and pair it with the Vandermonde χ_n in the inner product for which the Laurent monomials are orthonormal. An and Đoković showed that for a pattern of maximal size the single integer $\langle \chi_I, \chi_n \rangle$ decides whether every traceless complex $n \times n$ matrix is unitarily similar to one vanishing on I , and conjectured that for their Hessenberg-shaped patterns $J_{k,n}$ this integer is $(-1)^{n-1} \binom{n}{k} \binom{n-2}{k-1}$. They report the conjecture verified for $n \leq 10$ with one exception: at $(n, k) = (10, 5)$ their program ran out of memory. We compute that cell, $\langle \chi_{J_{5,10}}, \chi_{10} \rangle = -17640$, so the conjecture holds at all 45 cells with $n \leq 10$ and the pattern $J_{5,10}$ is 10-universal. We then compute the two rows past their range: the conjecture holds at $n = 11$ and at $n = 12$, so every $J_{k,11}$ is 11-universal and every $J_{k,12}$ is 12-universal. We also settle the first open case of their Problems 3.4 and 3.5, showing that $|\langle \chi_I, \chi_6 \rangle| \leq 6!$ and $\langle \chi_I, \chi_I \rangle \geq 6!$ over all 155 117 520 strict patterns of size 15, with equality exactly at $\chi_I = \pm \chi_6$. Finally we record that the conjectured values are n times Narayana numbers, an identity the source does not remark on and which names a concrete route to a proof for all n . Every computation below is machine-checked in Lean 4; the passage from a nonzero pairing to universality is two cited theorems of An and Đoković, and is not formalised.

1. INTRODUCTION

The objects. Fix $n \geq 2$ and write $\mathbb{Z}_n = \{1, \dots, n\}$ and $\mu_n = \binom{n}{2} = n(n-1)/2$. A *position* is an ordered pair of positive integers, a *pattern* is a finite set of positions, and a pattern is *strict* if it contains no diagonal position (i, i) . Let $\mathcal{P}_n$ be the set of patterns contained in $\mathbb{Z}_n \times \mathbb{Z}_n$ and $\mathcal{P}'_n \subset \mathcal{P}_n$ the strict ones of size exactly μ_n . For $I \in \mathcal{P}_n$ let $M_I(n)$ be the space of complex $n \times n$ matrices vanishing at every position of I and $L_I(n)$ its traceless part. The pattern I is *n -universal* when every traceless complex $n \times n$ matrix is unitarily similar to a matrix in $L_I(n)$. Schur's triangularization theorem is exactly the statement that $\text{NE}_n = \{(i, j) : i < j\}$ is n -universal, and the search for other universal patterns is the subject of An and Đoković's *Zero patterns and unitary similarity* [1].

That paper turns universality into a question about one integer. Put

$$\chi_I = \prod_{(i,j) \in I} (x_i - x_j), \quad \chi_n = \chi_{\text{NE}_n} = \prod_{1 \leq i < j \leq n} (x_i - x_j),$$

and let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$ for which the Laurent monomials are orthonormal, so that $\langle f, g \rangle = \sum_m f_m g_m$ is the sum of products of coefficients. Then a pattern $I \in \mathcal{P}'_n$ is *n -singular* if and only if $\langle \chi_I, \chi_n \rangle = 0$ ([1, Prop. 3.3]), and every nonsingular pattern in $\mathcal{P}_n$ is n -universal ([1, Thm. 3.8],

which that paper states as a generalisation of Schur's triangularization theorem and extracts from the companion paper [2]). A single integer therefore decides universality for a pattern of maximal size — and the integer is a signed sum over the 2^{μ_n} ways of choosing one of the two variables from each factor of χ_I , keeping only those choices whose exponent vector is a permutation of $(0, 1, \dots, n-1)$.

The questions the source leaves open. For $k \in \mathbb{Z}_{n-1}$ let $J_{k,n} \in \mathcal{P}'_n$ be the union of the translate $(0, 1) + \text{NE}_{n-1}$, the product $\{n\} \times \mathbb{Z}_k$, and $\{(i, 1) : k < i < n\}$: a Hessenberg-shaped pattern with a hook in the last row and a tail in the first column. Conjecture 3.6 of [1] asserts

$$(1) \quad \langle \chi_{J_{k,n}}, \chi_n \rangle = (-1)^{n-1} \binom{n}{k} \binom{n-2}{k-1}, \quad k \in \mathbb{Z}_{n-1}.$$

The source proves the column $k = 1$ (its Example 6.3, immediately after its Theorem 6.2, giving $(-1)^{n-1}n$) and reports, verbatim:

The conjecture has been verified for $n \leq 10$ and all $k \in \mathbb{Z}_{n-1}$, except for $(n, k) = (10, 5)$ in which case our program ran out of memory.

So one cell of a published table was missing, and with it the universality of one pattern of order 10. The same section poses two further questions, reported verified only for $n \leq 5$:

Problem 3.4. Is it true that $\langle \chi_I, \chi_n \rangle \leq n!$ for all $I \in \mathcal{P}_n$ and that equality holds iff $\chi_I = \chi_n$?

Problem 3.5. Is it true that $\langle \chi_I, \chi_I \rangle \geq n!$ for all $I \in \mathcal{P}'_n$ and that equality holds iff $\chi_I = \pm \chi_n$?

The computations carried out for $n \leq 5$ show that, in these cases, the answer is affirmative for both problems.

What is settled here.

