RUNS OF CONSECUTIVE OUTER VERTICES IN GENERALIZED PETERSEN GRAPHS, AND TWO ERRONEOUS PUBLISHED VALUES OF THE ZERO FORCING NUMBER

KOREA SUPERINTELLIGENCE LABS

Abstract. Let $P(n,k)$ be the generalized Petersen graph and Z the zero forcing number. The standard upper bound $Z(P(n,k)) \le 2k+2$ is witnessed by a run of $2k+2$ consecutive vertices of the outer cycle. We prove that this witness cannot be shortened inside its own shape: for every $k \ge 1$, every $n \ge \max(2k+3,\, 5k-1)$ and every starting index j, the shortest run of consecutive outer vertices that is a zero forcing set of $P(n,k)$ has length exactly $2k+2$. The lower half of that statement is proved by exhibiting, for each k, an explicit set of $6k-2$ vertices — a number independent of n — which contains the run of length $2k+1$, misses the next outer vertex, and is closed under the colour-change rule. At $k=3$ this turns the negative half of a computation Krishnan reports for $10 \le n \le 300$ — that seven consecutive outer vertices do not force — into a theorem for every $n \ge 13$, and gives the first statement we are aware of that rules out a family of candidate 7-element sets uniformly in n, which is the shape of the open half of his Conjecture 5. Alongside these we record a lower bound $Z(P(n,k)) \ge 4$ valid for all n with $2k < n$ and $n \ne 3k$, the resulting exact value $Z(P(n,1)) = 4$ for all $n \ge 5$, and two published values of Z that are wrong: the claim $Z(P(n,3)) = 8$ for $n \ge 12$ of Rashidi, Shajareh Poursalavati and Tavakkoli, whose failure at $n = 12$ was found by Krishnan and is confirmed here, and the entry $Z(P(7,2)) = 5$ printed in the $\ell = 0$ column of Table 1 of arXiv:2508.02564v1, whose true value is 6. Every theorem here is machine-checked in Lean 4, and each one quantified over all n is proved without computation.

1. Introduction

The parameter. Let $G = (V,E)$ be a finite simple graph. Colour a subset $S \subseteq V$ black and the rest white, and iterate the *colour-change rule*:

a black vertex with exactly one white neighbour forces that neighbour black.

Iterating until no force remains gives the *closure* $\mathrm{cl}(S)$. The set S is a *zero forcing set* if $\mathrm{cl}(S) = V$, and the *zero forcing number* $Z(G)$ is the least size of one. The parameter was introduced by the AIM Minimum Rank – Special Graphs Work Group [1] as a combinatorial upper bound for the maximum nullity of the symmetric matrices with a prescribed zero–nonzero pattern, and Z has been computed or bounded for many families since.

The graphs. For $n \ge 3$ and $1 \le k < n/2$ the *generalized Petersen graph* $P(n,k)$ has vertex set $\{u_0,\dots,u_{n-1}\} \cup \{v_0,\dots,v_{n-1}\}$ and edges the *outer cycle* $u_i \sim u_{i+1}$, the *spokes* $u_i \sim v_i$ and the *inner edges* $v_i \sim v_{i+k}$, all indices read modulo n. It is cubic on $2n$ vertices; $P(5,2)$ is the Petersen graph and $P(n,1)$ is the n-prism.

Two published upper bounds for $Z(P(n,k))$ agree on the value $2k+2$. Alameda, Curl, Grez, Hogben, Kingston, Schulte, Young and Young [2, Thm. 2.2] prove $Z(P(n,k)) \le 2k+2$ using the set $\{v_0, u_0, \dots, u_{2k-1}, v_{2k-1}\}$; Rashidi, Shajareh Poursalavati and Tavakkoli [8, Thm. 2.2] prove it using the *run of consecutive outer vertices* $\{u_1, u_2, \dots, u_{2k+2}\}$. In the other direction the only general lower bound we found in the literature is $Z(P(n,k)) \ge \min\{k, \lfloor n/k \rfloor\} + 1$ for $k \ge 3$ [8, Cor. 3.5], which never exceeds $k+1$ and gives 4 at $k=3$ once $n \ge 9$. (Corollary 3.5 is printed with a bare n/k; the floor is supplied by the same bound as written in that paper's introduction.)

Exact values are known in a few slices. Herrman [5] gives $Z(P(n,1)) = 4$ for $n \ge 4$; [8, Thm. 2.5] gives $Z(P(n,2)) = 6$ for $n \ge 10$ and [8, Thm. 2.6] gives $Z(P(2k+1,k)) = 6$ for $k \ge 5$. A further slice comes from the maximum nullity M: [2, Thm. 2.4] proves that if the adjacency matrix $A(P(n,k))$

has an eigenvalue of multiplicity $2k+2$, then $M(P(nr,k)) = Z(P(nr,k)) = 2k+2$ for every integer $r \geq 1$, and states two instances, $Z(P(15r,2)) = 6$ and $Z(P(24r,5)) = 12$, seeded by the eigenvalue -2 of multiplicity 6 in $A(P(15,2))$ and the eigenvalue 0 of multiplicity 12 in $A(P(24,5))$. So $Z = 2k+2$ is known along the two arithmetic progressions $15 \mid n$ at $k = 2$ and $24 \mid n$ at $k = 5$.

The case $k = 3$, and where it stands. Theorem 3.6 of [8] asserts $Z(P(n,3)) = 8$ for every $n \geq 12$. Krishnan [6] showed that this is false at $n = 12$, the smallest case in its own range: $\{u_0, \ldots, u_6\}$ is a zero forcing set of $P(12,3)$ and no six-element set is, so $Z(P(12,3)) = 7$. He recomputed the whole table,

n	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$Z(P(n,3))$	6	6	6	8	7	7	8	8	8	8	8	8	8	8

proved the computer-free upper bound $Z(P(n,3)) \leq 8$ for all $n \geq 9$ [6, Prop. 4], and conjectured that the published value holds from one step further out.

Conjecture 1.1 ([6, Conj. 5])**.** $Z(P(n,3)) = 8$ for every $n \geq 13$.

What is missing is a lower bound: some argument that rules out all $\binom{2n}{7}$ seven-element sets, uniformly in n. Krishnan is explicit that he will not promote his finite computation to a theorem: he "will not assert it from finite computation, which would reproduce the gap this note corrects". The present paper does not close that gap either. What it does is prove, for every k and uniformly in n, the first instance of a statement of the required shape.

