

TWO-NEIGHBORLY SIMPLICIAL 0/1-POLYTOPES

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ABSTRACT. A 0/1-polytope is the convex hull of a subset of the vertices of the unit cube. Ziegler proved that a simplicial 0/1-polytope of dimension d has at most $2d$ vertices and asked whether the equality case is always a cross polytope; Kaibel and Pokutta recently answered this in the negative with an explicit example in dimension 7, and asked in turn whether a simplicial 0/1-polytope can be 2-neighborly at all. We answer their question affirmatively by exhibiting simplicial 2-neighborly 0/1-polytopes in dimensions 4, 5, 6, 7, with 6, 8, 10, 12 vertices respectively; the three of dimension at least 5 have, in addition, no cube-antipodal pair of vertices, which is the stronger form in which the question is posed. Each example is certified twice and independently: through a complete facet enumeration, and through an explicit integer separating functional for every pair of vertices. We complement this with complete censuses of every subset of $\{0, 1\}^3$ and of $\{0, 1\}^4$, which show that in those dimensions the largest 2-neighborly simplicial example has $2d - 2$ vertices and that none of the $2d$ -vertex examples is 2-neighborly, and we recompute the invariants of Kaibel and Pokutta's dimension-7 examples from their definitions rather than from their tables. Every theorem and proposition below is machine-checked in Lean 4.

1. INTRODUCTION

A 0/1-*polytope* is a polytope $P = \text{conv } V$ with $V \subseteq \{0, 1\}^d$; every point of such a V is a vertex of P , so V is exactly the vertex set and $f_0(P) = |V|$. A d -polytope is *simplicial* if each of its facets is a $(d - 1)$ -simplex, and *2-neighborly* if every pair of its vertices spans an edge, that is, if its graph is complete. Write $\mathbf{1} = (1, \dots, 1)$ and call $v \mapsto \mathbf{1} - v$ the *cube antipodal map*; a pair $\{v, \mathbf{1} - v\} \subseteq V$ is a *cube-antipodal pair*, and V is *centrally symmetric* when $\mathbf{1} - v \in V$ for every $v \in V$.

Simplicial 0/1-polytopes are scarce. Ziegler [2, remark after Proposition 17] observed that a simplicial 0/1 d -polytope has at most $2d$ vertices: fix a coordinate j ; the hyperplanes $\{x_j = 0\}$ and $\{x_j = 1\}$ both support P , the two faces they cut out partition the vertex set, and each of the two is a simplex of dimension at most $d - 1$ and therefore carries at most d vertices. The 0/1-realizations of the d -dimensional cross polytope attain the bound, and Ziegler asked whether they are the only ones. Kaibel and Pokutta [1] state the question as follows.

Question 1.1 ([1, Question 1.1]). Is every simplicial d -dimensional 0/1-polytope with $2d$ vertices affinely isomorphic to the cross polytope $\text{conv}\{\pm e_1, \dots, \pm e_d\}$? Equivalently, is every such equality case centrally symmetric?

The answer is no: [1, Theorem 3.1] exhibits a set of 14 points of $\{0, 1\}^7$ whose hull is simplicial and not centrally symmetric, and [1, Theorem 4.3 and Table 1] report a complete classification of the equality case in dimensions up to 7, with exactly five exceptional B_7 -orbits. All five have five cube-antipodal pairs, and [1, Remark 3.3] notes that in the example of Theorem 3.1 those five pairs are exactly the five non-edges of the graph. This prompts the question we settle here.

Question 1.2 ([1, §5, Question 2]). All five examples with $d = 7$ have $d - 2$ cube-antipodal pairs. Which numbers of cube-antipodal pairs occur for $d \geq 8$? Are there simplicial 0/1-polytopes without any cube-antipodal pairs, i.e. are there any 2-neighborly simplicial 0/1-polytopes?

What is settled here. The last sentence of Question 1.2 has an affirmative answer, and not merely because a 0/1-simplex is trivially 2-neighborly. Theorem 3.1 produces simplicial 2-neighborly 0/1-polytopes of dimensions 4, 5, 6, 7 with 6, 8, 10, 12 vertices, and the three of dimension at least 5 have no cube-antipodal pair at all, so they answer both clauses of the question. The dimension-7 example

is worth singling out: it lives in the very dimension whose 14-vertex case [1] classifies completely, two vertices below that frontier.

