CONSECUTIVE ZECKENDORF–NIVEN TERMS IN A 2-AP: A TWO-PARAMETER FAMILY OF FIVE-TERM RUNS, AND SIX-TERM RUNS CONFINED TO TWO FIBONACCI SHAPES

KOREA SUPERINTELLIGENCE LABS

Abstract. A positive integer n is *Zeckendorf–Niven* when the number $s_Z(n)$ of summands in its Zeckendorf representation divides n. Lao, Miller, Rosa, Shiliaev, Tresch, Wong and Zhang recently studied how many consecutive terms of an arithmetic progression of common difference two can be Zeckendorf–Niven. They proved that the only runs of eight or more terms are subsequences of $2, 4, \dots, 18$, and constructed a five-term run starting at $27+F_{120j+17}$ for every j; between five and seven the maximum outside that block is left undetermined. We prove three things. First, their construction is one line of a two-parameter family: for $8 \le a$, $a+2 \le b$, $11 \le b$ and $F_a + F_b \equiv 58 \pmod{60}$, the number $n = 6 + F_a + F_b$ starts a run of five, and the run is exactly five on both sides. Second, every $n \ge 21$ that starts a run of six satisfies $n+3 = F_G$ or $n+5 = F_G$ for some $G \ge 21$; beyond $n \ge 21$ there is no hypothesis at all — none on the size of n, none on the position of its Zeckendorf digits. Third, neither of the two surviving one-parameter families produces a run of six for $G \le 501$, so no n with $21 \le n < F_{500} \approx 1.39 \cdot 10^{104}$ starts a run of six: throughout that range a run beginning outside $2, 4, \dots, 18$ has at most five terms, and five occurs. What is left of the question is a single Fibonacci-index divisibility problem, stated in Question 6.4. All three results, and every lemma they rest on, are machine-checked in Lean 4.

1. Introduction

Let $F_1 = F_2 = 1$ and $F_{i+2} = F_{i+1} + F_i$. Zeckendorf's theorem [2] says that every $n \in \mathbb{N}$ has a unique representation

$$n = \sum_{i \ge 2} \zeta_i(n) F_i, \qquad \zeta_i(n) \in \{0,1\}, \qquad \zeta_i(n)\zeta_{i+1}(n) = 0 \quad (i \ge 2), \tag{1}$$

and we write $s_Z(n) = \sum_{i \ge 2} \zeta_i(n)$ for the number of summands, $Z(n) = \{i : \zeta_i(n) = 1\}$ for the index set. Zeckendorf–Niven numbers, the n with $s_Z(n) \mid n$, go back to Ray and Cooper [3], who introduced the k-Zeckendorf Niven numbers — $k = 2$ is the case at hand — and proved that they have natural density zero; s_Z is entry A007895 of [7] and the Zeckendorf–Niven numbers are A328208. An *arithmetic progression of common difference* d, a d*-AP*, is a sequence $n, n+d, n+2d, \dots$; Grundman [4] showed that a run of consecutive Zeckendorf–Niven *integers* — the case $d = 1$ — all of whose terms exceed 6 has at most four terms, and that infinitely many runs of four occur.

The case $d = 2$ was taken up by Lao, Miller, Rosa, Shiliaev, Tresch, Wong and Zhang [1], whose §4 brackets the maximum run length from both sides. Their Theorem 4.3 reads, verbatim:

> *The only sequences of eight or more consecutive Zeckendorf-Niven terms in a* 2*-AP are subsequences of* $2, 4, 6, 8, 10, 12, 14, 16, 18$.

Their Theorem 4.4 supplies a matching construction, again verbatim:

> *For every* $j \in \mathbb{N}$*, let* $n = 27 + F_{120j+17}$*. Then* $n, n+2, n+4, n+6, n+8$ *is a sequence of five consecutive Zeckendorf-Niven terms in a* 2*-AP.*

Both quotations, and the numbering Theorem 4.3, Theorem 4.4, Lemma 4.1 and Lemma 4.2 used throughout, are those of `arXiv:2606.24006v1`, which is the only version at the time of writing.

Outside the block $2, 4, \dots, 18$ the two results leave the maximum somewhere between five and seven, and [1] determines it no further. It does not raise the gap as a named problem or conjecture either: §4 there is offered as "upper and lower bounds" on that maximum and stops at the two theorems above. Closing the gap is what §§5–6 below do, over a range of 10^{104}.

One reading of Theorem 4.3 has to be resisted at the outset, because it inverts the shape of the answer: the block the theorem names *is itself a nine-term run*. Indeed $2, 4, 6, 8, 10, 12, 14, 16, 18$ are all Zeckendorf–Niven while 20 is not, so the global maximum in a 2-AP is 9, and "the maximum is seven" and "the maximum is five" are both false as global statements even though five is the right answer outside the block (Proposition 4.2). Every statement below is therefore about starts $n \geq 21$, the first start beyond the block; 7, 19 and 20 are not Zeckendorf–Niven, so nothing is lost between 18 and 21.

This paper settles the following.