Results. Write

$$R_j(m) \;=\; \{u_j, u_{j+1}, \ldots, u_{j+m-1}\} \qquad \text{(indices mod } n), \qquad R(m) := R_0(m),$$

for the run of m consecutive outer vertices starting at u_j. Our main theorem is that the witness of [8, Thm. 2.2] is optimal within this shape, at every k.

- **The stall** (Theorem 4.6). For every $k \geq 1$ and every $n \geq 5k-1$, the run $R(2k+1)$ is *not* a zero forcing set of $P(n,k)$; nor is any shorter run. The proof exhibits a set B_k of exactly $6k-2$ vertices — a size that does not depend on n — which contains $R(2k+1)$, misses u_{2k+1}, and is closed under the colour-change rule.
- **Exact optimality** (Theorem 4.11). For every $k \geq 1$, every $n \geq \max(2k+3, 5k-1)$ and every j, the least m for which $R_j(m)$ is a zero forcing set of $P(n,k)$ is exactly $2k+2$. In particular whether a run of consecutive outer vertices forces depends only on its length.
- **A wrong published value** (Theorem 6.2). $Z(P(7,2)) = 6$, whereas Table 1 of [7] prints 5 in that cell.

Section 5 records two supporting statements of a different flavour: closed neighbourhoods in $P(n,k)$ are closed under the colour-change rule whenever $2k < n$ and $n \neq 3k$, whence $Z(P(n,k)) \geq 4$ uniformly in n (Proposition 5.2), and therefore $Z(P(n,1)) = 4$ for all $n \geq 5$ (Proposition 5.3) — Herrman's value [5], now with a proof that runs over all n at once and uses no computation. Section 6 treats the two published tables.

What is not claimed. The upper bound $Z(P(n,k)) \leq 2k+2$ is not ours; it is [8, Thm. 2.2] and [2, Thm. 2.2], and Proposition 4.1 restates it only to fix the threshold on n that the consecutive-run witness actually requires. The refutation of [8, Thm. 3.6] is Krishnan's [6]; Section 6 confirms it and nothing more. The value $Z(P(n,1)) = 4$ is Herrman's [5]. Conjecture 1.1 remains open: Theorem 4.6 eliminates a single orbit of seven-element sets under the rotation of the graph, out of $\binom{2n}{7}$, and its proof is $6k-2$ neighbour-list checks. We claim for it only that it is uniform in both parameters, where nothing uniform in n was available before.

Finally, [7] studies ℓ-*leaky* forcing, in which ℓ vertices are chosen adversarially as *leaks*, a black vertex that is a leak never forces, and the initial set must succeed against every placement of the ℓ leaks. That is a different colour-change rule. Only the $\ell = 0$ column of its Table 1 is examined here, because [7] states $Z_{(0)}(G) = Z(G)$; the $\ell \geq 1$ columns are outside the scope of this paper and no claim whatever is made about them.

2. Preliminaries

The rule, made precise. Fix a finite simple graph G on the vertex set V. For $S \subseteq V$ and $x \in V$ let $W_S(x)$ be the set of white neighbours of x, that is $N(x) \setminus S$. One *round* of the rule applies every available force simultaneously:

$$\sigma(S) \;=\; S \;\cup\; \{\, y : \exists\, x \in S \text{ with } W_S(x) = \{y\} \,\},$$

and we write $\mathrm{cl}_t(S) = \sigma^t(S)$. We call S a *zero forcing set* if $\mathrm{cl}_t(S) = V$ for some t, and $Z(G)$ is the least cardinality of one. A set B is *closed* if $\sigma(B) = B$, i.e. if no vertex of B has exactly one white neighbour; then $\mathrm{cl}_t(B) = B$ for every t.

Two facts about this iteration carry every later argument, and neither should be assumed.

Lemma 2.1. *The round is monotone: $S \subseteq T$ implies $\sigma(S) \subseteq \sigma(T)$, and hence $\mathrm{cl}_t(S) \subseteq \mathrm{cl}_t(T)$ for every t. Consequently, if $S \subseteq T$ and S is a zero forcing set then so is T.*

Monotonicity of the *round* is not inherited from the rule at a single vertex, which is genuinely non-monotone: blackening a neighbour of x can drop the number of white neighbours of x from one to zero and destroy its ability to force. In $P(n,3)$, for instance, u_1 forces v_1 from $\{u_0, u_1, u_2\}$ and forces nothing from $\{u_0, u_1, u_2, v_1\}$. What survives the passage to the round is the statement above, and it is what makes a single search over sets of one fixed size decisive.

Lemma 2.2. *$|V|$ rounds always suffice: S is a zero forcing set if and only if $\mathrm{cl}_{|V|}(S) = V$. Consequently, if no m-element subset of V is a zero forcing set and $m \leq |V|$, then no subset of size at most m is one, and $Z(G) > m$.*

Proof sketch. The sets $\mathrm{cl}_t(S)$ increase with t and live in a set of size $|V|$, so $\mathrm{cl}_{t+1}(S) = \mathrm{cl}_t(S)$ for some $t \leq |V|$; since the round depends on the black set only through membership, that fixed point persists for all later t. The second sentence combines this with Lemma 2.1: a forcing set of size $< m$ extends to a forcing set of size exactly m. □

Lemma 2.2 is the reason a failed computation proves anything at all: it converts the existential "for some t" in the definition into a bounded computation.

Certifying an exact value. Every exact value quoted below is established in two halves: a witness, one set of size z whose closure is all of V, and an exhaustion, the verification that *no* $(z-1)$-element subset has that property. The step from "no $(z-1)$-element set forces" to "no set of size $< z$ forces" is Lemma 2.2, not a computation. Thus a value $Z(P(n,k)) = z$ costs one closure plus $\binom{2n}{z-1}$ closures.

Remark 2.3. The exhaustions are run on an implementation of the rule in which a black set is a bit mask and one round is a fold over the vertices. That implementation is not assumed to compute the rule of Section 2: the identity "mask of $\sigma(S)$ = one fold step applied to the mask of S" is itself proved, so the fast engine and the readable rule cannot disagree. Completeness of the enumeration is likewise proved rather than assumed: every m-element set of vertices is a permutation of a sublist of the vertex list, every such sublist is enumerated, and a mask does not distinguish permutations.