The two clauses of Question 1.2 are joined by an “i.e.” that is licensed only at $|V| = 2d$. There the vertex set is forced to be balanced — each coordinate takes the value 1 on exactly d of the $2d$ vertices — so the barycentre is the interior point $\frac{1}{2}\mathbf{1}$, and a present cube-antipodal pair has an interior midpoint and cannot be an edge [1, §2]. Below $|V| = 2d$ this fails, and our four-dimensional example (Table 1) is the smallest illustration: it *has* a cube-antipodal pair and is 2-neighborly anyway.

Section 4 settles the equality-case reading in the two smallest interesting dimensions by brute force: a complete scan of all 2^8 subsets of $\{0, 1\}^3$ and all 2^{16} subsets of $\{0, 1\}^4$. No simplicial 0/1 d -polytope with $2d$ vertices is 2-neighborly for $d \in \{3, 4\}$; indeed every one of them is centrally symmetric there, so Question 1.1 has an affirmative answer in those dimensions, in agreement with the $d = 4$ row of [1, Table 1]. The censuses also locate the ceiling exactly: the largest number of vertices of a 2-neighborly simplicial 0/1 d -polytope is 4 at $d = 3$ and 6 at $d = 4$, i.e. $2d - 2$ in both cases — the value realized by our examples in dimensions 6 and 7. At $d = 3$ the scan itself covers every vertex number; at $d = 4$ it stops at $n = 2d$, above which Ziegler’s bound rather than the scan does the excluding (Remark 4.3). Section 6 records the resulting question, which we do not resolve.

Finally, Section 5 recomputes the data of [1, Theorem 3.1, Remarks 3.2 and 3.3, Tables 2 and 3] from the definitions. Nothing in that section is new; it is a reproduction, and it is reported because it is the control that makes the rest credible. The facet lists are not taken from [1]: they are recomputed by the exhaustive scan of Section 2, and only the bit strings are copied.

Relation to the literature. Two-neighborly 0/1-polytopes *without* the simpliciality hypothesis are a well-studied class. Maksimenko [3] enumerates all of them in dimension 7 and settles there the quantity $N_2(d)$, the maximum number of vertices of a 2-neighborly 0/1 d -polytope: $N_2(7) = 20$, with best examples of 28, 38 and 52 vertices in dimensions 8, 9, 10 against upper bounds 34, 68, 136. Since $N_2(7) = 20 > 14 = 2 \cdot 7$, the extremal 2-neighborly 0/1-polytopes of that dimension are far above Ziegler’s bound and hence are never simplicial. It is exactly the interaction of the two conditions that Question 1.2 asks about, and the examples below are small; a specialist who had asked the question would have found the four-dimensional one quickly. We report them because, as far as we could determine, nobody asked before [1].

2. PRELIMINARIES: CERTIFICATES FOR A FINITE POINT SET

Throughout, $V = \{v_1, \dots, v_n\} \subseteq \{0, 1\}^d$ is a set of distinct points and $P = \text{conv } V \subseteq \mathbb{R}^d$. We write $\hat{x} = (1, x) \in \mathbb{R}^{d+1}$ for the homogenization of $x \in \mathbb{R}^d$. All arithmetic below is over $\mathbb{Z}$; no floating point and no linear programming enter anywhere.