- **Five-term runs.** The construction of [1] is one line of a two-parameter family (Theorem 3.1): $n = 6 + F_a + F_b$ starts a five-term run whenever $8 \leq a$, $a + 2 \leq b$, $11 \leq b$ and $F_a + F_b \equiv 58 \pmod{60}$. The five Zeckendorf representations are forced, and the five divisibility conditions collapse to that single congruence. The run is *exactly* five — neither $n - 2$ nor $n + 10$ is Zeckendorf–Niven (Theorem 3.2) — so the only family [1] exhibits provably cannot be pushed to six. Its own line is $a = 8$, $b \equiv 17 \pmod{120}$, recovered in Corollary 3.3 together with the Fibonacci periodicity it rests on.
- **Six-term runs.** Every $n \geq 21$ that starts a six-term run has $n + 3 = F_G$ or $n + 5 = F_G$ for some $G \geq 21$ (Theorem 5.5). This is the main result. It has no hypothesis beyond $21 \leq n$: no bound on n, and — unlike the intermediate statement Proposition 5.1 — no bound on where the Zeckendorf digits of n first leave a gap. The mechanism is an identity we call the *alternation collapse*, $s_Z(F_{G+2} - w) = s_Z(F_G - w) + 1$ for $1 \leq w \leq F_G$ (Lemma 5.3), which turns the digit-sum profile of an arbitrarily high alternating block into a fixed small profile plus a constant, so that a 21×2 table decides all of them at once.
- **The range, and the residue.** The two surviving shapes are one-parameter families whose digit-sum profiles are known in closed form, so they can be tested far beyond the reach of a direct search: no $F_G - 3$ and no $F_G - 5$ with $G \leq 501$ starts a run of six (Proposition 6.1), whence no n with $21 \leq n < F_{500} \approx 1.39 \cdot 10^{104}$ starts one, and over that range the maximum outside the block is exactly five, attained there — and, by Theorem 3.1, infinitely often in all (Theorem 6.2). That the two shapes survive is not slack in an estimate (Remark 5.7); eliminating them needs an arithmetic input, $s_Z(F_G - 3) \nmid F_G - 3$ for all G, which is Question 6.4.

Three statements below rest on exhaustive computation, and each proof says which: a search over the 6765 possible low parts (Proposition 5.1), a table of 42 base profiles (Lemma 5.4), and a sweep of 242 parameter values in each of four one-parameter families (Proposition 6.1). All three are kernel computations depending on no axioms whatever; compiled evaluation is used in one place only, the two sweeps over $n < 10^7$ of Proposition 4.1, on which nothing in §§5–6 depends. Section 7 has the details.

2. Preliminaries

Throughout, n, x, G, w denote non-negative integers and F_i is as in §1, so that $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21$.

Definition 2.1. A finite list $l = (i_1 > i_2 > \cdots > i_r)$ of integers is *admissible* if $i_r \geq 2$ and $i_k \geq i_{k+1} + 2$ for $1 \leq k < r$. For $r \in \mathbb{N}$ we say that n *starts a run of length* r if $n + 2k$ is Zeckendorf–Niven for every $0 \leq k < r$; that is, the r terms $n, n+2, \ldots, n+2(r-1)$ of the 2-AP through n are all Zeckendorf–Niven. The *profile* of n is the vector $\big(s_Z(n), s_Z(n+2), \ldots, s_Z(n+10)\big)$.

By uniqueness in (1), an admissible list is the index set of the number it sums to. This is the only property of Zeckendorf representations that §3 uses.

Lemma 2.2. *If l is admissible then $s_Z\big(\sum_{i \in l} F_i\big) = |l|$.*

Proof. The list satisfies the two conditions of (1), so by uniqueness it is $Z(n)$ for $n = \sum_{i \in l} F_i$, and $s_Z(n) = |Z(n)| = |l|$. $\square$

The hypothesis is not decoration: $[4, 3]$ is a list of Fibonacci indices summing to $F_4 + F_3 = 5$, but it is not admissible and $s_Z(5) = 1 \neq 2$ (Proposition 3.4).

Definition 2.3. An index $G \geq 2$ is a *gap* of n when $\zeta_G(n) = \zeta_{G+1}(n) = 0$. The *low* and *high parts* of n at G are

$$\ell_G(n) = \sum_{i<G} \zeta_i(n) F_i, \qquad H_G(n) = \sum_{i \geq G} \zeta_i(n) F_i,$$

so that $n = \ell_G(n) + H_G(n)$ and $\ell_G(n) < F_G$.