- **Theorem 4.1.** $\langle \chi_{J_{5,10}}, \chi_{10} \rangle = -17640 = (-1)^9 \binom{10}{5} \binom{8}{4}$. This is the cell the source's program could not reach, so (1) holds at every one of the 45 cells with $n \leq 10$, with no exception (Corollary 4.2), and $J_{5,10}$ is 10-nonsingular, hence 10-universal: every traceless complex 10×10 matrix is unitarily similar to one vanishing at all 45 positions of $J_{5,10}$.
- **Theorem 4.3.** (1) holds at $n = 11$: the ten values are 11, 495, 5940, 27720, 58212, and back down. Every $J_{k,11}$ is 11-universal.
- **Theorem 4.5.** (1) holds at $n = 12$: the eleven values are -12 , -660 , -9900 , -59400 , -166320 , -232848 , and back down. Every $J_{k,12}$ is 12-universal. In both rows the $k = 1$ value and the universality at $k = 1, 2$ were already the source's; see Remark 4.6, which subtracts them.
- **Theorem 6.3.** Problems 3.4 and 3.5 have affirmative answers at $n = 6$, the first case the source does not cover, where $\mathcal{P}'_6$ has $\binom{30}{15} = 155\,117\,520$ elements.
- **Proposition 5.1** records that the right-hand side of (1) is $(-1)^{n-1}n N(n-1, k)$, n times a Narayana number, so that the conjectured rows are the rows of a triangle counting Grand Dyck paths by double rises. The source does not remark on this, and it names the bijective route to a proof of (1) for all n (Remark 5.3).

Theorems 4.1–6.3 are exhaustive finite computations, and §3 says precisely which searches were run and which of their steps are exact reductions rather than heuristics. Every computation is checked in Lean 4, with the three exceptions listed in §8. What the priority claims rest on — and where the literature search behind them stops — is set out in §9; the reader should weigh that section before reading any statement below as a priority claim. In particular the *right-hand side* of (1) is trivially evaluable and has been in the literature since 2006 as a combinatorial triangle (Remark 5.2); what is new here is the left-hand side.

Three limitations of method, stated at the front. First, universality itself is not formalised. What is machine-checked is the integer $\langle \chi_I, \chi_n \rangle$ and its vanishing or non-vanishing; the passage from “nonzero” to “ n -nonsingular” to “ n -universal” is two citations of [1] and is used as such. Second, at $n = 12$ the value is computed through a relabelled copy of the pattern, which is legitimate by a published equivariance of [1]; that identity is exercised at every size where both sides are affordable (Proposition 4.4) but is likewise not formalised. Third, the reduction from patterns to multisets that brings $n = 6$ into range is a two-line argument, not a formal one, and Proposition 6.2 is the empirical control on it. All three bridges are marked where they are used.

2. PRELIMINARIES

2.1. The pairing. Expanding $\chi_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_i x_{\sigma(i)}^{n-i}$ shows that χ_n is supported on the $n!$ monomials whose exponent vector is a permutation of the staircase $(0, 1, \dots, n-1)$, with coefficient ± 1 there. Consequently, for any pattern I of size μ_n ,

$$(2) \quad \langle \chi_I, \chi_n \rangle = (-1)^{\mu_n} \sum_{\omega} \text{sgn}(\omega) (-1)^{\text{inv}(\deg_{\omega})},$$

where ω runs over the *orientations* of I — one variable chosen from each factor $(x_i - x_j)$, contributing $+1$ and raising the exponent of x_i , or -1 and raising that of x_j — subject to the condition that the resulting exponent vector $\deg_{\omega}$ be a permutation of $0, 1, \dots, n-1$, and where inv counts the inversions of that permutation. Two standard values are used below as controls: $\langle \chi_n, \chi_n \rangle = n!$, stated in [1] immediately after the expansion of χ_n , and $\langle \chi_{\Pi_n}, \chi_n \rangle = n!!$ for the pattern

$$\Pi_n = \{(i, j) : i + j \leq n, i \neq j\} \cup \{(i, n - i + 1) : 2i \leq n\},$$

a proved proposition of [1, Prop. 6.4] about a family unrelated to $J_{k,n}$.

2.2. Singularity and universality. We use [1, Prop. 3.3] and [1, Thm. 3.8] only in the following combined form, and never unfold the definition of n -singularity, which reads there, verbatim: “We say that $I \in \mathcal{P}_n$ is *n-singular* if $\varphi_n(\chi_I) = 0$, and otherwise that it is *n-nonsingular*”, where φ_n is the quotient map of $\mathbb{R}[x_1, \dots, x_n]$ by the ideal generated by the elementary symmetric functions $\sigma_{k,n}$, $k \in \mathbb{Z}_n$. Nothing below touches that ideal.

Fact 2.1 (An–Đoković). Let $I \in \mathcal{P}'_n$. If $\langle \chi_I, \chi_n \rangle \neq 0$ then I is n -nonsingular, and hence n -universal: every traceless complex $n \times n$ matrix is unitarily similar to a matrix vanishing at every position of I .

We also use the equivariance displayed in [1, §4]: for $I \in \mathcal{P}'_n$ and $\sigma \in S_n$, writing $\sigma(I) = \{(\sigma i, \sigma j) : (i, j) \in I\}$,

$$(3) \quad \langle \chi_{\sigma(I)}, \chi_n \rangle = \langle \chi_I, \sigma \cdot \chi_n \rangle = \text{sgn}(\sigma) \langle \chi_I, \chi_n \rangle.$$

2.3. The patterns $J_{k,n}$. By construction $|J_{k,n}| = \binom{n-1}{2} + k + (n-1-k) = \mu_n$, and the three pieces are disjoint as sets of *ordered* pairs. Canonicalising each factor of $\chi_{J_{k,n}}$ to the form $(x_{\min} - x_{\max})$ flips exactly $k + (n-1-k) = n-1$ of them, which is where the sign $(-1)^{n-1}$ of (1) comes from; the conjectured magnitude $\binom{n}{k} \binom{n-2}{k-1}$ is positive.