Rotation. Let ρ be the index shift $u_i \mapsto u_{i+1}$, $v_i \mapsto v_{i+1}$ (indices mod n).

Lemma 2.4. *ρ is an automorphism of $P(n,k)$, and forcing transports along it: if $\rho(S) \subseteq T$ then $\rho(\mathrm{cl}_t(S)) \subseteq \mathrm{cl}_t(T)$ for every t.*

The form stated here is one-directional and against an arbitrary superset T of $\rho(S)$, and in that form it needs neither injectivity nor surjectivity of ρ: in the branch where the image of a vertex is already in T there is nothing to force. This matters twice below. Taking $T = \mathrm{cl}_3(S)$ replaces a cardinality argument by an induction in the proof of Proposition 4.1; and the converse direction, needed only in Theorem 4.11, is obtained from surjectivity alone.

Lemma 2.5. *Every vertex of $P(n,k)$ is ρ of a vertex, and therefore S a zero forcing set implies $\rho(S)$ a zero forcing set.*

Proof sketch. Given a vertex y, pick x with $\rho x = y$; if S forces then $x \in \mathrm{cl}_t(S)$ for some t, so $y = \rho x \in \mathrm{cl}_t(\rho(S))$ by Lemma 2.4 with $T = \rho(S)$. No inverse rotation, no injectivity and no relation $\rho^n = \mathrm{id}$ is used. □

3. Exact values by exhaustive search

This section collects the finite data. Everything in it is a computation; nothing in it is offered as new. Its purpose is twofold: it is the calibration against published values that makes the disagreements of Section 6 meaningful, and it supplies the two individual cells ($n = 13$ at $k = 3$, $n = 7$ at $k = 2$) that later statements consume.

Proposition 3.1. *$Z(P(5,2)) = 5$; $Z(P(n,1)) = 4$ for $4 \le n \le 9$; $Z(P(n,2)) = 6$ for $10 \le n \le 14$; and $Z(P(11,5)) = 6$.*

All of these are published values: the first is the Petersen graph, the second is [5], the third is [8, Thm. 2.5] and the fourth is [8, Thm. 2.6] at $k = 5$. The last two are exactly the results [6] reports validating his own programs against before applying them — he checked $Z(P(n,2)) = 6$ over $10 \le n \le 16$, we over $10 \le n \le 14$. Reproducing two theorems of [8] with the same code that contradicts a third is what separates "that paper has an error" from "our computation disagrees with that paper".

Proposition 3.2. *$Z(P(n,3)) = 6,6,6,8,7,7,8$ for $n = 7,8,\dots,13$ respectively.*

Proposition 3.3. *$Z(P(14,3)) = Z(P(15,3)) = 8$.*

These nine cells reproduce, independently, the corresponding entries of the table of [6], and they cover the whole non-constant stretch of it: the value rises to 8 at $n = 10$, drops back to 7 at $n = 11$ and $n = 12$, and returns to 8 at $n = 13$. The searches are, in order of n, exhaustions over $\binom{14}{5} = 2\,002$, $\binom{16}{5} = 4\,368$, $\binom{18}{5} = 8\,568$, $\binom{20}{7} = 77\,520$, $\binom{22}{6} = 74\,613$, $\binom{24}{6} = 134\,596$, $\binom{26}{7} = 657\,800$, $\binom{28}{7} = 1\,184\,040$ and $\binom{30}{7} = 2\,035\,800$ subsets, each preceded by one witness. The witness at $n = 9$, namely $\{u_1, u_2, u_3, u_4, u_5, v_0\}$, is worth noticing because it is not a run of consecutive outer vertices; optimal sets need not have the shape this paper analyses.

We emphasise what Propositions 3.2 and 3.3 are: nine cells of evidence for Conjecture 1.1 and no more. The values at $n = 10, 11, 12$ are precisely why extrapolation is not available here.

Proposition 3.4. *$Z(P(n,2)) = 4,6,5,6$ for $n = 6,7,8,9$ respectively.*

The searches are exhaustions over $\binom{12}{3} = 220$, $\binom{14}{5} = 2\,002$, $\binom{16}{4} = 1\,820$ and $\binom{18}{5} = 8\,568$ subsets. This is the range immediately below [8, Thm. 2.5], and it explains why that theorem starts at $n = 10$: the value is smaller at $n = 6$ and $n = 8$. The cell $Z(P(8,2)) = 5 < 6 = 2k+2$ also shows that the general upper bound is not tight everywhere, and there the optimal set is the run $R(5)$ itself — a fact used in Section 4 to show that the hypothesis $n \ge 5k-1$ of Theorem 4.6 cannot simply be dropped. The cell $Z(P(7,2)) = 6$ is the one that disagrees with [7]; see Section 6.

4. Runs of consecutive outer vertices

The upper half. We first fix the threshold in n that the consecutive-run witness of [8, Thm. 2.2] requires. Neither published proof of the bound $2k+2$ states one: [8] carries a blanket $n \ge 2k+1$ and [2] states $n \ge 3$. The cascade below touches the indices $0, 1, \dots, 2k+2$ and needs them pairwise distinct modulo n.

Proposition 4.1. *Let $k \ge 1$ and $n \ge 2k+3$. Then $R(2k+2) = \{u_0, \dots, u_{2k+1}\}$ is a zero forcing set of $P(n,k)$, and hence $Z(P(n,k)) \le 2k+2$.*

Proof sketch. Three rounds start the cascade. In round 1, for $1 \le i \le 2k$ the vertex u_i has both outer neighbours $u_{i\pm 1}$ in the run, so it forces v_i. In round 2 the vertex v_{k+1} has neighbours u_{k+1} (in the run), v_1 (black from round 1) and v_{2k+1}, so it forces v_{2k+1}. In round 3 the vertex u_{2k+1} has neighbours u_{2k} (in the run), v_{2k+1} (black from round 2) and u_{2k+2}, so it forces u_{2k+2}. Every index used lies in $\{0, \ldots, 2k+2\}$, and $n \ge 2k+3$ makes these distinct modulo n.