Complete facet enumeration. A facet of a d -polytope has affine dimension $d - 1$ and therefore contains d affinely independent vertices, so its supporting hyperplane is the affine hull of some d -subset of V . This makes the following scan complete. For each d -subset $S = \{p_1, \dots, p_d\} \subseteq V$ let

$$\text{val}_S(x) = \det \begin{pmatrix} 1 & x \\ 1 & p_1 \\ \vdots & \vdots \\ 1 & p_d \end{pmatrix},$$

an affine functional with integer coefficients, identically zero exactly when S is affinely dependent. Call S *facet-defining* if $\text{val}_S \not\equiv 0$ and val_S does not take both signs on V ; then, after orienting it so that $\text{val}_S \leq 0$ on V , the set $\{v \in V : \text{val}_S(v) = 0\}$ is the vertex set of a facet of P , and every facet arises this way. Expanding val_S along its first row expresses it by the $d + 1$ maximal minors of the $d \times (d + 1)$ matrix with rows $\hat{p}_i$; all of them are produced by a single division-free subset dynamic program over column masks, in $(d + 1)2^{d+1}$ integer operations, so a whole facet lattice costs $\binom{n}{d} \cdot (d + 1)2^{d+1}$ steps. For the 14-vertex example of [1] that is $3432 \cdot 2048 \approx 7 \cdot 10^6$ integer operations.

Once the facet list is known, P is simplicial precisely when every facet mask carries exactly d vertices; and for a simplicial P every subset of the vertex set of a facet is a face, so the numbers $f_0, \dots, f_{d-1}$ of faces of each dimension and the graph of P are read off as the sub-masks of the facet masks. The h -vector is then obtained from the f -vector by the usual

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}, \quad f_{-1} := 1.$$

Edge certificates. Two-neighborliness admits a second, independent certificate that does not use the facet list at all. An *edge certificate* for the pair $\{v_i, v_j\}$ is a vector $a \in \mathbb{Z}^{d+1}$ with

$$\langle a, \hat{v}_i \rangle = \langle a, \hat{v}_j \rangle = 0 \quad \text{and} \quad \langle a, \hat{v}_k \rangle < 0 \quad \text{for all } k \notin \{i, j\}.$$

Such an a is found by summing the oriented functionals of all facets containing both points — a face of a polytope is the intersection of the facets containing it — but its validity is then rechecked from the data alone, so the certificate is sound however it was produced. This matters: Proposition 2.1(i) below upgrades $\langle a, \hat{\cdot} \rangle \leq 0$ from V to P , so $F = P \cap \{x : \langle a, \hat{x} \rangle = 0\}$ is a face of P , and F is the convex hull of the points of V lying on it, namely the segment $[v_i, v_j]$. Thus each verified certificate exhibits one edge of P without appealing to the completeness of the d -subset scan.

What is and is not formalized. The arithmetic is formalized in full: every determinant, every count, every f - and h -vector is an exact integer computation on explicit data, checked inside Lean. Two statements of standard polytope theory are used but not formalized, and we flag them here rather than in passing.

- (F1) *A facet contains d affinely independent vertices.* This is what makes the d -subset scan a complete facet enumeration; without it the scan produces a list of genuine faces that might in principle be incomplete. All simpliciality claims below rest on it.
- (F2) *A coordinate slice of a simplicial 0/1 d -polytope has at most d points.* This is the pruning that makes the census of $\{0, 1\}^d$ affordable, and it is the same observation that gives Ziegler's bound. Proposition 4.1 runs the unpruned census in dimension 3 and Remark 4.3 records that the two agree there.

Three statements that do carry geometric content are formalized over $\mathbb{R}$, without any appeal to computation.

Proposition 2.1. *Let $d \geq 1$.*

- (i) *Let $a \in \mathbb{R}^d$, $b \in \mathbb{R}$ and $s \subseteq \mathbb{R}^d$ with $\sum_i a_i v_i \leq b$ for all $v \in s$. Then $\sum_i a_i x_i \leq b$ for every $x \in \text{conv } s$.*
- (ii) *Let $s \subseteq \mathbb{R}^d$ consist of points with all coordinates in $\{0, 1\}$, and let $v \in \{0, 1\}^d$ with $v \notin s$. Then $v \notin \text{conv } s$. Consequently every point of a 0/1-set is a vertex of its convex hull, and $f_0(\text{conv } V) = |V|$.*
- (iii) *Let $V \subseteq \{0, 1\}^d$ be finite and suppose that $|\{v \in V : v_j = b\}| \leq d$ for every coordinate j and every $b \in \{0, 1\}$. Then $|V| \leq 2d$.*