Since the representation (1) is finite, every n has a gap of index at least 8, and hence a least such gap, which we call its *first gap.*

Lemma 2.4. *Let $G, H, m \in \mathbb{N}$ with $m < F_{G+1}$ and $i \geq G + 2$ for every $i \in Z(H)$. Then $s_Z(H + m) = s_Z(H) + s_Z(m)$. Consequently, if $G \geq 8$ is a gap of n then*

$$s_Z(n + x) = s_Z\big(\ell_G(n) + x\big) + s_Z\big(H_G(n)\big) \qquad \textit{for every } 0 \leq x \leq 10. \tag{2}$$

Proof. From $m < F_{G+1}$, every index of $Z(m)$ is at most G, so concatenating the decreasing lists $Z(H)$ and $Z(m)$ leaves a jump of at least two at the junction and the concatenation is admissible; Lemma 2.2 gives the first claim. For the second, G being a gap puts every index of $Z(H_G(n))$ at $G + 2$ or above, while $F_{G+1} = F_{G-1} + F_G > 10 + F_G$ for $G \geq 8$ because $F_{G-1} \geq F_7 = 13$, so $\ell_G(n) + x < F_{G+1}$ for $x \leq 10$. $\square$

Additivity of the Zeckendorf digit sum across a jump of two is classical; Jamet, Popoli and Stoll [5] call two such numbers *non-interfering* and record both the identity and the reason the required jump is 2 rather than 1. The particular split (2), at the first index j with $\zeta_j(n) = \zeta_{j+1}(n) = 0$, is the device the proof of [1, Theorem 4.3] uses by hand; there it appears with the threshold $j \geq 7$, switching to $j \geq 8$ in the branches where $\zeta_7(n) = 1$. The next proposition explains that switch: every hypothesis of Lemma 2.4 fails without warning if it is relaxed.

Proposition 2.5. *Each hypothesis of Lemma 2.4 is load-bearing.*

(i) *A jump of one is not enough: for $G = 8$, $H = F_9 = 34$ and $m = F_8 = 21 < F_9$, one has $s_Z(55) = 1$ while $s_Z(34) + s_Z(21) = 2$.*
(ii) *The threshold $G \geq 8$ in (2) cannot be relaxed to $G \geq 7$: $n = 46$ has a gap at 7 with $\ell_7(46) = 12 < F_7$ and $H_7(46) = 34$, all other hypotheses hold, and at $x = 10$ one has $s_Z(56) = 2$ while $s_Z(22) + s_Z(34) = 3$.*
(iii) *The bound $x \leq 10$ cannot be dropped: $n = 75$ has a gap at 8 with $\ell_8(75) = 20$ and $H_8(75) = 55$, and at $x = 14$ one has $s_Z(89) = 1$ while $s_Z(34) + s_Z(55) = 2$.*

Failure in case (ii) is not exotic: among the $n < 200\,000$ having a gap at index 7, 5.9% break (2) at some $x \leq 10$. (This last count is a computation outside the formal development.)

The second ingredient is the divisibility step, which is what the proofs of [1, Lemmas 4.1 and 4.2] run on, isolated and stated for an arbitrary even distance.

Lemma 2.6. *Let $d \geq 1$ and suppose m and $m + 2d$ are both Zeckendorf–Niven with $s_Z(m) = s_Z(m + 2d)$. Then $s_Z(m) \mid 2d$; in particular $s_Z(m) \leq 2d$.*

Proof. $s_Z(m)$ divides m and, by the hypothesis of equality, also $m + 2d$, hence their difference. $\square$

So a run cannot afford a repeated digit sum at two nearby positions unless that digit sum is small. Everything below is an elaboration of this one observation.

3. Five-term runs: a two-parameter family

Write $6 = F_2 + F_5$. For $n = 6 + F_a + F_b$ with a and b far enough apart, the addition of $2, 4, 6, 8$ rewrites only the digits below index 7, and the five representations are forced.

Theorem 3.1. *Let a, b satisfy $8 \le a$, $a + 2 \le b$, $11 \le b$ and*

$$F_a + F_b \equiv 58 \pmod{60}.$$

Then $n = 6 + F_a + F_b$ starts a run of five: $n, n+2, n+4, n+6, n+8$ are all Zeckendorf–Niven.

Proof. The five index sets are read off Lemma 2.2:

$$\begin{array}{llll}
n & = F_2 + F_5 + F_a + F_b & s_Z = 4 & \text{condition } 4 \mid n \\
n+2 & = F_6 + F_a + F_b & s_Z = 3 & \text{condition } 3 \mid n+2 \\
n+4 & = F_3 + F_6 + F_a + F_b & s_Z = 4 & \text{implied by } 4 \mid n \\
n+6 & = F_2 + F_4 + F_6 + F_a + F_b & s_Z = 5 & \text{condition } 5 \mid n+6 \\
n+8 & = F_2 + F_7 + F_a + F_b & s_Z = 4 & \text{implied by } 4 \mid n \quad (a \ge 9)
\end{array}$$

Each list is admissible: the largest index used below a is 7, and $a \ge 8$ keeps the jump to a at least 2, while $b \ge a + 2$ does the same at the top. The branch $a = 8$ is genuinely different and is where $11 \le b$ is spent: there the last line carries, since $F_7 + F_8 = F_9$, and $n + 8 = F_2 + F_9 + F_b$ has three summands rather than four, so that the condition at $n + 8$ becomes $3 \mid n + 8$, which follows from $3 \mid n + 2$. In both branches the conditions reduce to $4 \mid n$, $3 \mid n + 2$ and $5 \mid n + 6$, that is to $F_a + F_b \equiv 2 \pmod 4$, $\equiv 1 \pmod 3$ and $\equiv 3 \pmod 5$; by the Chinese remainder theorem this is $F_a + F_b \equiv 58 \pmod{60}$. □

