THE BORDERLINE CASE OF THE MIGLIORE–MIRÓ-ROIG–NAGEL CONJECTURE FOR LEVEL MONOMIAL ALMOST COMPLETE INTERSECTIONS: COMPLETE INTEGER ELIMINATIONS UP TO $a = 13$

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let $\Bbbk$ have characteristic zero, let $R = \Bbbk[x, y, z]$, and let I be generated by x^{a_1+t}, y^{a_2+t}, z^{a_3+t} and $x^{a_1}y^{a_2}z^{a_3}$, so that R/I is a level monomial almost complete intersection. Migliore, Miró-Roig and Nagel conjectured a characterisation of when such an algebra fails the weak Lefschetz property; in the form Cook and Nagel restated it, failure occurs — apart from the two quadruples $(2, 9, 13, 9)$ and $(3, 7, 14, 9)$, which are exceptions — exactly when t is even, $a_1 + a_2 + a_3$ is odd and a_1, a_2, a_3 are not all distinct. One implication is a theorem and the other is open. Booth and Vraciu reduce the first unresolved value $t = (a_1 + a_2 + a_3)/3 + 1$ to the integer zeros of an explicit polynomial $\mathcal{F}_a(a_1, a_2)$, where a is defined by $a_3 = 2(a_1 + a_2) - 3a$. They settle $a \le 3$, report a *Macaulay2* verification for $a \le 6$ under the extra hypothesis $a_1 \ge a$, and stop: for $a \ge 7$, "the list of values that need to be checked is too large to be practical."

We determine the zero set of $\mathcal{F}_a$ on the admissible region $0 < a_1 \le a_2 \le a_3$ completely, for $2 \le a \le 13$ and in both regimes. In the regime $a \le a_1$ the zeros are the diagonal $a_1 = a_2$ for odd a and none for even a, apart from the single triple $(3, 7, 14)$ at $a = 2$; the levels $7 \le a \le 13$ are new. In the regime $a_1 = r < a$ — which the published *Macaulay2* code excludes by construction — the zeros are $a_2 = a_3 = 3a - 2r$ for odd r and none for even r, apart from the single triple $(2, 9, 13)$ at $a = 3$; this is new for $4 \le a \le 13$, and all 78 pairs (a, r) are settled without any appeal to compiled evaluation. The two exceptions are exactly the two quadruples the conjecture already excepts, and every remaining zero is a triple on which failure of the weak Lefschetz property is already a theorem: no analogue of the exceptional quadruples appears at any level above $a = 3$ in this range. Granting Booth and Vraciu's reduction, the conjecture therefore holds at $t = (a_1 + a_2 + a_3)/3 + 1$ for every admissible triple with $2 \le a \le 13$, and no exceptional quadruple beyond the two known ones occurs at that t in that range. Up to $a = 8$ the method replaces divisor lists by a scan reaching only $\sqrt{N(a)}$ together with one modulus certificate per surviving value of $S = a_1 + a_2$; past $a = 8$ the scan is itself the obstruction — it would need $1.5 \cdot 10^{10}$ trial divisions at $a = 12$ — and is replaced by a divisor list *built* from the factorisation of $N(a)$, with each certificate evaluated at S reduced modulo its own modulus. Two small corrections to the source are recorded. Every theorem below about the polynomials $\mathcal{F}_a$ is machine-checked in Lean 4; the consequence for the conjecture, being a statement about Lefschetz properties, is not.