A pattern is *doubly laced* if it contains some position together with its transpose, and *simple* otherwise. Simple patterns are nonsingular for trivial reasons ([1, §3]), so the conjecture would be empty if the $J_{k,n}$ were simple. They are not.

Proposition 2.2. *For every n with $2 \leq n \leq 12$ and every $k \in \mathbb{Z}_{n-1}$, the pattern $J_{k,n}$ lies in $\mathcal{P}'_n$: it is contained in $\mathbb{Z}_n \times \mathbb{Z}_n$, is strict, has pairwise distinct positions, and has exactly μ_n of them. Moreover $J_{5,10}$ is doubly laced. For $2 \leq n \leq 10$ one has $\langle \chi_n, \chi_n \rangle = n!$ and $\langle \chi_{\Pi_n}, \chi_n \rangle = n!!$.*

Proof and scope. Four finite checks, run over the stated ranges by the evaluators of §3. The last two statements are the source's own values, recomputed here from the definitions rather than quoted; they are what licenses the use of the same evaluators on the cells the source could not reach. The double-lacing of $J_{5,10}$ can also be seen by hand: $(j, 10)$ lies in $(0, 1) + \text{NE}_9$ for $j \leq 8$ while $(10, j)$ lies in $\{10\} \times \mathbb{Z}_5$ for $j \leq 5$. $\square$

2.4. Problems 3.4 and 3.5 concern $\mathcal{P}'_n$ only. Both problems are posed over all of $\mathcal{P}_n$, respectively $\mathcal{P}'_n$, but only patterns of maximal strict size can be extremal.

Lemma 2.3. *If $I \in \mathcal{P}_n$ contains a diagonal position then $\chi_I = 0$; and if I is strict with $|I| \neq \mu_n$ then $\langle \chi_I, \chi_n \rangle = 0$. Hence Problem 3.4 over $\mathcal{P}_n$ is equivalent to the same inequality over $\mathcal{P}'_n$.*

Proof. A diagonal position contributes the factor $x_i - x_i = 0$. If I is strict then χ_I is homogeneous of degree $|I|$, while χ_n is homogeneous of degree μ_n ; two homogeneous Laurent polynomials of different degrees have disjoint supports, so they pair to 0. A pattern with $\chi_I = 0$ or $\langle \chi_I, \chi_n \rangle = 0$ satisfies the inequality of Problem 3.4 strictly, since $n! > 0$. $\square$

This lemma is elementary and is not part of the machine-checked development; the computations of §6 range over $\mathcal{P}'_n$.

3. TWO EVALUATIONS OF THE PAIRING, AND THE REPRODUCTION GATE

Everything below rests on evaluating $\langle \chi_I, \chi_n \rangle$ exactly, so we describe the two independent evaluators and then the published data they were required to reproduce before anything new was claimed.

3.1. The unfolded definition. The first evaluator multiplies the μ_n linear forms of χ_I out into a sparse polynomial — keys are packed exponent vectors, six bits per variable — does the same for χ_n , and pairs the two coefficient vectors. It is the definition with nothing added: it never mentions orientations, staircases, masks or signs. It is affordable up to $n = 7$.

3.2. The block-closing dynamic program. The second evaluator folds the sum in (2) factor by factor, with two exact collapses.

- (C1) An exponent that reaches n can never be part of a staircase, so the state is dropped.
- (C2) The factors are processed in blocks by their smaller endpoint, so that after block v the exponent of x_v is final. A final exponent matters only through *which* staircase value it consumes and through the parity of the inversions it makes with the values already consumed — and since blocks run in increasing v , those are exactly the consumed values above it. So the exponent of x_v is replaced by one bit of an n -bit mask together with a sign.

Collapse (C1) alone leaves $\#\{e \in [0, 9]^{10} : \sum e = 45\} = 432\,457\,640$ live monomials at the middle layer of $n = 10$, which does not fit ten gigabytes with eight-byte keys and values. The source reports only that its program ran out of memory at that cell; this is the obvious way that happens. With (C2) the same cell peaks at about $1.38 \cdot 10^6$ live states, a reduction by a factor of some 314, and that is the whole reason the missing cell is now computable. The program is exhaustive: no orientation is discarded except by (C1), and (C2) is a change of state variable, not a pruning.

Proposition 3.1. *The two evaluators return the same integer on $J_{k,n}$, on NE_n and on Π_n , for every n with $2 \leq n \leq 7$ and every $k \in \mathbb{Z}_{n-1}$.*

Proof and scope. An exhaustive comparison over the stated range — every pattern in it, and agreement on every one. Counted out, the checked predicate makes $\sum_{n=2}^7 (2 + (n-1)) = \sum_{n=2}^7 (n+1) = 33$ comparisons: for each of the six values of n , one at NE_n , one at Π_n and one at each of the $n-1$ patterns $J_{k,n}$. (They are 33 comparisons at 32 distinct patterns, NE_2 and Π_2 being the same pattern.) The two code paths share only the constructors of the patterns. Outside the formal development the same agreement was checked once more, between the C implementation of the dynamic program and a literal enumeration of all 2^{μ_n} orientations, at the twenty cells with $3 \leq n \leq 7$. $\square$

Proposition 3.2. *Conjecture (1) holds at all 36 cells with $2 \leq n \leq 9$ and $k \in \mathbb{Z}_{n-1}$.*

Proof and scope. An exhaustive evaluation of the 36 cells. This is the range the source reports verified; the point of recomputing it from the definitions is that without it the new cells would rest on an evaluator that had never been tested against anything the authors published. Together with Proposition 2.2 it is the gate through which the rest of this paper passes. $\square$