Proof. (i) is the minimality of the convex hull together with the convexity of a closed half-space. For (ii), consider $\varphi(x) = \langle 2v - \mathbf{1}, x \rangle$. On a 0/1 point u the value $\varphi(u)$ falls short of $\varphi(v)$ by exactly the Hamming distance from u to v : each coordinate where u and v agree contributes the same amount to both, and each coordinate where they differ contributes one less. Hence $\varphi(u) \leq \varphi(v) - 1$ for every $u \in s$, and by (i) the same bound holds on $\text{conv } s$, which excludes v . (iii) is a count: for a fixed coordinate j the two slices $\{v_j = 0\}$ and $\{v_j = 1\}$ partition V , and each has at most d elements. With the (unformalized) input (F2) this is Ziegler's bound $f_0(P) \leq 2d$. $\square$

Part (ii) is why the vertex count of a 0/1-polytope can be read off its defining data at all, and part (i) is what turns a certified functional into a supporting functional of the polytope rather than of the finite point list.

3. TWO-NEIGHBORLY SIMPLICIAL 0/1-POLYTOPES

We write 0/1 points as bit strings, so that 1001 denotes $(1, 0, 0, 1)$.

d	W_d	$ W_d $	facets	antipodal pairs
4	0000, 1000, 0100, 0010, 1001, 0111	6	9	1
5	00000, 10000, 01000, 10100, 11010, 11101, 11011, 00111	8	20	0
6	101011, 000100, 110001, 011011, 110101, 011010, 111110, 101110, 000011, 001001	10	44	0
7	0011100, 1100100, 1100001, 1000010, 0110110, 0001111, 1001011, 0010011, 1010011, 1111010, 0010100, 0111100	12	88	0

TABLE 1. The four witnesses of Theorem 3.1.

Theorem 3.1. *For $d \in \{4, 5, 6, 7\}$ let $W_d \subseteq \{0, 1\}^d$ be the set listed in Table 1. Then $\text{conv } W_d$ is a 2-neighborly simplicial 0/1-polytope of dimension d with $|W_d|$ vertices and the stated number of facets, and*

$$\begin{aligned}
 f(\text{conv } W_4) &= (6, 15, 18, 9), & h(\text{conv } W_4) &= (1, 2, 3, 2, 1), \\
 f(\text{conv } W_5) &= (8, 28, 52, 50, 20), & h(\text{conv } W_5) &= (1, 3, 6, 6, 3, 1), \\
 f(\text{conv } W_6) &= (10, 45, 114, 167, 132, 44), & h(\text{conv } W_6) &= (1, 4, 10, 14, 10, 4, 1), \\
 f(\text{conv } W_7) &= (12, 66, 208, 400, 468, 308, 88), & h(\text{conv } W_7) &= (1, 5, 15, 23, 23, 15, 5, 1).
 \end{aligned}$$

None of the four is centrally symmetric. The sets W_5, W_6, W_7 contain no cube-antipodal pair; W_4 contains exactly one, namely $\{1000, 0111\}$.

Corollary 3.2. *Question 1.2 has an affirmative answer in both of its clauses: there are 2-neighborly simplicial 0/1-polytopes, and there are simplicial 0/1-polytopes with no cube-antipodal pair. Neither conclusion holds at $|W| = 2d$; see Section 4.*

Proof of Theorem 3.1. Each assertion is a finite exact computation, run inside the Lean kernel by the scheme of Section 2. For each d the complete facet enumeration scans all $\binom{|W_d|}{d}$ subsets — 15, 56, 210 and 792 subsets respectively — computes the $d + 1$ maximal minors of each by the subset dynamic program, keeps the subsets whose functional does not change sign on W_d , and deduplicates by the tight-vertex mask. Full dimensionality is a separate scan over $(d + 1)$ -subsets for a nonvanishing homogeneous determinant. Simpliciality is then the statement that every one of the 9, 20, 44, 88 facet masks has exactly d bits set; the f -vector is the count of sub-masks of facet masks by cardinality, over all $2^{|W_d|}$ masks; the graph is the same count restricted to pairs, and it has no non-edge in any of the four cases; the h -vector is the displayed linear transform of the f -vector. Central symmetry and the cube-antipodal pair count are direct checks on the point sets. The completeness of the facet enumeration is (F1); everything else is certified. $\square$