1. INTRODUCTION

1.1. **The conjecture.** Let $\Bbbk$ be a field of characteristic zero and $R = \Bbbk[x, y, z]$. An Artinian graded quotient R/I has the *weak Lefschetz property* (WLP) if multiplication by a general linear form ℓ has maximal rank as a map $(R/I)_j \to (R/I)_{j+1}$ for every j. For $0 < a_1 \le a_2 \le a_3$ and $t \ge 1$ put

$$I = \big(x^{a_1+t},\, y^{a_2+t},\, z^{a_3+t},\, x^{a_1}y^{a_2}z^{a_3}\big), \tag{1}$$

a monomial almost complete intersection. Writing $d_i = a_i + t$ for the three pure-power degrees, the three differences $d_i - a_i$ agree, and this is exactly the shape of a *level* monomial almost complete intersection in the normal form of Migliore, Miró-Roig and Nagel [2, Proposition 6.1(iii)]. Set $\Sigma = a_1 + a_2 + a_3$.

> **Conjecture** ([3, Conjecture 4.13]: a restatement of [2, Conjecture 6.8(b)] incorporating the two exceptional quadruples recorded in [2, Conjecture 7.13]). Assume $0 < a_1 \le a_2 \le a_3 \le 2(a_1 + a_2)$, $3 \mid \Sigma$ and $t \ge \Sigma/3$. Then, apart from the two quadruples $(a_1, a_2, a_3, t) = (2, 9, 13, 9)$ and $(3, 7, 14, 9)$, the algebra R/I of (1) fails WLP if and only

> if t is even, Σ is odd, and $a_1 = a_2$ or $a_2 = a_3$; and it fails WLP in those two cases as well.

Since $a_1 \le a_2 \le a_3$, the last condition says that a_1, a_2, a_3 are not all distinct, which is how [1] phrases it and how we phrase it below. The two exceptional quadruples are not part of [2, Conjecture 6.8(b)] as printed; they were found after that paper was first circulated and are recorded in [2, Conjecture 7.13] and in [3, Conjecture 4.13].

The "if" implication is a theorem [2, Corollary 7.4]. The forward implication — *failure of WLP forces that parity pattern* — is open, and it is what a counterexample would have to violate. Two families of cases remain open in general, as [1] lists them: Σ odd, t even, $a_1 < a_2 < a_3$; and Σ even, t odd, with $a_1 < a_2$ or $a_2 < a_3$.

1.2. **The borderline value of t, and where the source stops.** The smallest admissible t, namely $t = \Sigma/3$, has $t \equiv \Sigma \pmod 2$ and so is settled by Cook and Nagel [3, Proposition 4.14(i)], whose hypothesis is exactly that t and Σ have the same parity; the first unresolved value is therefore

$$t = \frac{\Sigma}{3} + 1.$$

Booth and Vraciu [1] attack exactly that value. Introducing the integer $a \ge 0$ by

$$a_3 = 2(a_1 + a_2) - 3a, \tag{2}$$

one has $\Sigma = 3(a_1 + a_2 - a)$ — so $3 \mid \Sigma$ is automatic — and $t = a_1 + a_2 - a + 1$, while $a_3 \le 2(a_1 + a_2)$ is exactly $a \ge 0$ and $a_3 \equiv a \pmod 2$. As [1] record, pairing [3, Theorem 4.10] with [2, Lemma 7.1] makes failure of WLP at that t equivalent to the existence of a homogeneous relation

$$FL = Ax^{t+a_1} + By^{t+a_2} + Cz^{t+a_3} + D \cdot x^{a_1} y^{a_2} z^{a_3}$$

of degree Σ with $F \notin I$; there $\deg C = a - 1$, $\deg D = 0$ and $D \neq 0$, so one may take $D = 1$. Applying the operator $\Delta = \partial/\partial x - \partial/\partial y$ exactly a times annihilates the term in C, and after the substitution $z \mapsto -(x + y)$ what remains is the assertion that a single explicit integer — one binomial-coefficient sum, written out in §2 — vanishes. Clearing the binomial ratios turns it into an integer polynomial $\mathcal{F}_a(a_1, a_2)$, and the conjecture at that t becomes a Diophantine question:

> *find every integer zero of $\mathcal{F}_a$ in the admissible region $0 < a_1 \le a_2 \le a_3$.*