Theorem 3.2. *Under the hypotheses of Theorem 3.1, neither $n-2$ nor $n+10$ is Zeckendorf–Niven. The run is therefore exactly five terms long, on both sides.*

Proof. $n - 2 = F_2 + F_4 + F_a + F_b$ has four summands, and $4 \mid n$ gives $n - 2 \equiv 2 \pmod 4$. For $a \ge 9$, $n + 10 = F_4 + F_7 + F_a + F_b$ has four summands and $n + 10 \equiv 2 \pmod 4$; for $a = 8$, $n + 10 = F_4 + F_9 + F_b$ has three and $n + 10 \equiv 8 \equiv 2 \pmod 3$, since $3 \mid n + 2$. □

The smallest members are $184 = 6 + F_9 + F_{12}$, $1624 = 6 + F_8 + F_{17}$ and $4564 = 6 + F_{14} + F_{19}$. The construction of [1] is the line $a = 8$: there $F_8 = 21$ and one needs $F_b \equiv 37 \pmod{60}$, which $b \equiv 17 \pmod{120}$ supplies because 120 is a period of the Fibonacci sequence modulo 60.

Corollary 3.3. *For every m, $F_{m+120} \equiv F_m \pmod{60}$, hence $F_{120j+17} \equiv F_{17} = 1597 \equiv 37 \pmod{60}$ for every j. Consequently* [1, Theorem 4.4] *holds: $n = 27 + F_{120j+17}$ starts a run of five for every $j \in \mathbb{N}$. Moreover $n + 10$ is not Zeckendorf–Niven, so that run too is exactly five terms long.*

Proof. The addition formula $F_{k+m+1} = F_k F_m + F_{k+1} F_{m+1}$ at $k = 119$ gives $F_{m+120} = F_{119} F_m + F_{120} F_{m+1}$, and $F_{119} \equiv 1$, $F_{120} \equiv 0 \pmod{60}$; the first claim follows, and the second by induction on j from $F_{17} = 1597 = 26 \cdot 60 + 37$. Now $27 = 6 + F_8$ and $F_8 + F_{120j+17} \equiv 21 + 37 = 58 \pmod{60}$, so Theorems 3.1 and 3.2 apply with $a = 8$, $b = 120j + 17 \ge 17$. □

The exponent $120 = \operatorname{lcm}\{\pi(3), \pi(4), \pi(5)\}$ of Pisano periods is quoted in [1]; here it is proved, from the addition formula alone. The remaining hypotheses of Theorem 3.1 are likewise sharp.

Proposition 3.4. *Each of the following witnesses that a hypothesis above cannot be dropped.*

(i) *Admissibility in Lemma 2.2: $F_4 + F_3 = 5$ but $s_Z(5) = 1$, not 2.*
(ii) *The congruence in Theorem 3.1: at $(a, b) = (8, 10)$ one has $F_8 + F_{10} = 76 \equiv 16 \pmod{60}$, and the candidate start $6 + F_8 + F_{10} = 82$ is not even Zeckendorf–Niven.*
(iii) *The bound $11 \le b$: at $b = 10$ the carry $F_9 + F_{10} = F_{11}$ collapses the representation of $n + 8$, and $s_Z(90) = 2$ where the proof of Theorem 3.1 asserts 3.*
(iv) *The bound $8 \le a$: at $a = 7$ the summand F_6 produced by $+2$ is adjacent to F_7, and $6 + F_7 + F_9 + 2 = 55 = F_{10}$ has one summand, not three.*

4. The exhaustive record

The next two propositions collect what direct enumeration says. They are not needed for §§5–6, but they fix the shape of the answer and they are the reason the two-parameter family of §3 deserves to be called *the* family of five-term runs rather than *a* family.

Proposition 4.1. (i) $2, 4, 6, 8, 10, 12, 14, 16, 18$ *are nine consecutive Zeckendorf–Niven terms of a* 2*-AP, and* 20 *is not Zeckendorf–Niven, so this run stops at nine. Neither* 7 *nor* 19 *is Zeckendorf–Niven.*

(ii) *The only* $n < 10^7$ *that start a run of six are* $2, 4, 6, 8$*, all inside the block.*

(iii) *The* $n < 10^7$ *that start a run of five are exactly*

$$2,\ 4,\ 6,\ 8,\ 10,\ 184,\ 1624,\ 4564,\ 17710,\ 28684,\ 46984,\ 128164,$$
$$514324,\ 1028464,\ 6020704,\ 9227464,\ 9227704.$$

Proof. Part (i) and the range $n < 21$ of part (ii) are finite checks on explicit numbers. Parts (ii) and (iii) over $21 \le n < 10^7$ are exhaustive sweeps: for each such n the six values $s_Z(n + 2k)$ are computed by one downward greedy pass, and the filter returns the empty list in case (ii) and the displayed list in case (iii). These two sweeps are the only claims in this paper delegated to compiled evaluation rather than to kernel reduction (§7); each was recomputed independently in C. ☐

Proposition 4.2. *"No* n *starts a run of eight" and "no* n *starts a run of six" are both false, by Proposition 4.1(i) at* $n = 2$*. "Four is the maximum outside the block" is also false:* $184 = 6 + F_9 + F_{12}$ *starts a run of five, by Theorem 3.1. Finally, in the* 1*-AP of Grundman* [4]*, the only five-term runs of consecutive Zeckendorf–Niven* integers *below* 30 *start at* 1 *and* 2*, and the four-term runs at* $1, 2, 3$ *— consistent with* [4]*, whose bound of four applies above* 6*.*

The last sentence is a cross-check on the definitions rather than a new statement: a disagreement with a published 1-AP bound would have exposed a wrong s_Z.