Proposition 3.3. *For each $d \in \{4, 5, 6, 7\}$ and each of the $\binom{|W_d|}{2}$ pairs of points of W_d — 15, 28, 45 and 66 pairs respectively — there is an integer edge certificate in the sense of Section 2. Consequently $\text{conv } W_d$ is 2-neighborly.*

Proof. The certificate for a pair is produced by summing the oriented integer functionals of the facets containing both points, and is then verified from the data: its value is zero at the two points of the pair and strictly negative at every other point of W_d . All $15 + 28 + 45 + 66 = 154$ certificates are exhibited and verified. Soundness is the argument of Section 2, which uses Proposition 2.1(i) and (ii) and does not use (F1). $\square$

Remark 3.4. Proposition 3.3 is therefore logically independent of Theorem 3.1's facet route, and its 2-neighborliness conclusion does not depend on the unformalized completeness statement (F1). What still depends on (F1) is simpliciality.

Remark 3.5. The f - and h -vectors of $\text{conv } W_4$ are those of the cyclic polytope $C(6, 4)$ [4], but the agreement is forced rather than informative. In a simplicial 4-polytope every facet is a tetrahedron, so $4f_3 = 2f_2$, and Euler's relation $f_0 - f_1 + f_2 - f_3 = 0$ then gives $f_3 = f_1 - f_0$ and $f_2 = 2(f_1 - f_0)$: every 2-neighborly simplicial 4-polytope with 6 vertices has $f = (6, 15, 18, 9)$ and $h = (1, 2, 3, 2, 1)$, and $C(6, 4)$ is one such polytope. We claim no combinatorial isomorphism: only the vectors were compared.

Remark 3.6. The four sets were found by randomized local search over subsets of $\{0, 1\}^d$, outside the formal development; the two examples at $2d - 2$ vertices were found quickly, in about one second at $d = 6$ and about nine minutes at $d = 7$. The search plays no role in the verification: what is checked is the exhibited point set.

4. COMPLETE CENSUSES IN DIMENSIONS 3 AND 4

The intended reading of Question 1.2 places it in the equality case $|V| = 2d$, where by [1, §2] a cube-antipodal pair is automatically a non-edge. In the two smallest dimensions where a simplicial 0/1-polytope with $2d$ vertices exists at all, the question can be answered by brute force. For a subset $V \subseteq \{0, 1\}^d$ write the triple

($\text{conv } V$ is a simplicial d -polytope, and 2-neighborly, and V centrally symmetric),

and let the census in dimension d record, for each n , how many n -element subsets satisfy the first, the first two, and the first and third conditions.

Proposition 4.1. *The census of all 256 subsets of $\{0, 1\}^3$, with no pruning whatever, is*

n	4	5	6	7	8
<i>simplicial</i>	58	8	4	0	0
<i>of those 2-neighborly</i>	58	0	0	0	0
<i>of those centrally symmetric</i>	0	0	4	0	0

and the entries for $n \leq 3$ vanish.

Three consequences, all inside dimension 3. There is no simplicial 0/1 3-polytope with more than $2d = 6$ vertices, so Ziegler's bound is attained and cannot be beaten — and here that is a theorem about the scan, not an application of (F2). All four examples with 6 vertices are centrally symmetric, so Question 1.1 has an affirmative answer at $d = 3$; these four are the 0/1-octahedra. And no example with 6 vertices is 2-neighborly, while the largest n admitting a 2-neighborly simplicial example is $n = 4 = 2d - 2$.

Proposition 4.2. *Among all 65 536 subsets of $\{0, 1\}^4$, the census for $5 \leq n \leq 8$ is*

n	5	6	7	8
<i>simplicial</i>	3008	1088	528	58
<i>of those 2-neighborly</i>	3008	912	0	0
<i>of those centrally symmetric</i>	0	0	0	58

So all 58 simplicial examples with the maximal $2d = 8$ vertices are centrally symmetric: Question 1.1 has an affirmative answer at $d = 4$ as well, in agreement with the $d = 4$ row of [1, Table 1], which reports 3 classes, all of them cross polytopes. None of the 58 is 2-neighborly, and the largest n admitting a 2-neighborly simplicial example is $n = 6 = 2d - 2$, realized 912 times.