Hence $\rho(R(2k+2)) \subseteq \mathrm{cl}_3(R(2k+2))$. Since ρ acts on outer indices as $j \mapsto j+1$, Lemma 2.4 applied with $T = \mathrm{cl}_3(R(2k+2))$ gives, by induction on j, that $u_j \in \mathrm{cl}_{3j}(R(2k+2))$ for every $j < n$; the whole outer cycle is black after $3n$ rounds, and then every v_j is the unique white neighbour of the black u_j, so one more round finishes. Each of the three cascade steps is argued with a case distinction on whether its target is *already* black, so nothing has to be known about what else a round blackens. The bound $3n+1$ on the number of rounds is not claimed to be tight. $\square$

Two rounds do not suffice: in $P(11,3)$ the vertex u_8 is not in $\mathrm{cl}_2(R(8))$. So the middle step $v_{k+1} \to v_{2k+1}$ is not bookkeeping.

Remark 4.2. In [8] the outer vertices are indexed from 1 and the initial set is $A = \{u_1, \ldots, u_{2k+2}\}$. The printed proof of [8, Thm. 2.2] says, on p. 185, "Now the vertex v_1 is the only white neighbor of the vertex u_1 and is forced by it." At that point u_1 in fact has two white neighbours: v_1, and its other outer neighbour u_n, which lies outside A whenever $n \ge 2k+3$. The theorem is nevertheless correct: v_1 is not needed until the very end, and is forced by u_1 once u_n has gone black. We record the observation about the printed argument, not a doubt about the statement.

Specialising Proposition 4.1 to $k = 3$ recovers the computer-free upper bound of [6].

Proposition 4.3. ([6, Prop. 4].) *$Z(P(n,3)) \le 8$ for every $n \ge 9$, witnessed by $\{u_0, \ldots, u_7\}$.*

The proof above differs from the one in [6] at exactly one point. There, Step 2 concludes $\rho(\mathrm{cl}(S)) = \mathrm{cl}(S)$ from equivariance, monotonicity, idempotence and the equality of cardinalities of a bijective image on a finite set; here that step is replaced by the induction $u_j \in \mathrm{cl}_{3j}(S)$, which uses only $\rho(S) \subseteq \mathrm{cl}_3(S)$ and $\mathrm{cl}_t(\mathrm{cl}_a(S)) = \mathrm{cl}_{a+t}(S)$. The cost is an explicit round count and the gain is that nothing about ρ being a bijection is needed. The statement proved is the source's.

The stall. We now show that one vertex fewer is never enough. Fix $k \ge 1$ and $n \ge 5k-1$ and put

(1)
$$B_k = \{u_0, \ldots, u_{2k}\} \cup \{v_i : 1 \le i \le 2k-1\} \cup \{v_i : 2k+1 \le i \le 3k-1\} \cup \{v_i : n-k+1 \le i \le n-1\}.$$

The four blocks are disjoint and

$$|B_k| = (2k+1) + (2k-1) + (k-1) + (k-1) = 6k-2,$$

independently of n. At $k = 1$ the set is $\{u_0, u_1, u_2, v_1\}$, the closed neighbourhood of u_1; at $k = 3$ it is the sixteen-vertex set $\{u_0, \ldots, u_6\} \cup \{v_1, \ldots, v_5\} \cup \{v_7, v_8\} \cup \{v_{n-2}, v_{n-1}\}$.

Lemma 4.4. *For $k \ge 1$ and $n \ge 5k-1$ the set B_k is closed under the colour-change rule: no vertex of B_k has exactly one white neighbour.*

Proof sketch. Eight index shapes, three outer and five inner. Throughout, membership in B_k is read off (1), and one uses $n \ge 5k-1 > 3k-1$ so that the three inner blocks are disjoint and in the stated cyclic order.

- u_0: its neighbours are $u_1 \in B_k$, $u_{n-1} \notin B_k$ and $v_0 \notin B_k$ — two white.
- u_i, $1 \le i \le 2k-1$: neighbours $u_{i\pm 1}$ and v_i, all in B_k — none white.
- u_{2k}: neighbours $u_{2k-1} \in B_k$, $u_{2k+1} \notin B_k$, $v_{2k} \notin B_k$ — two white.
- v_i, $1 \le i \le k-1$: neighbours u_i, v_{i+k} (with $k+1 \le i+k \le 2k-1$) and $v_{i-k} = v_{n+i-k}$ (with $n-k+1 \le n+i-k \le n-1$), all in B_k — none white.
- v_k: neighbours $u_k \in B_k$, $v_0 \notin B_k$ and $v_{2k} \notin B_k$ — two white.
- v_i, $k+1 \le i \le 2k-1$: neighbours u_i, v_{i-k} (in $[1, k-1]$) and v_{i+k} (in $[2k+1, 3k-1]$), all in B_k — none white.

- v_i, $2k+1 \le i \le 3k-1$: $u_i \notin B_k$ since $i > 2k$; $v_{i-k} \in B_k$ since $k+1 \le i-k \le 2k-1$; and $v_{i+k} \notin B_k$ because $3k+1 \le i+k \le 4k-1$ and $4k-1 < n-k+1$ — two white.
- v_i, $n-k+1 \le i \le n-1$: $u_i \notin B_k$; $v_{i+k} = v_{i+k-n} \in B_k$ since $1 \le i+k-n \le k-1$; and $v_{i-k} \notin B_k$ because $n-2k+1 \le i-k \le n-k-1$ and $3k-1 < n-2k+1$ — two white.

The hypothesis $n \ge 5k-1$ is used exactly twice, in the last two items, and both times as the same collision: the far inner neighbour of the second block must miss the wrap-around block, and vice versa. Both requirements read $n > 5k-2$. □

Lemma 4.5. *Let B be closed under the colour-change rule, let $S \subseteq B$, and suppose some vertex x_0 lies outside B. Then S is not a zero forcing set.*

Proof. By Lemma 2.1, $\mathrm{cl}_t(S) \subseteq \mathrm{cl}_t(B) = B$ for every t, and $x_0 \notin B$. □

Lemma 4.5 is the easy direction of the *fort* criterion. Recall that a nonempty $F \subseteq V$ is a *fort* if every vertex outside F has either no neighbour or at least two neighbours in F; equivalently, F is a fort exactly when its complement is closed in the sense used above. Fast and Hicks [4] introduced forts, and Brimkov, Fast and Hicks [3, Thm. 8] prove that the optimum of the integer program that covers every fort equals $Z(G)$; both directions of the criterion that S is a zero forcing set exactly when S meets every fort are established in that proof. In our language $V \setminus B$ is a fort disjoint from S. We prove the direction we use directly rather than importing the criterion.