Remark 4.3. Outside the formal development, a direct enumeration in increasing Zeckendorf order covers every $n < F_{54} = 86\,267\,571\,272$ and finds exactly one run of six or more terms, namely the block, and exactly 23 maximal runs of five: 20 with index set $\{b, a, 5, 2\}$, i.e. members of Theorem 3.1, and 3 of the form $F_m - 1$ with $m = 4, 22, 35$, the first of which is $n = 2$, inside the block. Nothing else occurs there. For $m \ge 8$ the second shape visibly stops at five, its profile being $(\lfloor (m-1)/2 \rfloor, 2, 2, 2, 3, 3)$: the last two entries repeat with value 3 and $3 \nmid 2$, so Lemma 2.6 forbids a sixth term.

5. Six-term runs are confined to two Fibonacci shapes

Fix $n \ge 21$ starting a run of six and let G be its first gap, so $G \ge 8$. By (2) the whole profile of n is the profile of the low part $\ell_G(n)$ shifted by the constant $s_Z(H_G(n))$; by Lemma 2.6 a repeat in that profile at two positions $2i < 2j$ within the window forces

$$s_Z\big(\ell_G(n) + 2i\big) + s_Z\big(H_G(n)\big) \le 2(j - i). \tag{3}$$

Everything turns on finding such a repeat with a large enough value. For small G the low part can simply be enumerated.

Proposition 5.1. *Every* $\ell < F_{20} = 6765$ *has two adjacent positions of its profile carrying the same value, and that value can be taken* ≥ 2*. Consequently:*

(i) *no* $n \ge 21$ *having a gap at some index* $8 \le G \le 20$ *starts a run of six, so a run beginning at such an* n *has at most five terms;*

(ii) *among any six consecutive terms of a* 2*-AP whose start has a gap at some* $8 \le G \le 20$*, two adjacent terms have the same digit sum.*

Five terms would not be enough for (ii): the number 27 *occurring in* [1, Theorem 4.4] *has profile* $(3, 2, 3, 4, 2, 2)$*, whose first five entries have all adjacent pairs distinct.*

Proof. The first sentence is an exhaustive kernel computation over the 6765 values $\ell < F_{20}$, each a comparison of six digit sums; it depends on no axioms. Given it, (3) with $j = i+1$ forces $s_Z(H_G(n)) = 0$, i.e. $H_G(n) = 0$ and $n = \ell_G(n) < F_G \leq 6765$; a second exhaustive computation over $n < 6765$ returns $2, 4, 6, 8$ as the only starts of a run of six there, and all four are below 21. Part (ii) is the same repeat, transported back through (2). The profile of 27 is a direct computation. □

What Proposition 5.1 does not reach is an n whose first gap sits above 20, i.e. whose Zeckendorf digits alternate without interruption from position 8 upwards. The number $17710 = F_{22} - 1$, with $Z(17710) = \{21, 19, 17, 15, 13, 11, 9, 7, 5, 3\}$, is such an n. The rest of this section removes the restriction. Three lemmas: the shape of the low part, the transport of its profile to a bounded index, and a finite table.

Lemma 5.2. *Let $G \geq 8$ be the* first *gap of n. Then*

$$F_G - 21 \;\leq\; \ell_G(n) \;<\; F_G,$$

that is, $\ell_G(n) = F_G - w$ for a single parameter $1 \leq w \leq 21$.

Proof. Minimality of G says that no $8 \leq j < G$ is a gap, i.e. $\zeta_j(n) = 1$ or $\zeta_{j+1}(n) = 1$; with the Zeckendorf condition $\zeta_j\zeta_{j+1} = 0$ this makes the digits on positions $8, \ldots, G-1$ alternate, and the alternating sum $F_{G-1} + F_{G-3} + \cdots$ comes within $F_8 = 21$ of F_G. Formally one proves $F_G + \ell_m(n) \leq \ell_G(n) + F_m$ for $8 \leq m \leq G$ by downward induction on m, using only $F_{m+1} \leq 2F_m$ and the fact that raising the split index by one adds exactly the summand at that index; the step for $m \notin Z(n)$ moves two indices at once, because then $m+1 \in Z(n)$ and $m+1 < G$. Taking $m = 8$ and $\ell_8(n) \geq 0$ gives $F_G \leq \ell_G(n) + 21$. The upper bound is Definition 2.3. □

The bound is attained: all 21 values $w = 1, \ldots, 21$ occur among $n < 200\,000$, so the case list of Lemma 5.4 cannot be shortened.