Remark 4.3. The dimension-4 census is run over all 65 536 subsets, but a subset is passed to the facet enumeration only when $5 \leq n \leq 8$ and every coordinate slice has at most 4 points; both filters are necessary conditions for simpliciality by (F2), and the upper limit $n \leq 2d$ is Ziegler's bound. Accordingly Proposition 4.2 is a complete classification for $5 \leq n \leq 8$ and says nothing about $n > 8$, whereas Proposition 4.1 scans without any filter and therefore does settle $n \in \{7, 8\}$ in dimension 3. As a check that the filter loses nothing, the pruned and unpruned dimension-3 censuses were computed separately and agree in every entry with $n \leq 6$.

Remark 4.4. The count 58 at $n = 4$, $d = 3$ is the number of affinely independent quadruples of cube vertices: $\binom{8}{4} = 70$ minus the 12 coplanar quadruples, namely the 6 facets and the 6 diagonal rectangles.

Remark 4.5. The computations in this remark were run in C, outside the formal development. They are *not* machine-checked, they are reported as such here and in Section 7, and no statement of this paper depends on them. We ran the analogous complete exhaustion in dimension 5, over all $\binom{32}{n}$ subsets for $6 \leq n \leq 10$ — about $1.07 \cdot 10^8$ subsets in total. The largest n admitting a 2-neighborly simplicial example is again $2d - 2 = 8$, all 3008 examples with $n = 10 = 2d$ are centrally symmetric, and none of them is 2-neighborly. The same computation reproduced the orbit counts 3 and 7 of [1, Table 1] in dimensions 4 and 5, under the hyperoctahedral groups B_4 and B_5 .

Negative controls. An exhaustive census is only as good as the predicate it evaluates, so we record that the predicate is neither vacuously true nor vacuously false on genuine 0/1-polytopes.

Proposition 4.6. (i) *The 3-cube $\text{conv}\{0, 1\}^3$ has 6 facets, the largest carrying $4 > 3$ vertices, and 4 non-edges; it is neither simplicial nor 2-neighborly.*
(ii) *The 4-cube $\text{conv}\{0, 1\}^4$ has 8 facets, the largest carrying $8 > 4$ vertices; it is not simplicial.*
(iii) *Let $X = \{0000, 1000, 0100, 0010, 1101, 1011, 0111, 1111\}$. Then $\text{conv } X$ is a simplicial 0/1 4-polytope with the maximal $2d = 8$ vertices, centrally symmetric, with $16 = 2^4$ facets, $f = (8, 24, 32, 16)$ and $h = (1, 4, 6, 4, 1)$: a 0/1-realization of the 4-dimensional cross polytope. It is not 2-neighborly; its four non-edges are its four cube-antipodal pairs.*

Part (iii) is the non-vacuity control for Question 1.1 in dimension 4: the class the census pronounces on is nonempty. Together with W_4 of Theorem 3.1 — simplicial and 2-neighborly with $6 < 2d$ vertices — and part (ii) — 2^d vertices, not simplicial — the three conditions “simplicial”, “ $2d$ vertices” and “2-neighborly” are pairwise separated.

5. THE KAIBEL–POKUTTA EXAMPLES, RECOMPUTED

Nothing in this section is new. It is reported because the same enumeration code produces both these numbers and those of Section 3, and every quantity here can be compared against [1]. The 98 bits of the vertex set of [1, Theorem 3.1] are the only thing copied; each bit string is additionally re-read as a binary numeral, so a transcription error would have to survive two independent encodings.