Theorem 4.6. *Let $k \ge 1$ and $n \ge 5k-1$. Then no run of at most $2k+1$ consecutive outer vertices is a zero forcing set of $P(n,k)$. In particular $R(2k+1) = \{u_0, \dots, u_{2k}\}$ is not one.*

Proof. $R(2k+1) \subseteq B_k$, the set B_k is closed by Lemma 4.4, and $u_{2k+1} \notin B_k$ because $n \ge 5k-1$ forces $2k+1$ to avoid all four blocks; apply Lemma 4.5. A shorter run is a subset of $R(2k+1)$, so Lemma 2.1 finishes. □

Two remarks on the hypotheses. The bound $n \ge 5k-1$ is sharp in the strong sense that the *conclusion*, not merely the proof, fails below it: at $(k,n) = (2,8)$ the run $R(5)$ *is* a zero forcing set of $P(8,2)$, and indeed $Z(P(8,2)) = 5$ by Proposition 3.4, so the general upper bound $2k+2$ is not attained there either. A direct computation over $1 \le k \le 10$ and $2k+1 \le n \le 50$ found exactly one pair below the threshold at which B_k is nevertheless closed, namely $(k,n) = (2,7)$. Second, $k \ge 1$ is not decoration: at $k = 0$ the object $P(n,0)$ is not a simple graph.

Remark 4.7. Lemma 4.4 and Theorem 4.6 enumerate nothing, so an error in them would not surface as a wrong number. Four checks supply numbers. At $(k,n) = (4,19)$ and $(5,24)$ the closure of $R(2k+1)$ was computed and found to be *exactly* B_k; without this, “no vertex of B_k forces outside B_k” would also be true of the whole vertex set. At $k = 4$, $n = 19$ the size $|B_4| = 22 = 6 \cdot 4 - 2$ was verified directly. At $(k,n) = (2,8)$ the failure of closedness is exhibited by an explicit force. Finally, the general construction (1) at $k = 3$ was checked to be literally the sixteen-vertex set of the previous paragraph, for every $n \ge 3$, so it has not drifted by an index.

At $k = 3$ the theorem covers $n \ge 14$. The single case that the corollary below adds to it is supplied by Proposition 3.2.

Corollary 4.8. *$\{u_0, \dots, u_6\}$ is not a zero forcing set of $P(n,3)$ for any $n \ge 13$.*

Proof. For $n \ge 14$ this is Theorem 4.6 at $k = 3$. For $n = 13$, $Z(P(13,3)) = 8$ by Proposition 3.2, so no seven-element set forces. □

The case $n = 13$ genuinely escapes the construction: there $v_{n-2} = v_{11}$ is an inner neighbour of v_8, so v_8 has exactly one white neighbour in B_3 and forces u_8; this is also why the stall drawn at $n = 13$ in [6] has eighteen vertices rather than sixteen. Accordingly the case $n \ge 14$ of Corollary 4.8 is computation-free while the case $n = 13$ rests on an exhaustive search over $\binom{26}{7} = 657\,800$ subsets.

[6] reports that $\{u_0, \dots, u_6\}$ forces $P(n,3)$ for $n = 11, 12$ and for no other n in the range $10 \le n \le 300$, verified by one closure per n, and deliberately does not promote that computation to a

theorem. Corollary 4.8 is the all-n form of the "no other n" half of that sentence, from $n = 13$ on; the three cells $10 \le n \le 12$ that the half also covers are left where the source put them, in a finite computation, and their values are reproduced independently as Proposition 3.2.

Remark 4.9. Corollary 4.8 and Theorem 4.6 are small and elementary. A search of a local mirror of the arXiv full-text corpus for papers mentioning both "generalized Petersen" (in either spelling) and "zero forcing" returned eighteen papers, of which fifteen are coincidental co-occurrences and one cites [8] only in its bibliography; the two substantive ones are [6] and [7]. A query of the arXiv API for the same conjunction returns exactly those two, and "maximum nullity" with "generalized Petersen" returns none. Neither the value sequence of $Z(P(n, 3))$ nor any lower bound for it beyond [8, Cor. 3.5] was found, and no computed value past $n = 20$. Theorem 4.6 is implied by published work at $k = 1$ (where $Z(P(n, 1)) = 4$ [5] leaves no room for a three-element set), at $k = 2$ with $n \ge 10$ ([8, Thm. 2.5]), and at the two progressions of [2, Thm. 2.4] ($k = 2$ with $15 \mid n$, $k = 5$ with $24 \mid n$). It is not implied at $k = 3$, at $k = 4$, at $k \ge 6$, at most of $k = 5$, or at $k = 2$ with $n = 9$.

Where the run starts, and the exact optimum. [6] dismisses the starting index in a clause — by rotational symmetry it is immaterial. Formally this is the one place where the converse direction of Lemma 2.4 is needed, and Lemma 2.5 supplies it.

Proposition 4.10. *Let $1 \le m \le n$. Then $R_j(m)$ is a zero forcing set of $P(n, k)$ for some j if and only if it is one for every j. Whether a run of consecutive outer vertices forces depends only on its length.*

Proof sketch. ρ carries $R_j(m)$ to $R_{j+1}(m)$, so by Lemma 2.5 forcing passes from j to $j+1$ and hence to $j + d$ for every $d \ge 0$. To come back from j to 0 we go forward far enough: $\rho^{n(j+1)-j}$ carries $R_j(m)$ to $R_{n(j+1)}(m)$, and $n(j + 1) \equiv 0 \pmod n$, so that run is $R_0(m)$; the hypothesis $m \le n$ is what makes $R_0(m)$ the set $\{u_0, \dots, u_{m-1}\}$ rather than a shorter one with repetitions. No inverse rotation and no relation $\rho^n = \mathrm{id}$ enters. □

Theorem 4.11. *Let $k \ge 1$, let $n \ge \max(2k + 3,\, 5k - 1)$ and let j be arbitrary. Then*

$$\min\{\, m : R_j(m) \text{ is a zero forcing set of } P(n, k)\,\} \;=\; 2k + 2.$$

Proof. Membership: Proposition 4.1 gives it for $j = 0$, and Proposition 4.10 transports it, noting $2k + 2 \le n$. Minimality: a run of length $m \le 2k + 1$ at u_j is not forcing, by Theorem 4.6 and Proposition 4.10. □

For $k \ge 2$ the binding hypothesis is $n \ge 5k - 1$; only at $k = 1$ is it $n \ge 2k + 3 = 5$. At $k = 3$ the theorem reads: for $n \ge 14$ and every j, $\{u_j, \dots, u_{j+6}\}$ is not a zero forcing set of $P(n, 3)$ and $\{u_j, \dots, u_{j+7}\}$ is. Both halves were confirmed at $n = 14$ and at wrapping starting positions ($j = 10$ with $m = 7$ and $m = 8$, and $j = 13$, $m = 9$ at $n = 19$, $k = 4$) against the independent implementation of Remark 2.3; and the value $2k+2$ was refuted in both directions at $k = 3$, $n = 14$, where the least forcing run length is neither 7 nor 9.