Lemma 5.3 (alternation collapse and profile transport)**.** *For all G, w with $1 \leq w \leq F_G$,*

$$s_Z\big(F_{G+2} - w\big) = s_Z\big(F_G - w\big) + 1. \tag{4}$$

Consequently, for $8 \leq \mathrm{base}$, $1 \leq w \leq 21$, $x \leq 10$ and every $t \in \mathbb{N}$,

$$s_Z\big(F_{\mathrm{base}+2t} - w + x\big) = s_Z\big(F_{\mathrm{base}} - w + x\big) + \begin{cases} t, & x < w, \\ 0, & x \geq w. \end{cases} \tag{5}$$

Proof. For (4): $F_{G+1} \leq F_{G+2} - w < F_{G+2}$, so the greatest Fibonacci number below $F_{G+2} - w$ has index $G+1$; peeling it leaves $F_{G+2} - w - F_{G+1} = F_G - w$ and costs one summand. For (5) it suffices to treat $t = 1$ and iterate. If $x < w$ then both sides read $F_\bullet - (w - x)$ with $1 \leq w - x \leq 21 \leq F_G$, so (4) applies and the step costs one summand. If $x \geq w$ then both sides read $F_\bullet + (x - w)$ with $x - w \leq 10 < F_{G-1}$, so peeling the leading Fibonacci number on each side leaves the same $s_Z(x - w)$ and the step costs nothing. □

The point of (5) is that the offset depends on x only through whether $x < w$: it is *uniform on each side of w*. So the profile of an arbitrarily high alternating low part is a fixed small profile plus a constant, and since every $G \geq 21$ is $18 + 2t$ or $19 + 2t$ with $t \geq 1$, two base indices carry all of them.

Lemma 5.4. *Write $P_i = s_Z(F_{\mathrm{base}} - w + 2i)$ for the base profile.*

(a) *For each* base $\in \{18, 19\}$ *and each w with $1 \leq w \leq 21$ and $w \notin \{3, 5\}$ there are positions $i < j < 6$ with $P_i = P_j$ lying on the same side of w, and with room to spare against the bound of Lemma 2.6: either $2j < w$ and $2(j - i) \leq P_i$, or $w \leq 2i$ and $2(j - i) < P_i$.*

(b) *For $w \in \{3, 5\}$ and both base indices there are positions $i < j < 6$ with $w \leq 2i$, $P_i = P_j$ and $2(j - i) \leq P_i$.*

Proof. A table of 2×21 profiles of six entries each, searched over the 15 position pairs: an exhaustive kernel computation depending on no axioms. Parts (a) and (b) partition $\{1, \dots, 21\}$, so the exceptional set $\{3, 5\}$ is produced by the computation rather than chosen. Both tables were recomputed from independently written code. $\square$

Theorem 5.5. *Let $n \geq 21$ start a run of six consecutive Zeckendorf–Niven terms in a 2-AP. Then there is $G \geq 21$ with*

$$n + 3 = F_G \qquad \text{or} \qquad n + 5 = F_G.$$

Proof. Let $G \geq 8$ be the first gap of n. If $G \leq 20$, Proposition 5.1(i) already excludes n, so assume $G \geq 21$ and write $G = \text{base} + 2t$ with base $\in \{18, 19\}$ and $t \geq 1$. By Lemma 5.2, $\ell_G(n) = F_G - w$ for some $1 \leq w \leq 21$, and by (5) the profile of $\ell_G(n)$ is the profile of $F_{\text{base}} - w$ raised by t at the positions $2i < w$ and unchanged at the positions $2i \geq w$.

Suppose first $w \notin \{3, 5\}$ and take the positions $i < j$ of Lemma 5.4(a). They lie on the same side of w, so the transport adds the same amount at both and the two values stay equal; (3) therefore applies to them. In the case $2j < w$ the common transported value is $P_i + t \geq 2(j - i) + 1$, because $t \geq 1$; in the case $w \leq 2i$ it is $P_i > 2(j - i)$. Either way the value exceeds $2(j - i)$, contradicting (3). So no such n exists in this case.

Now suppose $w \in \{3, 5\}$ and take the positions of Lemma 5.4(b). Both lie at or above w, so the transport leaves them unchanged, and (3) reads $P_i + s_Z(H_G(n)) \leq 2(j - i) \leq P_i$. Hence $s_Z(H_G(n)) = 0$, i.e. $H_G(n) = 0$ and $n = \ell_G(n) = F_G - w$ with $w \in \{3, 5\}$. $\square$

Corollary 5.6. *If $n \geq 21$ and neither $n + 3$ nor $n + 5$ is a Fibonacci number, then n does not start a run of six; in particular, if such an n starts a run of five, that run is maximal.*

Remark 5.7. The two exceptional values of w are forced by the arithmetic and not by a loose estimate. For $w = 3$ at base 18 the profile is $(7, 8, 2, 2, 2, 3)$, and after transport by t it is $(7 + t, 8 + t, 2, 2, 2, 3)$; for $w = 5$ at base 19 it is $(7 + t, 8 + t, 9 + t, 2, 2, 2)$. In both, the large entries differ by one, so they never repeat, and the only repeats sit at the value 2, which is exactly the bound $2(j - i)$ available at $j = i + 1$ — enough for Lemma 5.4(b), never enough for a contradiction. Widening the search over position pairs, or raising the base index inside its parity class, therefore cannot help: what is needed is arithmetic input about the numbers $F_G - 3$ and $F_G - 5$ themselves. Section 6 supplies it for $G \leq 501$ and Question 6.4 states what a complete answer requires.