Theorem 5.1 ([1, Theorem 3.1 and Remark 3.3]). *Let*

$$V = \{0010110, 1011101, 1000100, 1001010, 0111000, 1100001, 0010001, \\ 0001100, 0100010, 1001111, 1101110, 0110101, 1110011, 0111011\}.$$

Then $P = \text{conv } V$ is a simplicial 0/1-polytope of dimension 7 with $14 = 2 \cdot 7$ vertices and 136 facets, each facet carrying exactly 7 vertices, and P is not centrally symmetric: the four points 0010110, 0111000, 1100001, 1001111 have no antipode in V . Moreover

$$f(P) = (14, 86, 292, 590, 712, 476, 136), \quad h(P) = (1, 7, 23, 37, 37, 23, 7, 1),$$

exactly five cube-antipodal pairs are present in V , and the graph of P has exactly five non-edges. In particular P is not 2-neighborly.

The proof is one run of the enumeration of Section 2: all $\binom{14}{7} = 3432$ seven-subsets are scanned, 136 of them are facet-defining after deduplication, and the remaining assertions are counted from the resulting masks. Since P is not centrally symmetric, it answers Question 1.1 in the negative. The equivalence of the two readings of that question is quoted from [1] and is not reproved here; the direction we use is [1, Lemma 4.2], that a centrally symmetric d -polytope with exactly $2d$ vertices is affinely isomorphic to the d -dimensional cross polytope.

Proposition 5.2 ([1, Remark 3.2, Tables 2 and 3]). *Let $V_1, \dots, V_5 \subseteq \{0, 1\}^7$ be the five orbit representatives of [1, Table 3]. Each $\text{conv } V_i$ is a simplicial 0/1 7-polytope with 14 vertices that is not*

centrally symmetric, and each V_i has exactly five cube-antipodal pairs and exactly five non-edges — the same five pairs, since at $|V| = 2d$ a present cube-antipodal pair cannot be an edge. They fall into the two combinatorial types of [1, Table 2]: $\text{conv } V_1$ has 144 facets, $f = (14, 86, 296, 610, 748, 504, 144)$ and $h = (1, 7, 23, 41, 41, 23, 7, 1)$, while $\text{conv } V_2, \text{conv } V_3, \text{conv } V_4, \text{conv } V_5$ each have 136 facets and the f - and h -vectors of Theorem 5.1.

Furthermore, let V' be obtained from the set V of Theorem 5.1 by exchanging the fourth coordinates of 0111000 and 1100001, as in [1, Remark 3.2]. Then $\text{conv } V'$ is a simplicial 0/1 7-polytope with 14 vertices that is centrally symmetric, with $128 = 2^7$ facets, $f = (14, 84, 280, 560, 672, 448, 128)$ and $h = (1, 7, 21, 35, 35, 21, 7, 1)$ — the 7-dimensional cross polytope — and its graph has seven non-edges, one for each of its seven cube-antipodal pairs.

Every number above was recomputed by the scan and none disagrees with [1]. The exception is the classification itself: we did not re-run the enumeration of *all* simplicial 0/1 7-polytopes with 14 vertices, so the counts 3, 7, 63, 1631 of [1, Table 1] in dimensions 4 to 7, and its splitting of the 1631 into 1626 cross polytopes and 5 B_7 -orbits that are not centrally symmetric, are used here only as quoted from [1].

Remark 5.3. Combining the quoted classification with the observation that every one of the 1631 examples has five or seven cube-antipodal pairs — seven for the 1626 centrally symmetric ones, and five for the other five by [1, §4] — hence at least five non-edges, gives: no simplicial 0/1 7-polytope with 14 vertices is 2-neighborly. This is a consequence of [1] that the source does not draw, and it is what makes the twelve-vertex example W_7 of Theorem 3.1 sharp-looking rather than merely small: in dimension 7 the answer is no at 14 vertices and yes at 12.

6. WHAT REMAINS OPEN

The equality case is where Question 1.2 was posed and where it is still open. Collecting what is known, let $\nu(d)$ denote the largest n for which there is a 2-neighborly simplicial 0/1 d -polytope with n vertices. Propositions 4.1 and 4.2 give $\nu(3) = 4$ and $\nu(4) = 6$; the dimension-5 exhaustion of Remark 4.5 gives $\nu(5) = 8$; and Theorem 3.1 gives $\nu(6) \geq 10$ and $\nu(7) \geq 12$. In every case the value is exactly $2d - 2$. (In the $d = 4$ and $d = 5$ scans the range $n > 2d$ is excluded by Ziegler's bound rather than scanned; the $d = 3$ scan covers it.)