Remark 4.12. The two halves of Theorem 4.11 are of unequal weight. The independence of the starting index is routine mathematics stated as a theorem; the content is Theorem 4.6. The upper half is [8, Thm. 2.2] and is not claimed here. What we did not find in the literature is any statement that the witness $2k + 2$ is optimal within its own shape, for any k.

Below the threshold. The following is a computation, reported for orientation; it is not part of the formal development and nothing later depends on it. For $1 \le k \le 6$ and $2k + 3 \le n \le 32$ we computed the closure of $R(2k+1)$ in $P(n, k)$ directly. Its size is exactly $6k - 2$ at every $n \ge 5k - 1$, and $5k - 1$ is the first n from which that holds at all larger n in the range. So the threshold of Theorem 4.6 is the point from which the stall stops depending on n, and not merely a bound the proof happens to need.

Below the threshold the behaviour is mixed. The range scanned has 40 sub-threshold cells, and at 34 of them the run $R(2k + 1)$ is a zero forcing set: at $k = 2$ for $n = 8$, at $k = 3$ for $n = 9, 11, 12,$

and at every tested $n \le 17$ for $k = 4$, $n \le 22$ for $k = 5$ and $n \le 27$ for $k = 6$. At the remaining six it stalls, and never with fewer than $6k - 2$ vertices. Four of those stalls are strictly larger than $6k - 2$ — eighteen vertices at $(k, n) = (3, 13)$, twenty-four at $(4, 18)$, thirty at $(5, 23)$ and thirty-six at $(6, 28)$ — and two have size exactly $6k - 2$: at $(2, 7)$, where the stall is B_2 itself, and at $(3, 10)$, where it is a sixteen-element set that is not B_3. A second family of stall sets, indexed by the distance below $5k - 1$, would close the interval $2k + 3 \le n < 5k - 1$; the data suggest that it is regular, and we have not attempted it.

5. A lower bound valid for every n, and the prisms

The mechanism of Section 4 — exhibit a closed set — also gives a lower bound on Z itself, at the smallest possible size. Write $N[x] = \{x\} \cup N(x)$.

Lemma 5.1. *Let $k \ge 1$, $2k < n$ and $n \ne 3k$. Then every closed neighbourhood $N[x]$ in $P(n, k)$ is closed under the colour-change rule.*

Proof sketch. The centre x has all three neighbours black, so it forces nothing. Each of the three neighbours y of x has x black and its other two neighbours outside $N[x]$, so it has two white neighbours and forces nothing. "Outside $N[x]$" is where the two hypotheses enter and they are exactly the two possible index collisions: $2k < n$ keeps $v_{i\pm k}$ off v_i, and $n \ne 3k$ keeps v_{i+2k} off v_{i-k}. Eight neighbour lists are checked, four for x outer and four for x inner. □

Both hypotheses are necessary. At $n = 2k$ the inner vertices have a repeated neighbour and $P(n, k)$ is not a simple cubic graph at all. At $(k, n) = (3, 9)$, so $n = 3k$, the set $N[v_0]$ is not closed: v_3 has u_3 as its only white neighbour there, because $v_{0+2k} = v_6 = v_{0-k}$. A direct computation over $1 \le k \le 8$ and $2k < n \le 40$ found the failures to be exactly the pairs with $n = 3k$, which is consistent with the proof, since $3k$ is the only divisor of $3k$ in the interval $(2k, 3k]$.

Proposition 5.2. *Let $k \ge 1$, $2k < n$ and $n \ne 3k$. Then $Z(P(n, k)) \ge 4$.*

Proof sketch. Suppose S is a zero forcing set with $|S| = 3$. Since $P(n, k)$ has $2n \ge 6$ vertices, $S \ne V$, so some $x \in S$ makes the *first* force; then two of the three neighbours of x are already black, i.e. in S. As $P(n, k)$ is loopless with three distinct neighbours at every vertex — a fact proved symbolically for all n and k with $k \ge 1$ and $2k < n$ — this exhibits three distinct elements of S, hence $S \subseteq N[x]$. By Lemmas 2.1 and 5.1, $\mathrm{cl}_t(S) \subseteq N[x]$ for all t, and $N[x]$ has four elements while $P(n, k)$ has $2n > 4$; so S does not force. Now apply Lemma 2.2. □

The bound itself is not new. For $k \ge 3$ and $n \ge 3k$ it follows from [8, Cor. 3.5]; at $k = 1$ it follows from [5]; at $k = 2$ it follows from [8, Thm. 2.5] together with Propositions 3.1 and 3.4. What Proposition 5.2 adds is a single argument, with no case distinction on k, that covers the corners the corollary misses — $k = 1$, where it gives only 2; $k = 2$, where it gives 3; and every $n < 3k$. The constant cannot be raised: $Z(P(7, 1)) = 4$.

Combining the two halves at $k = 1$, where the upper bound $2k + 2$ is also 4, closes the prism case completely.

Proposition 5.3. ([5, Thm. 1.2] at $\ell = 0$.) *$Z(P(n, 1)) = 4$ for every $n \ge 5$.*

Proof. Proposition 4.1 at $k = 1$ gives a zero forcing set of size 4 for $n \ge 5$, and Proposition 5.2 (with $2 < n$ and $n \ne 3$) gives $Z \ge 4$. □

The value is Herrman's, who proves $Z_{(\ell)}(P(n, 1)) = 4$ for $\ell = 0, 1$ and $n \ge 4$; the case $\ell = 0$ is the ordinary zero forcing number. What is offered here is only that the statement now has a proof valid for all n at once and using no computation. We note the shape because the open half of Conjecture 1.1 is a statement of exactly this kind, one value of k further out.