4. THE MISSING CELL, AND THE CONJECTURE THROUGH $n = 12$

Theorem 4.1. $\langle \chi_{J_{5,10}}, \chi_{10} \rangle = -17640 = (-1)^9 \binom{10}{5} \binom{8}{4} = -252 \cdot 70$. *In particular Conjecture (1) holds at $(n, k) = (10, 5)$, the cell at which the source reports its program ran out of memory.*

Proof and scope. The value is the output of the block-closing dynamic program of §3 on the 45 factors of $\chi_{J_{5,10}}$: an exhaustive traversal of the $2^{45} \approx 3.5 \cdot 10^{13}$ orientations, collapsed by (C1) and (C2) to a table peaking at about $1.38 \cdot 10^6$ live states. The run carries an additional structural check: after the last factor every vertex has been closed and every staircase value $0, \dots, 9$ consumed, so exactly

one state survives, with key $(2^{10} - 1) \cdot 16^{12}$ — a run that lost or duplicated a state could not pass this. The evidence that the evaluator is right at this size is Propositions 2.2, 3.1 and 3.2; the evidence that the answer does not depend on the traversal is Remark 7.1, where sixty-five different vertex orders, hence sixty-five different state spaces, all return -17640 . $\square$

Corollary 4.2. *The value $\langle \chi_{J_{5,10}}, \chi_{10} \rangle$ is nonzero, so the pattern $J_{5,10}$ is 10-nonsingular and hence 10-universal: every traceless complex 10×10 matrix is unitarily similar to a matrix vanishing at all 45 positions of $J_{5,10}$. Moreover Conjecture (1) holds at every one of the 45 cells with $2 \leq n \leq 10$ and $k \in \mathbb{Z}_{n-1}$, with no exception.*

Proof. The non-vanishing and the 45 cells are finite checks. The passage from non-vanishing to universality is Fact 2.1, i.e. two published theorems of [1]; it is not formalised here. $J_{5,10}$ was the one pattern inside the source’s own verification range whose universality those two results could not settle, because the integer was not known. $\square$

Theorem 4.3. *For every $k \in \mathbb{Z}_{10}$,*

$$\langle \chi_{J_{k,11}}, \chi_{11} \rangle = \binom{11}{k} \binom{9}{k-1},$$

that is, the row 11, 495, 5940, 27720, 58212, 58212, 27720, 5940, 495, 11. So Conjecture (1) holds at $n = 11$, one row past the range the source reports verified, and every $J_{k,11}$ is 11-nonsingular, hence 11-universal.

Proof and scope. Ten exhaustive runs of the dynamic program, one per cell, over the 2^{55} orientations of a 55-factor product; the largest table holds $1.50 \cdot 10^7$ live states. The non-vanishing of each value is read off the ten equalities, and universality is again Fact 2.1. Outside the formal development the whole row was recomputed by an independent C implementation of the same recurrence (291 seconds, 1.07 gigabytes peak), and the conjectured right-hand side — the arithmetic side only, never the pairing — was matched against the published triangle of Remark 5.2. $\square$

At $n = 12$ the pattern’s own vertex order is too expensive, and the computation is run through a relabelled copy. That is licensed by (3), which we check rather than assume.

Proposition 4.4. *The identity (3) holds for the patterns and permutations at which both of its sides were affordable: at every cell with $3 \leq n \leq 8$, for all n cyclic rotations of the vertices and for the reversal, and at $n = 10$ for three further permutations across all nine cells. Moreover, for the permutation σ used at $n = 12$ below, $\sigma(J_{k,12}) \in \mathcal{P}'_{12}$ for every $k \in \mathbb{Z}_{11}$.*

Proof and scope. Exhaustive checks over the stated ranges. Each compares the quantity $\text{sgn}(\sigma) \langle \chi_{\sigma(I)}, \chi_n \rangle$, computed on the relabelled pattern, with $\langle \chi_I, \chi_n \rangle$ computed on the original. Counted out, the first range is $\sum_{n=3}^8 (n+1)(n-1) = \sum_{n=3}^8 (n^2 - 1) = 193$ cell/permutation pairs — the $n-1$ cells against each of the n rotations and the reversal — and the $n = 10$ range adds $3 \cdot 9 = 27$ more. The identity itself is the source’s, and is published mathematics; what is verified here is that this evaluator obeys it, which is what makes Theorem 4.5 an assertion about $J_{k,12}$ rather than about a relabelled copy. The identity is not formalised. $\square$

Theorem 4.5. *For every $k \in \mathbb{Z}_{11}$,*

$$\langle \chi_{J_{k,12}}, \chi_{12} \rangle = -\binom{12}{k} \binom{10}{k-1},$$

that is, the row $-12, -660, -9900, -59400, -166320, -232848, -166320, -59400, -9900, -660, -12$. So Conjecture (1) holds at $n = 12$, two rows past the range the source reports verified, and every $J_{k,12}$ is 12-nonsingular, hence 12-universal.

Proof and scope. Eleven exhaustive runs, one per cell, over the 2^{66} orientations of a 66-factor product — each carried out on $\sigma(J_{k,12})$ for the fixed permutation $\sigma = (11, 6, 3, 9, 0, 4, 8, 1, 10, 2, 7, 5)$ of the twelve vertices (written as the list of 0-indexed images), the result being multiplied by $\text{sgn}(\sigma) = -1$. The passage from that number to $\langle \chi_{J_{k,12}}, \chi_{12} \rangle$ is (3), checked in Proposition 4.4 but not formalised. Why the relabelling is needed, and what it buys, is Remark 7.1. Outside the formal development the whole row was recomputed by the independent C implementation (283 seconds, 1.06 gigabytes peak), and the cells $k = 1, 2, 11$ were recomputed a second time in the pattern's own, unrelabelled vertex order, returning $-12, -660, -12$; that is the check that the relabelling does not quietly change the answer. As before, universality is Fact 2.1. $\square$

The values now known, in absolute value (the sign is $(-1)^{n-1}$ throughout), are

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11
8	8	168	840	1400	840	168	8				
9	9	252	1764	4410	4410	1764	252	9			
10	10	360	3360	11760	17640	11760	3360	360	10		
11	11	495	5940	27720	58212	58212	27720	5940	495	11	
12	12	660	9900	59400	166320	232848	166320	59400	9900	660	12

with the bold entry the cell of Theorem 4.1 and the last two rows past the source's range.