Booth and Vraciu answer it for $a \le 3$ by hand. Their closing remark then reports a *Macaulay2* computation and states its limit; we quote its operative part, because both of the gaps this paper closes are visible in it (emphasis ours, here and in the next quotation; the ellipsis replaces the description of the structural shape of $\mathcal{F}_a$ that is Theorem 3.1 below):

> "We have implemented the construction for the polynomial obtained *in the case* $a_1 \ge a$ in *Macaulay2* … Unfortunately, for values of $a \ge 7$ the list of values that need to be checked is too large to be practical. Using *Macaulay2*, we were able to verify for values of a up to 6 that there are no positive integer solutions for $\mathcal{F}(a_1, a_2) = 0$, except for the known exceptions when failure of WLP has been established. For larger values of a, we were not able to finish checking all possible values even with the help of *Macaulay2* (although we have verified all possible values of S up to 1000 for a up to 40)."

and the appendix that carries the code opens: "Here we provide the *Macaulay2* code extending the proof of [Theorem 3.6] up to $a = 6$ *under the condition* $a_1 \ge a$." So two things are left open at the borderline t: the levels $a \ge 7$ altogether, and — for every $a \ge 4$ — the regime $a_1 < a$, which the published code excludes by construction. The second gap is not empty; §2.3 exhibits admissible triples inside it, and §5 shows that for $a \ge 4$ every zero other than the diagonal $a_1 = a_2$ lies in it.

1.3. **Results.** Throughout, $a \ge 2$ is an integer, a_3 is given by (2), and a pair (a_1, a_2) of positive integers is called *a-admissible* when $a_1 \le a_2 \le a_3$. The polynomial $\mathcal{F}_a$ has two shapes, because the form Δ^a produces has degree $\min(a, a_1)$: the *generic regime* $a \le a_1$, where $\mathcal{F}_a(a_1, a_2)$ is a genuine two-variable polynomial, and the *boundary regime* $a_1 = r < a$, where it collapses to a univariate polynomial $\mathcal{F}_{a,r}(a_2)$. The two regimes together cover every admissible pair.

- **Generic regime, $a = 7$ and $a = 8$** (Theorem 4.3). On $a \le a_1 \le a_2 \le a_3$ the equation $\mathcal{F}_7 = 0$ holds exactly on the diagonal $a_1 = a_2$, and $\mathcal{F}_8$ has no zero at all. These are the two levels the source names as past its practical limit.
- **Generic regime, $9 \le a \le 13$** (Theorem 4.4). The same answer at the next five levels: the diagonal for odd a, nothing for even a. The enumeration behind $a = 13$ reaches $S = a_1 + a_2 = 1\,440\,680\,120\,598\,548\,517\,206$; the source's reported partial check reaches $S \le 1000$.
- **Boundary regime, $4 \le a \le 8$** (Theorem 5.2). For every r with $1 \le r < a$ and every a_2 in the admissible range $a_2 \ge 3a - 2r$: if r is odd, the unique zero of $\mathcal{F}_{a,r}$ is $a_2 = 3a - 2r$, which gives the triple $(r, 3a - 2r, 3a - 2r)$; if r is even, there is no zero. This regime is outside the source's published computation for every $a \ge 4$.
- **Boundary regime, $9 \le a \le 13$** (Theorem 5.3). The same rule at all 50 further pairs (a, r). With the previous item and Theorem 5.1, all 78 pairs with $2 \le a \le 13$ are settled, and none of them by compiled evaluation.
- **No further exceptional quadruple** (Theorem 6.3). For $9 \le a \le 13$ and $1 \le a_1 \le a_2 \le 40$ — with no hypothesis relating a_1 to a, and for the un-normalised coefficient, so that no clearing of denominators is involved — the zeros are the diagonal at odd a and the boundary line at odd $r < a$, and nothing else. The pattern "one exceptional quadruple at each of the two smallest levels" does not continue.
- **The consequence** (Corollaries 6.1 and 6.4). Granting the reduction of [1], the conjecture stated above holds at $t = \Sigma/3 + 1$ for every admissible triple with $2 \le a \le 13$, and no exceptional quadruple beyond $(2, 9, 13, 9)$ and $(3, 7, 14, 9)$ occurs at that t in that range.