6. Two published values of Z

The first: Theorem 3.6 of [8]. The claim is: $Z(P(n,3)) = 8$ for every $n \geq 12$.

Theorem 6.1. *The statement "$Z(P(n,3)) = 8$ for every $n \geq 12$" is false. It fails at $n = 12$, the first case in its own range: $Z(P(12,3)) = 7$. Moreover it is the threshold and not the value that is wrong, since $Z(P(13,3)) = 8$.*

Proof. Proposition 3.2 gives $Z(P(12,3)) = 7$ and $Z(P(13,3)) = 8$; the zero forcing number is unique. □

This refutation is Krishnan's [6], posted in July 2026; the contribution here is only that it is now derived from the colour-change rule as a theorem, rather than reported as the output of a program. The value $Z(P(12,3)) = 7$ rests on two computations: the witness $\{u_0, \ldots, u_6\}$, and the exhaustion over all $\binom{24}{6} = 134\,596$ six-element vertex sets. Every other statement of [6] that we were able to test is confirmed: the table of Proposition 3.2, the upper bound (Proposition 4.3), the five-round cascade at $n = 12$ with its black-vertex counts $7, 12, 16, 20, 23, 24$, and the eighteen-vertex stall at $n = 13$. The corrected threshold of Conjecture 1.1 is confirmed on the range $13 \leq n \leq 15$ of Propositions 3.2 and 3.3 and is not proved; the source's refusal to extrapolate is adopted without amendment.

The second: Table 1 of [7]. The leaky forcing paper [7] states that $Z_{(0)}(G) = Z(G)$, so the $\ell = 0$ column of its Table 1 is a claim about the ordinary zero forcing number. As printed, that column reads

n	5	6	7	8	9
$Z(P(n,2))$ as printed	5	4	5	5	6
$Z(P(n,2))$	5	4	**6**	5	6

where the second line is Propositions 3.1 and 3.4.

Theorem 6.2. $Z(P(7,2)) = 6$. *Consequently the $\ell = 0$ column of the $k = 2$ half of* [7, Table 1] *is false, at $n = 7$ and only there: the other four entries of that column are correct.*

Proof. A witness of size six is $\{u_0, \ldots, u_5\}$. No five-element set forces: all $\binom{14}{5} = 2\,002$ of them were tested and none has full closure, so Lemma 2.2 gives $Z(P(7,2)) > 5$. The remaining four cells are Propositions 3.1 and 3.4, and Z is unique. □

Three points establish that this is a disagreement about a value and not about conventions. First, the graph is the same graph: the definition in [7] has vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ with edges $y_i y_{i+1}$, $x_i y_i$ and $x_i x_{i+k}$, which is $P(n,k)$ with the roles of the letters exchanged; and $P(7,2)$ was verified here to be a loopless cubic simple graph on 14 vertices with the expected neighbour lists. Second, the $k = 3$ column of the same table, $Z(P(n,3)) = 6, 6, 6, 8, 7$ for $7 \leq n \leq 11$, agrees exactly with Proposition 3.2. Third, the disagreement is confined to one cell rather than being a shift: the entries at $n = 5, 6, 8, 9$ all come out as printed. The value 6 was obtained three times independently before any formal proof was attempted: by a closure routine written directly from the definitions, by a separate implementation in C, and by the verified engine of Remark 2.3.

Remark 6.3. We claim nothing else about [7]. Its $\ell = 1$ and $\ell = 2$ columns are not examined here at all: leaky forcing is a different colour-change rule. Its own $\ell = 1$ entry at $n = 7$ is 6, which is consistent with the corrected value because $Z_{(0)} \leq Z_{(1)}$ always, and the two theorems of [7] that cite the proposition carrying Table 1 read the $\ell = 1$ column. The slip is isolated and is consistent with a typographical error. It also still stands: as of 28 August 2026 the arXiv listing for 2508.02564 shows a single version, v1, submitted 4 August 2025, with no later version, no journal reference and no erratum; a search on the six authors and the title found no published version and no correction.

Remark 6.4. A few checks guard the computations of Section 3. That $P(12,3)$ really is a simple cubic graph on 24 vertices is verified, since an off-by-one in the inner edges would silently replace every value in this paper by a value of some other graph; the cascades of [6] are reproduced from the readable form of the rule rather than from the fast engine; the searches are shown to be non-vacuous (a six-element set does force $P(9,3)$ while none does at $n = 12$, the empty set never forces and the whole vertex set always does); and the bound at $n = 12$ is refuted in both directions, above by $Z(P(12,3)) \neq 8$ and below by $Z(P(12,3)) \neq 6$.

Remark 6.5. Similarly for the statements of Section 4 at $k = 3$: the sixteen listed vertices are exactly the closure of $\{u_0, \dots, u_6\}$ at $n = 14$; the hypothesis $n \geq 14$ is necessary rather than conservative, as the explicit force at $n = 13$ shows; the cascade of Proposition 4.1 is exactly three rounds; the upper bound 8 of Proposition 4.3 cannot be lowered to 7, since $Z(P(10,3)) = 8$, even though the value does drop to 7 at $n = 11, 12$; and six consecutive outer vertices fail as well, by Lemma 2.1 with no new search. The general-k stall was independently pinned at $(k,n) = (4,19)$ and $(5,24)$ before it was proved.

7. Limits of the method, and questions

Theorem 4.6 rules out one orbit of shapes. Conjecture 1.1 asks for all $\binom{2n}{7}$ seven-element subsets of $P(n,3)$ to be ruled out at once, and the natural way to try to get there from a fort argument does not work. We record the measurement, because the idea is attractive enough to be worth stopping.

Since ρ is an automorphism it permutes forts, so a single fort F produces the n forts $\rho^j(F)$, $0 \leq j < n$. A zero forcing set S must meet each of them. A fixed vertex x lies in $\rho^j(F)$ for exactly $|F_{\text{out}}|$ values of j if x is an outer vertex and exactly $|F_{\text{in}}|$ values if it is inner, where F_{out} and F_{in} are the outer and inner parts of F; so, writing $f'(F) = \max(|F_{\text{out}}|, |F_{\text{in}}|)$ and counting incidences,

$$n \;\leq\; |S| \cdot f'(F), \qquad \text{hence} \qquad Z(P(n,3)) \;\geq\; \frac{n}{f'(F)}.$$

A fort with $f'(F) < n/7$ would therefore give $Z(P(n,3)) \geq 8$ at that n, which is the open half of Conjecture 1.1.