Remark 4.6 (two columns of every row are already the source's). This should be said before the rows above are read as wholly new. The source's Example 6.3 evaluates $\langle \chi_{J_{1,n}}, \chi_n \rangle = (-1)^{n-1}n$ for every $n > 1$, so the $k = 1$ entry of every row, including the two new ones, is a published theorem, and with Fact 2.1 it makes $J_{1,n}$ n -universal there. And [1, Thm. 2.7] — “an easy consequence of a result of Košir and Sethuraman proven in” [6], it says — states that $J_{2,n}$ is n -universal for every $n \geq 3$, without evaluating the pairing. So what Theorems 4.3 and 4.5 add is the values at $2 \leq k \leq n-1$ and the universality at $3 \leq k \leq n-1$; at $n = 10$ the one cell in question, $k = 5$, lies in that range.

Proposition 4.7 (negative controls). *Replacing $\binom{n-2}{k-1}$ in (1) by the strictly larger $\binom{n-1}{k-1}$, or by the strictly smaller $\binom{n-3}{k-1}$, gives a claim that fails already at $n = 9$. The conjectured row is not a property of every pattern in $\mathcal{P}'_n$: $\langle \chi_{\Pi_9}, \chi_9 \rangle = 945 = 9!!$, which is none of the eight conjectured values at $n = 9$. And $\langle \chi_{J_{5,10}}, \chi_{10} \rangle$ is none of $-17639, -17641, +17640$.*

Proof and scope. Finite checks, computed by the same evaluators as the results they guard — a control recalled from memory is not a control. The first two refute a too-large and a too-small claim, so the computed numbers pin down the second binomial coefficient; the third shows the conjecture is not vacuous over $\mathcal{P}'_n$; the last excludes the two nearest integers and the wrong sign at the new cell. $\square$

5. THE CONJECTURED VALUES ARE n TIMES NARAYANA NUMBERS

Write $N(m, j) = \frac{1}{m} \binom{m}{j} \binom{m}{j-1}$ for the Narayana numbers.

Proposition 5.1. *For $2 \leq n \leq 20$ and $k \in \mathbb{Z}_{n-1}$,*

$$(n-1) \binom{n}{k} \binom{n-2}{k-1} = n \binom{n-1}{k-1} \binom{n-1}{k},$$

equivalently $\binom{n}{k} \binom{n-2}{k-1} = n N(n-1, k)$; and $\sum_{k=1}^{n-1} \binom{n}{k} \binom{n-2}{k-1} = \binom{2n-2}{n-1}$.

Proof and scope. Both are finite checks over $2 \leq n \leq 20$ in the division-free form displayed on the left. The identity is elementary binomial algebra and holds for all n ; what is checked here is the range in which it is used. The row sum follows from the first identity and $\sum_j N(m, j) = \text{Cat}(m) = \binom{2m}{m}/(m+1)$. $\square$

Remark 5.2. The triangle $\binom{n}{k} \binom{n-2}{k-1}$ is not new, and it would be wrong to present the numbers on the right of (1) as such. By Proposition 5.1 the row at n is n times Narayana row $n-1$, which is row $n-1$ of the triangle counting Grand Dyck paths of semilength $n-1$ by their number of double rises — entry A118963 of the OEIS [10], contributed in May 2006, two years before [1] was posted, with no reference and no link field and with no mention of zero patterns or unitary similarity. That entry has offset 1 and its rows are printed $T(m, 0), \dots, T(m, m-1)$, so its row m is the semilength- m row and has m entries; the identification above is $\binom{n}{k} \binom{n-2}{k-1} = T(n-1, k-1)$, which is its stated formula $T(m, j) = \frac{m+1}{m} \binom{m}{j} \binom{m}{j+1}$. Its row 9 is 10, 360, 3360, 11760, 17640, 11760, 3360, 360, 10, which contains the number of Theorem 4.1. Conversely [1] never writes “Narayana”, “Catalan” or “Dyck” — although it does use the OEIS to identify other sequences it meets, naming A006882 for the double factorials of the proposition quoted in §2. So the arithmetic value -17640 was available to anyone who evaluated the right-hand side; what was not available, and is Theorem 4.1, is that the signed sum over the 2^{45} orientations of $\chi_{J_{5,10}}$ equals it.

Remark 5.3. The identification also names a route to a proof of (1) for all n , which no further computation can supply: a sign-reversing involution on the orientations of $J_{k,n}$ surviving the staircase filter of (2), leaving n copies of the set of Grand Dyck paths of semilength $n-1$ with k double rises, would prove the conjecture outright. The machinery of [1] does not reach it: its Theorem 6.2 evaluates the pairing for the patterns $J(\sigma, \mathbf{i})$, which are assembled from the $n-1$ row sets $I_k = \{(r_k, \sigma(j)) : k < j \leq n\}$ and their transposes; and writing $J_{k,n} = A \sqcup (B \cup C)$ with $A = (0, 1) + \text{NE}_{n-1}$, the rows of A are such sets, but $B \cup C = \{(n, j) : j \leq k\} \cup \{(i, 1) : k < i < n\}$ is one of them only when $k = 1$ — exactly the one column the source proves.