Two supporting layers are stated as well, because they are what makes the above a statement about the source's objects rather than about a transcription of them: the source's own $a \le 3$ theorem and its printed intermediate formulas, rebuilt symbol for symbol from the recursion (Theorems 4.1 and 5.1); the levels $a = 4, 5, 6$ of the generic regime, recomputed by a different elimination and agreeing with the source's *Macaulay2* result (Theorem 4.2); the structural shape of $\mathcal{F}_a$ that the source observed experimentally, here proved (Theorem 3.1); and the comparison of the cleared polynomials with the un-normalised binomial coefficient (Propositions 2.2 and 2.4). The two lemmas that carry the range past $a = 8$ — the divisor list built from a factorisation, and the certificate read modulo its own modulus — are stated as Lemmas 3.5 and 3.6 and combined in Proposition 3.7; they are elementary, and are given because they are what moves the range. Section 7 records the negative controls, including two small corrections to the source, neither of which affects any of its results.

1.4. **What is not claimed.** Four limits belong at the front.

First, the reduction is *cited, not reproved*. Nothing here is a statement about Lefschetz properties, about graded Artinian algebras, or about syzygy bundles; the theorems below are statements about integer polynomials. The implication "R/I fails WLP at $t = \Sigma/3 + 1 \Rightarrow \mathcal{F}_a(a_1, a_2) = 0$" is due to Booth and Vraciu and is taken as given. Corollaries 6.1 and 6.4 are therefore conditional on it, and on [2, Corollary 7.4] for the converse direction of the conjecture. We claim nothing about the converse implication "$\mathcal{F}_a = 0 \Rightarrow$ failure"; the reduction gives necessity only.

Second, the range. Everything below is the single value $t = \Sigma/3 + 1$ and the levels $2 \le a \le 13$. The remaining small levels $a \in \{0, 1\}$ lie inside the source's own theorem, which is stated for $a \le 3$ and covers $a = 0$ by an earlier result of [1] and $a = 1$ by the one-line observation that the coefficient is then a nonzero multiple of $a_1 - a_2$. They are not reproved here, and that is why Corollaries 6.1 and 6.4 are stated from $a = 2$ upward. The level $a = 14$ is not reached; §8 records what it would cost.

Third, the identification of $\mathcal{F}_a$ with the source's polynomial. For $a = 2, 3$ it is proved here, symbol for symbol, from the source's own displayed formulas (Theorems 4.1, 5.1). For $4 \le a \le 11$ the normalising factors used below were checked to reproduce the constant appearing in the source's *Macaulay2* listing, as a polynomial identity in $\mathbb{Z}[a_1, a_2]$; that check is an exact integer computation carried out *outside* the formal development, and is reported as such. That comparison was not run at $a = 12, 13$, so nothing about those two levels rests on it. Proposition 2.2 is the independent safety net: over a finite window it compares the cleared polynomials with the un-normalised binomial-coefficient sum, computed from the

definition; Proposition 2.4 is that same comparison at $9 \le a \le 13$, and it is what ties the polynomials used at the two highest levels to the coefficient the reduction is about.

Fourth, the source text. The e-print [1] was read here in its version v2 of 6 July 2026 — the LaTeX source of that version, from the live *arXiv* record on 3 September 2026; that record shows exactly two versions. Both displayed quotations above, every formula of [1] reproduced below, and the number "Theorem 3.6" we use for its borderline theorem are v2's, the last obtained by compiling that source. The one place where we do not follow v2 verbatim is the branch condition of (3)–(4), for the reason given in Proposition 2.3(2). The version of record appeared online in *Experimental Mathematics* on 10 August 2026; it has no volume, issue or final page numbers assigned, its text is behind a paywall and was not read here, and no numbering of *that* version is quoted anywhere. A reader working from the published version should locate the borderline theorem by its statement rather than by our number.