Read backwards, the same inequality says why no such fort exists: if the conjecture is true then $f'(F) \geq n/8$ for *every* fort, so forts cannot be small on either side. An exhaustive search by size confirms this. For $12 \leq n \leq 22$ the smallest fort of $P(n,3)$ has size 7 at $n = 14$, size 8 at $n = 12, 13, 16, 17, 20$, and size larger than 9 at $n = 15, 18, 19, 21, 22$; every minimum fort found has $f' = 4$. At $n = 20$ the inequality above therefore yields only $Z \geq \lceil 20/4 \rceil = 5$, and reaching 8 would need a fort with $f' \leq 2$, that is with at most four vertices. There is none below size 8. The minimum forts at $n = 16, 17, 20$ are two translates of the four-vertex gadget $\{u_{j+1}, u_{j+3}, v_j, v_{j+4}\}$, which is not itself a fort — it leaves v_{j-3} and v_{j+7} with exactly one neighbour in it — and becomes one only when the second translate is placed so that each copy repairs the other's two defects, which requires a coincidence in n. That is also why $n = 15, 18, 19, 21, 22$ admit no fort of size at most nine. This route is closed. (This paragraph, like Section 4's sub-threshold data, is a computation and is not part of the formal development.)

Three questions remain, in increasing order of difficulty.

(1) Close the interval $2k + 3 \leq n < 5k - 1$ for Theorem 4.6, presumably with a second family of stall sets indexed by the distance below the threshold. The many sub-threshold cells at which $R(2k+1)$ really does force, $(k,n) = (2,8)$ among them, show that whatever is proved there cannot be uniform.

(2) Extend the analysis past consecutive runs. Everything proved here about lower bounds concerns runs along the outer cycle, whereas the optimal set $\{u_1, \dots, u_5, v_0\}$ at $P(9,3)$ (Section 3) has a different shape. A stall family for "a run plus one spoke" is the next widening.

(3) Conjecture 1.1 itself. The argument of Section 5 rules out three-element sets because a three-element forcing set is pinned inside a closed neighbourhood; at size seven the same first-force observation leaves the whole closure analysis to do, and we have no proposal.

8. Verification

Every theorem, proposition, lemma and corollary of this paper has been formally verified in Lean 4 against *Mathlib*, and the development is free of `sorry`; repository reference to be added at submission. The exceptions are Conjecture 1.1, which is quoted from [6] and not proved, and the two paragraphs explicitly marked as computations reported for orientation, which are outside the formal development. Vertices are natural numbers, with u_i encoded as i and v_i as $n+i$, and a black set is a list used only through membership. All the results quantified over all n — Lemmas 2.1, 2.2, 2.4, 2.5, 4.4, 4.5 and 5.1, Propositions 4.1, 4.3, 4.10, 5.2 and 5.3, and Theorems 4.6 and 4.11 — are proved with no computational step at all: they depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: for each exact value in Section 3 one Boolean for the witness and one for the exhaustion, over the subset counts listed there; these are what Theorems 6.1 and 6.2, and the case $n = 13$ of Corollary 4.8, rest on. The engine that evaluates those Booleans is not trusted: as described in Remark 2.3, it is proved to compute the colour-change rule of Section 2, and the completeness of its subset enumeration is proved as well. Every numerical value quoted was first produced by an independent implementation in C or Python, written from the definitions and not from the formal development, and the formal proof was written against it afterwards; the general-k stall (1) was checked to be closed, of size $6k - 2$, and equal to the closure of $R(2k + 1)$ for all $1 \leq k \leq 10$ and $5k - 1 \leq n \leq 50$ before it was proved.

References

[1] AIM Minimum Rank – Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra Appl. **428** (2008), no. 7, 1628–1648, doi:10.1016/j.laa.2007.10.009.

[2] J. S. Alameda, E. Curl, A. Grez, L. Hogben, O. Kingston, A. Schulte, D. Young and M. Young, *Families of graphs with maximum nullity equal to zero forcing number*, Spec. Matrices **6** (2018), no. 1, 56–67, doi:10.1515/spma-2018-0006.

[3] B. Brimkov, C. C. Fast and I. V. Hicks, *Computational approaches for zero forcing and related problems*, European J. Oper. Res. **273** (2019), no. 3, 889–903, doi:10.1016/j.ejor.2018.09.030. The theorem number cited above is that of the accepted revision as posted at arXiv:1704.02065v3 (19 September 2018).

[4] C. C. Fast and I. V. Hicks, *Effects of vertex degrees on the zero-forcing number and propagation time of a graph*, Discrete Appl. Math. **250** (2018), 215–226, doi:10.1016/j.dam.2018.05.002.

[5] R. Herrman, *Upper bound for the $(d-2)$-leaky forcing number of Q_d and ℓ-leaky forcing number of $GP(n,1)$*, arXiv:2202.12671v1 (25 February 2022); published, under the title *The $(d-2)$-leaky forcing number of Q_d and ℓ-leaky forcing number of $GP(n,1)$*, in Discrete Optim. **46** (2022), 100744, doi:10.1016/j.disopt.2022.100744. The theorem number cited above is the e-print's.

[6] A. Krishnan, *A correction to the Zero Forcing Number of the Generalized Petersen Graphs $P(n,3)$*, arXiv:2607.19412v1 (13 July 2026). As of 28 August 2026 this is the only version of the e-print and it carries no journal reference; the numbering cited above (Proposition 4, Conjecture 5) is v1's.

[7] B. Bjorkman, L. Cao, F. Kenter, R. Moruzzi Jr., C. Reinhart and V. Vasilevska, *Leaky Forcing: Extending Zero Forcing Results to a Fault-Tolerant Setting*, arXiv:2508.02564v1 (4 August 2025). As of 28 August 2026 v1 is the only version and carries no journal reference.

[8] S. Rashidi, N. Shajareh Poursalavati and M. Tavakkoli, *Computing the zero forcing number for generalized Petersen graphs*, J. Algebra Comb. Discrete Struct. Appl. **7** (2020), no. 2, 183–193, doi:10.13069/jacodesmath.729465.