The carve-out of Lemma 5.4 is symmetric in the two base indices, and by one case larger than it needs to be. Outside the formal development, the search behind part (a) was run at all 42 pairs (base, w) with base $\in \{18, 19\}$ and $1 \leq w \leq 21$: it succeeds at 39 of them and fails at exactly three, namely $(18, 3)$, $(19, 3)$ and $(19, 5)$. At base $= 18$ the value $w = 5$ is thus settled like the rest, so the family $n = F_G - 5$ in fact survives only for *odd* G. Nothing below uses this: Lemma 5.4, Theorem 5.5 and the sweep of Proposition 6.1 keep both parities for both values of w, which is the conservative choice.

Proposition 5.8. (i) *The hypothesis $1 \leq w$ in (4) is load-bearing: at $w = 0$ both sides are Fibonacci numbers and $s_Z(F_{12}) = 1$, not $s_Z(F_{10}) + 1 = 2$. So is $w \leq F_G$: at $G = 4$, $w = 4 > F_4 = 3$ the right side truncates to $s_Z(0) + 1 = 1$ while $s_Z(F_6 - 4) = s_Z(4) = 2$.*

(ii) *Theorem 5.5 cannot drop $21 \leq n$: the number 2 starts a run of six, and neither 5 nor 7 is a Fibonacci number of index ≥ 21.*

(iii) *The gain over Proposition 5.1 is real: $17710 = F_{22} - 1$ has no gap at any $8 \leq G \leq 20$, so Proposition 5.1 says nothing about it, whereas Theorem 5.5 does, because 17713 and 17715 lie strictly between $F_{22} = 17711$ and $F_{23} = 28657$.*

(iv) *The exceptional shape is inhabited. The digits of $17708 = F_{22} - 3$ alternate without interruption, $Z(17708) = \{5, 7, 9, \dots, 19, 21\}$, so it has no gap at any $8 \leq G \leq 20$, and $17708 + 3$ is a Fibonacci number: both statements above are silent about it. It is not a counterexample — it starts no run of six, and Proposition 6.1 covers its family — but the carve-out does name a real set of integers.*

6. The two exceptional families, and the range below F_{500}

Theorem 5.5 reduces the question to two one-parameter families, and (5) gives their profiles in closed form. That is what makes them testable far beyond the reach of any direct search: to decide whether $F_G - w$ starts a run of six one never evaluates s_Z at a 105-digit number, but only divides such a number by six integers already known from the closed form.

Proposition 6.1. *For* $w \in \{3, 5\}$, $\text{base} \in \{18, 19\}$ *and every* $t < 242$*, the number* $F_{\text{base}+2t} - w$ *does not start a run of six. Since* $18 + 2t$ *runs over the even and* $19 + 2t$ *over the odd indices, this says: no* $F_G - 3$ *and no* $F_G - 5$ *with* $18 \leq G \leq 501$ *starts a run of six.*

Proof. By (5) the six digit sums $s_Z(F_{\text{base}+2t} - w + 2i)$, $i < 6$, are the six base values raised by t below w and unchanged above; "$F_{\text{base}+2t} - w$ starts a run of six" therefore becomes six explicit divisibility tests with known divisors. Running all 4×242 parameter pairs is an exhaustive kernel computation and depends on no axioms; it returns no survivor. The whole sweep was recomputed independently, evaluating s_Z of each term directly rather than through the closed form, and out to $t < 492$; that wider run also returns no survivor and no discrepancy with the closed form. □

Theorem 6.2. *No* n *with* $21 \leq n < F_{500}$ *starts a run of six consecutive Zeckendorf–Niven terms in a* 2*-AP, where*

$$F_{500} = 1394232245616978 \ldots 4125 \approx 1.39 \cdot 10^{104}.$$

Equivalently: a run of consecutive Zeckendorf–Niven terms of a 2*-AP beginning at any* n *in that range has at most five terms, and five is attained there — by Theorem 3.1, which supplies infinitely many starts in total and many below* F_{500}*.*

Proof. Let $21 \leq n < F_{500}$ start a run of six. By Theorem 5.5, $n + 3 = F_G$ or $n + 5 = F_G$ for some $G \geq 21$; since $n + 5 < F_{500} + 5 \leq F_{501}$, necessarily $G \leq 500$. Writing $G = \text{base} + 2t$ with $\text{base} \in \{18, 19\}$ gives $t < 242$, and Proposition 6.1 excludes it. □

This is what [1, §4] leaves undetermined, answered over a range of 10^{104}: the maximum outside the block $2, 4, \ldots, 18$ is exactly five there. For comparison, the sweep of Proposition 4.1 reaches 10^7, and the enumeration in Remark 4.3 reaches $8.6 \cdot 10^{10}$.