6. PROBLEMS 3.4 AND 3.5 AT $n = 6$

By Lemma 2.3 both problems are questions about $\mathcal{P}'_n$, and $|\mathcal{P}'_n| = \binom{n(n-1)}{\mu_n}$ is 924, 184 756 and 155 117 520 at $n = 4, 5, 6$.

Proposition 6.1. *Both problems have affirmative answers at $n = 4$ and at $n = 5$: for every one of the $\binom{12}{6} = 924$ patterns of $\mathcal{P}'_4$ and every one of the $\binom{20}{10} = 184 756$ patterns of $\mathcal{P}'_5$, one has $\langle \chi_I, \chi_n \rangle \leq n!$ and $\langle \chi_I, \chi_I \rangle \geq n!$, with equality in the first exactly when $\chi_I = \chi_n$ and in the second exactly when $\chi_I = \pm \chi_n$.*

Proof and scope. Every pattern is enumerated as a bitmask over the $n(n-1)$ off-diagonal positions, and both pairings are computed from the unfolded polynomial definition of §3, with no reduction of any kind; the scan is additionally checked to have visited exactly 924 and 184 756 patterns, so it is exhaustion rather than a sample. This is the range the source reports verified, recomputed from its definitions. $\square$

Two exact reductions bring $n = 6$ into range. First, a pattern is a set of *ordered* positions, so each unordered pair $\{i, j\}$ carries a multiplicity $m_{ij} \in \{0, 1, 2\}$ and

$$(4) \quad \chi_I = (-1)^{r(I)} \prod_{i < j} (x_i - x_j)^{m_{ij}}, \quad r(I) = \#\{(i, j) \in I : i > j\}.$$

Hence $\langle \chi_I, \chi_I \rangle$ depends on I only through m , and $\langle \chi_I, \chi_n \rangle$ only through m and one sign; and since $\sum m_{ij} = \mu_6 = 15$ is odd, some pair has $m_{ij} = 1$, so both signs are realised by patterns with the same m and $\max_I \langle \chi_I, \chi_6 \rangle = \max_m |\langle \chi_m, \chi_6 \rangle|$. That replaces 155 117 520 patterns by $\#\{m \in \{0, 1, 2\}^{15} : \sum m = 15\} = 1\,787\,607$ multisets. Second, relabelling by $\sigma \in S_6$ multiplies $\langle \chi_I, \chi_6 \rangle$ by $\text{sgn}(\sigma)$ and fixes $\langle \chi_I, \chi_I \rangle$, so only orbit representatives need evaluating: 3012 of them.

Proposition 6.2. *At $n = 4$ and $n = 5$, where the reduction-free scan of Proposition 6.1 is also available, the multiset/orbit certificate — 12 and 110 representatives respectively — returns the same verdict.*

Proof and scope. An exhaustive check of the certificate at both sizes, against the reduction-free answer. The step (4) from patterns to multisets is the two-line argument above and is *not* formalised; this proposition is the empirical control on it. $\square$

Theorem 6.3. *For every $I \in \mathcal{P}'_6$,*

$$|\langle \chi_I, \chi_6 \rangle| \leq 720 = 6! \quad \text{and} \quad \langle \chi_I, \chi_I \rangle \geq 720,$$

and in either case equality holds exactly when $\chi_I = \pm \chi_6$. Hence Problems 3.4 and 3.5 both have affirmative answers at $n = 6$, the first case the source leaves open. Moreover $\#\{m \in \{0, 1, 2\}^{15} : \sum m = 15\} = 1\,787\,607$.

Proof and scope. The search is over $\mathcal{P}'_6$, of size $\binom{30}{15} = 155\,117\,520$, reduced by (4) to 1 787 607 multisets and then by the S_6 -action to 3012 orbit representatives. What is verified is the certificate consisting of those 3012 representatives, presented as base-3 codes: each has digit sum 15; each is the lexicographic minimum of its own S_6 -orbit, checked against all 720 permutations; the list is strictly increasing, so no representative repeats; the orbit sizes $6!/|\text{stab}|$ sum to 1 787 607, which is computed independently as the coefficient of q^{15} in $(1 + q + q^2)^{15}$ rather than asserted, so the 3012 orbits provably exhaust the multisets; and each representative satisfies both inequalities, with equality in either exactly at the all-ones multiset, i.e. exactly at $\chi_I = \pm \chi_6$. Nothing about how the 3012 codes were found is trusted. For Problem 3.4 as posed, the equality clause with $\chi_I = \chi_6$ follows because $\langle \chi_6, \chi_6 \rangle = 720 > 0$ selects the sign, and the extension from $\mathcal{P}'_6$ to $\mathcal{P}_6$ is Lemma 2.3. The reduction (4) is argued, not formalised; Proposition 6.2 is the control on it. Outside the formal development the reduced scan takes 1.67 seconds. $\square$

Proposition 6.4 (negative controls at $n = 6$). *Among the 3012 representatives, equality is attained exactly once on each side: $\#\{m : |\langle \chi_m, \chi_6 \rangle| = 720\} = 1$ and*

$\#\{m : \langle \chi_m, \chi_m \rangle = 720\} = 1$. So neither bound can be strengthened — ≤ 719 and ≥ 721 are both false — and the equality clause is about a case that occurs, the all-ones multiset being one of the listed representatives. Furthermore: dropping a single representative makes the orbit sizes fail to sum to 1 787 607; the same codes are rejected at $n = 5$, where $\mu_5 = 10$ makes every one of them fail the digit-sum test; and the same codes in reverse order are rejected, although every one of them is still canonical.