Question 6.1. Is $\nu(d) = 2d - 2$ for every $d \geq 3$?

An affirmative answer would settle Question 1.2 negatively in its equality-case reading for every d , since $2d - 2 < 2d$. We emphasize that nothing here proves the upper half. Our evidence for it above dimension 5 is only that randomized searches at $n = 2d - 1$ failed: in dimension 6 at $n = 11$ a search of about $1.25 \cdot 10^6$ evaluations never produced fewer than three non-edges, and in dimension 7 at $n = 13$ a shorter search stalled at four. A failed search is not an exhaustion. The smallest untouched case is a complete exhaustion in dimension 6 at $n = 11$ or $n = 12$, which the slice condition (F2) brings within reach even though $\binom{64}{12} \approx 3.3 \cdot 10^{12}$ is not.

Question 1.2 also asks which numbers of cube-antipodal pairs occur for $d \geq 8$, and we add nothing to that beyond the equality-case observation of [1]: at $|V| = 2d$ a 2-neighborly example must be cube-antipodal-free. A search in dimension 8 at $n = 16$ over balanced, cube-antipodal-free configurations, run outside the formal development for about an hour, found none; the same search in dimension 7 at $n = 14$, where [1]'s classification proves none exists, behaves identically, which is all one can say. Dimension 8 at 16 vertices is open.

Finally, one structural reduction is worth restating because it makes Question 1.2 a question about pairs of 0/1-simplices rather than about polytopes. By [1, §4], a simplicial 0/1 d -polytope with $|V| = 2d$ splits along its last coordinate as $V = (A \times \{0\}) \cup (B \times \{1\})$, where $A, B \subseteq \{0, 1\}^{d-1}$ are the vertex sets of the two coordinate faces and each is an affinely independent set of d points; writing $B' = \mathbf{1} - B$, balance says exactly that A and B' have the same column-sum vector, and V is centrally symmetric if and only if $A = B'$. The cube antipode of $(a, 0)$ is $(\mathbf{1} - a, 1)$, which lies in $B \times \{1\}$ precisely when $a \in B'$, so the cube-antipodal pairs of V correspond to the elements of $A \cap B'$, and a

2-neighborly example at $|V| = 2d$ requires $A \cap B' = \emptyset$. The data available says that $|A \cap B'|$ is never small: it equals d in every example of dimensions 3, 4 and 5, it equals 6 in dimension 6, and it is 5 or 7 in dimension 7 [1, Table 1 and §4]. A theorem forcing $A \cap B' \neq \emptyset$ for every d would settle the equality-case reading of Question 1.2 outright.

7. VERIFICATION

Every theorem and proposition above has been formally verified in Lean 4; the development is free of **sorry**, and the repository reference is to be added at submission. Proposition 2.1 is proved against *Mathlib* by ordinary mathematical reasoning over $\mathbb{R}$ and depends on no computation: its axiom set is the standard **propext**, **Classical.choice**, **Quot.sound**. The remaining statements are finite exact computations over $\mathbb{Z}$ and are checked by compiled evaluation (**native_decide**), so each of them additionally depends on native reduction axioms, one per compiled evaluation; this is a deliberate trade, since the same computations run through the kernel evaluator would take hours rather than minutes. The certificate layer imports no library at all and is written in core Lean. Total elapsed verification time for the whole development is under nine minutes on one core: about five and a half minutes for Theorem 5.1 and Proposition 5.2, two minutes for the censuses of Section 4, twenty seconds for Theorem 3.1 and Proposition 3.3, and half a minute for Proposition 2.1. The two statements (F1) and (F2) of Section 2 are the only unformalized inputs, and Section 2 records exactly which conclusions rest on them. The computations reported in Remark 4.5 and in Section 6 were run outside the formal development and are not machine-checked.

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