Remark 6.3. The bound $G \leq 501$ is a resource decision, not an obstruction: the same computation was also compiled at $G \leq 1001$, i.e. $n < F_{1000} \approx 4.3 \cdot 10^{208}$, and returned no survivor there either; it was dialled back to keep the memory footprint of a single check inside a fixed budget. Splitting the computation into chunks removes the constraint entirely.

Nothing in this method reaches all G at once, and Remark 5.7 explains why. The missing input is arithmetic. Take $w = 3$ and G even, and set $c = (G - 4)/2$, so that the closed form of Lemma 5.3 gives the profile $(c,\, c + 1,\, 2,\, 2,\, 2,\, 3)$ at $F_G - 3, F_G - 1, F_G + 1, F_G + 3, F_G + 5, F_G + 7$. The six Zeckendorf–Niven conditions then read

$$c \mid F_G - 3, \quad c + 1 \mid F_G - 1, \quad 2 \mid F_G + 1, \quad 2 \mid F_G + 3, \quad 2 \mid F_G + 5, \quad 3 \mid F_G + 7. \tag{6}$$

The last four are elementary conditions on the index: F_G is odd exactly when $3 \nmid G$, and $F_G \equiv 2 \pmod 3$ exactly when $G \equiv 3, 5, 6 \pmod 8$. The first two are not. The other three combinations of $w \in \{3, 5\}$ with the parity of G produce conditions of the same kind, with $c = (G - 3)/2$ for $w = 3$ and G odd, and $c = (G - 5)/2$ resp. $c = (G - 4)/2$ for $w = 5$ and G odd resp. even.

Question 6.4. Is there a $c \geq 9$ with $F_{2c+4} \equiv 3 \pmod c$ and $F_{2c+4} \equiv 1 \pmod{c + 1}$? That is: can the first two conditions of (6) hold together? The analogous question for $w = 5$ and G odd asks for a $c \geq 8$ with $F_{2c+5} \equiv 5 \pmod c$, $F_{2c+5} \equiv 3 \pmod{c + 1}$ and $F_{2c+5} \equiv 1 \pmod{c + 2}$.

A negative answer in all four parameter combinations would remove the bound $n < F_{500}$ from Theorem 6.2 and settle the maximum outside the block at five for every n, five being attained by Theorem 3.1. As numerical evidence, and outside the formal development: among the 990 even G

with $22 \le G \le 2000$, exactly 165 satisfy the two index congruences above, and for every one of them $c \nmid F_G - 3$; the corresponding search for odd $G \le 2001$ in the $w = 5$ family is likewise empty.

7. Verification

Every theorem, proposition, lemma and corollary of this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [6]; the development contains no `sorry` and no added axiom. The counts and enumerations in the remarks explicitly flagged as *outside the formal development* are computations only, and nothing formal depends on them. Zeckendorf's theorem is not reproved here: s_Z is *defined* as the length of the Zeckendorf representation supplied by the Zeckendorf development in Mathlib [6] (the module `Mathlib.Data.Nat.Fib.Zeckendorf`, due to Y. Dillies), whose existence and uniqueness that library carries, and Lemma 2.2 is one rewrite by its characterisation of admissible index lists. The statements of §§2–6 — Lemmas 2.2, 2.4, 2.6, 5.2 and 5.3, Theorems 3.1, 3.2, 5.5 and 6.2, Corollaries 3.3 and 5.6, Propositions 5.1 and 6.1, and all four propositions of negative controls — depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, while the three exhaustive computations they invoke (the 6765 low parts of Proposition 5.1, the 2×21 table of Lemma 5.4, and the 4×242 parameter sweep of Proposition 6.1) are kernel reductions depending on no axioms at all. Compiled evaluation, through Lean's `native_decide`, appears in exactly one place — the two sweeps over $n < 10^7$ in Proposition 4.1(ii),(iii) — and no statement of §5 or §6 uses it. Every number quoted was produced twice from independently written code before the formal proof was written against it: the enumeration of Remark 4.3 in C, the profiles and sweeps of §§5–6 in Python by a greedy peel sharing no code with the Lean evaluators, with no discrepancy in any of them. Repository reference to be added at submission.

References

[1] K. Lao, S. J. Miller, N. Rosa, M. Shiliaev, G. Tresch, T. W. H. Wong and H. Zhang, *On Zeckendorf-Niven numbers and arithmetic progressions*, arXiv:2606.24006v1 (2026).

[2] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège **41** (1972), 179–182.

[3] A. Ray and C. Cooper, *On the natural density of the k-Zeckendorf Niven numbers*, J. Inst. Math. Comput. Sci., Math. Ser. **19** (2006), no. 1, 83–98; Zbl 1205.11020.

[4] H. G. Grundman, *Consecutive Zeckendorf-Niven and lazy-Fibonacci-Niven numbers*, Fibonacci Quart. **45** (2007), no. 3, 272–276; Zbl 1214.11016.

[5] D. Jamet, P. Popoli and T. Stoll, *Maximum order complexity of the sum of digits function in Zeckendorf base and polynomial subsequences*, Cryptogr. Commun. **13** (2021), no. 5, 791–814; arXiv:2106.09959.

[6] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; arXiv:1910.09336.

[7] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A007895 and A328208.