Proof and scope. Finite checks on the same certificate, computed by the same evaluators as the result they guard. The last three show in turn that the covering count is testing exhaustion rather than passing on any list, that canonicity is not vacuously true, and that the strict-increase requirement is what stops a list from repeating a representative and passing the covering count by luck. $\square$

7. COMPUTATIONAL NOTES

The two remarks in this section are measurements taken outside the formal development. They carry no theorem and are recorded because they are what made $n = 12$ reachable, and because one of them is an independent confirmation of Theorem 4.1.

Remark 7.1 (the vertex order is a knob worth turning). The cost of the block-closing dynamic program depends on the order in which the vertices are closed, because that order decides how early each exponent stops being tracked separately and collapses into the staircase mask (collapse (C2)); and the order in which the pattern happens to be written is not a good one. Sweeping 65 relabellings at $(n, k) = (10, 5)$, the peak number of live states ran from 151 208 to 1 379 171, a spread of 9.1, with the pattern’s own order second worst at about $1.38 \cdot 10^6$ states;¹ every one of the 65 returned -17640 , so the value of Theorem 4.1 came back from 65 different block decompositions and 65 different state spaces. At $n = 12$ the effect is decisive. In the pattern’s own vertex order the three cells that were run, $k = 1, 2, 11$, cost $8.4 \cdot 10^7$, $1.0 \cdot 10^8$ and $2.2 \cdot 10^7$ live states at 8.40, 8.41 and 2.10 gigabytes, and the interior cells were projected at 13–17 gigabytes; with the relabelling σ of Theorem 4.5 the largest cell of the whole row costs $8.3 \cdot 10^6$ states and 1.06 gigabytes, and the eleven cells together take 283 seconds. That is a twelvefold reduction in states, and it is what brought $n = 12$ inside the budget — at the price of a few-minute sweep run at a size that already finished.

Remark 7.2 (where the next cell stops). At $n = 13$ the obstruction is no longer the size of the search but the encoding of its states: the packed key needs $13 \cdot 4 + 13 = 65$ bits, one bit past a machine word, so it needs a 128-bit key — about 1.5 times the memory — together with its own relabelling sweep. With a good relabelling the state count is plausibly around $6 \cdot 10^7$, which is affordable. We have not run it.

8. VERIFICATION

Every statement of §§2–6 that carries a proposition, theorem or corollary number, except Lemma 2.3 and Fact 2.1, is formalised and machine-checked in Lean 4 [7];

¹Two runs of the pattern’s own order recorded that peak as 1 377 601 and as 1 377 615. We have not re-run the sweep and quote neither figure as exact; nothing here or below turns on the difference, which is one part in 10^5 .

there is no `sorry` anywhere in the development. The objects are defined in a module that imports nothing at all — not even `Mathlib` [8] — so that it compiles to a shared library and the finite checks run as compiled code rather than under an interpreter; the two evaluators of §3, the patterns, the certificate machinery of §6 and the range predicates all live there. Each finite check is discharged by Lean’s compiled evaluator (`native_decide`) and contributes one additional axiom naming that computation; the statements of §2, §3 and §4 other than Theorem 4.5 and Proposition 4.4, together with the reduction-free scans of Proposition 6.1, depend on nothing else; those two — which are the contents of the $n = 12$ module — additionally use `propext`, which enters through a `decide` in the permutation-sign code; and the certificate statements of §6 additionally use `propext`, `Classical.choice` and `Quot.sound`, all three entering through library functions in the certificate parser and the stabiliser count. The non-vanishing statements in Theorems 4.3 and 4.5 are derived from the displayed values by rewriting and kernel reduction and add no axiom. Checking the five result modules — the ranges $n \leq 9$, $n = 10$, $n = 11$, $n = 12$ and the $n = 6$ certificate — takes roughly 24, 132, 750, 1020 and 50 seconds respectively, at peak resident sizes between 1.6 and 4.0 gigabytes. These are wall times on a shared machine and are deliberately rounded: two adjacent runs of the first two modules differed by some 15% under different loads, and the $n = 6$ module was recorded at 45.8 and at 53.4 seconds in two runs. We have not re-run them and quote no figure here as exact. Three things are deliberately *not* formalised, and each is flagged where it is used: Fact 2.1 (the passage from a nonzero pairing to universality, two published theorems of [1]); the identity (3) used at $n = 12$, which is exercised in Proposition 4.4; and the pattern-to-multiset reduction (4), for which Proposition 6.2 is the control. Repository reference to be added at submission.

9. PROVENANCE, AND THE LIMITS OF THE SEARCH

Theorems 4.1, 4.3, 4.5 and 6.3 are offered as new, with the two columns of Remark 4.6 subtracted; the searches behind that claim were run in August 2026, each channel with a positive control, and their edges are as follows.

The arXiv listing for [1] was checked to have no version past v2, and all quotations above are read against v2. The citation lists of the arXiv and journal records were taken from two independent aggregators and agree: there are exactly two citing papers, [9] and [3]; both were read in full, each cites [1] in a single see-also line, and neither mentions Conjecture 3.6, Problem 3.4 or Problem 3.5. The lists of papers citing the three predecessors on zero patterns from 2007 — [4], [5] and [6], with 15, 2 and 4 citing papers respectively — were also read; none of them touches the pairing $\langle \chi_I, \chi_n \rangle$. A zbMATH search for the two phrases of the title returns exactly four documents, the source and three from 2007. Both authors’ subsequent output was scanned; neither returned to the topic. A local full-text corpus of some 881 000 arXiv papers contains the identifier of [1] exactly once, as the corpus’s own marker for the paper itself, and its title four times: two title lines and the bibliography entries of the two citing papers. Four general web searches returned nothing further.

Two limits are worth stating plainly. MathSciNet is paywalled and was not queried; and a paper proving Conjecture 3.6 while citing neither [1] nor using its vocabulary would evade every channel run. Read the negative result as “not found”, not as “new”.

The strongest argument against novelty is the one recorded in Remark 5.2, and we repeat it here rather than bury it: the right-hand side of (1) is trivially evaluable, and the whole triangle has been in the OEIS since 2006. The number -17640 was therefore never out of reach. What Theorem 4.1 adds is the left-hand side — that this particular signed sum over 2^{45} orientations of a 45-factor product of linear forms evaluates to -17640 and not to something else — and the reason that is not a routine computation is that the authors of [1] could not perform it: the naive route carries $4.32 \cdot 10^8$ live monomials at its middle layer, which is the obvious candidate for the memory wall they report, and the value came out only after the collapse (C2) cut that to about $1.38 \cdot 10^6$.

10. WHAT REMAINS

Proving Conjecture (1). This is the only thing that beats another row of data, and Remark 5.3 names a concrete route: a sign-reversing involution matching the surviving orientations of $J_{k,n}$ with n copies of the Grand Dyck paths of semilength $n - 1$ with k double rises. The source’s own Theorem 6.2 covers the column $k = 1$ (its Example 6.3) and no more.

The next cell. $n = 13$ is priced in Remark 7.2: it is blocked by the width of the packed state key rather than by the size of the search, and any attempt should sweep the vertex labelling before deciding a cell is out of range, the spread at $(10, 5)$ being a factor of 9.1.

Problems 3.4 and 3.5 at $n = 7$. Here the reduction that settled $n = 6$ does not carry over unchanged: $\#\{m \in \{0, 1, 2\}^{21} : \sum m = 21\} = 1\,105\,350\,729$ and $|\mathcal{P}_7| = \binom{42}{21} \approx 5.38 \cdot 10^{11}$, so the multiset reduction alone is not enough. The S_7 -orbit count is only about 219 000, but reaching it by canonicalising $1.1 \cdot 10^9$ multisets against 5040 permutations is on the order of 10^{12} operations; what is needed instead is orderly generation of the orbit representatives, extending a partial multiset only while it stays canonical. Even then each representative costs a 21-factor expansion over $\binom{27}{6} = 296\,010$ monomials, so Problem 3.5, which needs the whole coefficient vector, stays out of range; Problem 3.4 alone could use the collapse of §3 instead and is the case worth pricing properly.

Two bridges. Defining the pairing as an operation on multivariate polynomials inside a proof library, and proving the unfolded evaluator equal to it, would turn Proposition 3.1 from an agreement of two programs into a theorem; and formalising (4) would remove the one unformalised step under Theorem 6.3.

REFERENCES

- [1] Jinpeng An and Dragomir Ž. Đokovićoković, *Zero patterns and unitary similarity*, Journal of Algebra **324** (2010), no. 1, 51–80, doi:10.1016/j.jalgebra.2009.12.014; arXiv:0807.3580 (v1, 23 July 2008; v2, 17 December 2009). All quotations here are read against v2, which is the latest version, and every section, theorem, proposition, problem and example number cited above is v2’s.
- [2] Jinpeng An and Dragomir Ž. Đokovićoković, *Universal subspaces for compact Lie groups*, Journal für die reine und angewandte Mathematik **647** (2010), 151–173, doi:10.1515/crelle.2010.076; arXiv:0802.1915. The companion paper: [1] states that its Theorem 3.8 and Proposition 3.7 are extracted from Theorem 4.2 and Proposition 4.3 here.
- [3] Saurav Bhaumik and Arunava Mandal, *On universal subspaces for Lie groups*, Geometriae Dedicata **219** (2025), no. 6, paper no. 94, 14 pp., doi:10.1007/s10711-025-01050-1; arXiv:2204.01566. It is one of exactly two papers citing [1], and cites it in a single see-also line.

- [4] Dragomir Ž. Đokovićoković and Charles R. Johnson, *Unitarily achievable zero patterns and traces of words in A and A^** , Linear Algebra and its Applications **421** (2007), no. 1, 63–68, doi:10.1016/j.laa.2006.03.002.
- [5] Dragomir Ž. Đokovićoković, *Universal zero patterns for simultaneous similarity of several matrices*, Operators and Matrices **1** (2007), no. 1, 113–119, doi:10.7153/oam-01-07.
- [6] John Holbrook and Jean-Pierre Schoch, *Moving zeros among matrices*, Linear Algebra and its Applications **424** (2007), no. 1, 83–95, doi:10.1016/j.laa.2006.04.010. [1] cites this paper as carrying an appendix by T. Košir and B. Al. Sethuraman, and takes its Theorem 2.7 from a result proved there.
- [7] Leonardo de Moura and Sebastian Ullrich, *The Lean 4 Theorem Prover and Programming Language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, Cham, 2021, pp. 625–635, doi:10.1007/978-3-030-79876-5_37.
- [8] The mathlib Community, *The Lean Mathematical Library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381, doi:10.1145/3372885.3373824.
- [9] Vladimir Müller and Yuri Tomilov, *Matrix representations of arbitrary bounded operators on Hilbert spaces*, Journal für die reine und angewandte Mathematik **808** (2024), 111–141, doi:10.1515/crelle-2023-0095; arXiv:2206.14266. The other of the two papers citing [1]; it cites [1] once, adding that “we feel that this setting is rather different”.
- [10] The On-Line Encyclopedia of Integer Sequences, <https://oeis.org>. Sequence A118963, *Triangle read by rows: $T(n, k)$ is the number of Grand Dyck paths of semilength n that have k double rises*, contributed by Emeric Deutsch, 7 May 2006; also A001263 (Narayana) and A000984 (central binomial coefficients).