The numbering used for [2] — Proposition 6.1(iii), Conjecture 6.8(b), Lemma 7.1, Corollary 7.4, Conjecture 7.13 — was read off the published *Transactions* version. The numbering used for [3] — Theorem 4.10, Conjecture 4.13, Proposition 4.14(i) — was read off the e-print arXiv:1606.01809, in which those results carry exactly those numbers.

1.5. **Method.** The elimination has the shape the source's closing remark describes, with two changes that are what make $a = 7, 8$ affordable.

Write $S = a_1 + a_2$ and $P = a_1 a_2$. In the generic regime $\mathcal{F}_a$ is symmetric for even a and antisymmetric for odd a, so after dividing out $(a_1 - a_2)$ in the odd case and taking the primitive part one gets a polynomial $f_a(S, P)$ of degree $\lfloor a/2 \rfloor$ in P with

$$f_a(S, P) = \mathcal{P}_a(S) + \big(2S - c(a)\big)\, G_a(S, P),$$

$\mathcal{P}_a$ an explicit product of linear factors and $c(a) \in \{3a - 2, 3a - 1\}$ according to the parity of a. This is the shape the source reports having observed experimentally; here it is an identity proved by expansion (Theorem 3.1). It forces $2S - c(a) \mid \mathcal{P}_a(S)$ and hence $2S - c(a) \mid N(a)$ for the explicit integer $N(a) = 2^{\deg \mathcal{P}_a} \mathcal{P}_a(c(a)/2)$, so only finitely many S survive.

Change one: do not list the divisors of $N(a)$, scan for them. A scan of trial divisors $d \le \sqrt{N}$ that records both d and N/d is complete, because a divisor exceeding $\sqrt{N}$ has cofactor below it. At $a = 8$ this replaces a search of size $3.2 \cdot 10^{11}$ by one of size $5.6 \cdot 10^5$.

Change two: kill each surviving S with a modulus, not with a factorisation. For fixed S, $f_a(S, \cdot)$ is a univariate integer polynomial in P; a single modulus m on which it has no zero rules out every integer P at once. The moduli needed are tiny: at most 151 over all of $2 \le a \le 8$, and at most 37 in the boundary regime.

The boundary regime needs neither device. There $\mathcal{F}_{a,r}$ is univariate of degree at most 15; splitting off its integer roots leaves a cofactor with none, certified by one modulus. All 28 pairs (a, r) with $2 \le a \le 8$ and $1 \le r < a$ are handled that way, and the whole of that half of the development is free of compiled evaluation.

Past $a = 8$ the first of those two devices becomes the obstruction in its turn: the scan is $\sqrt{|N(a)|}$ trial divisions, which is $2.6 \cdot 10^8$ at $a = 11$ and about $1.5 \cdot 10^{10}$ at $a = 12$. The way out is not a faster scan but a different argument. By Theorem 3.1, $N(a) = \prod_i (c(a) - 2i)$ is a product of about $3a/2$ odd integers smaller than $3a$, so its prime factorisation costs nothing and the divisor list can be *built* from it — complete because a divisor of a product of coprime prime powers splits accordingly (Lemma 3.5), not because the list has the right length. The moduli need one more turn as well: at $a = 13$ the surviving values of S run to $1.4 \cdot 10^{21}$ and $f_a(S, \cdot)$ has degree 18 in S, so a certificate evaluated at the true S involves integers of more than 380 decimal digits, while the certificate is only a statement modulo m and S may be reduced first (Lemma 3.6). With both replacements the levels $9 \le a \le 13$ come within reach, and one of the three exhaustive Boolean evaluations per level disappears together with the scan. In the boundary regime the degree of $\mathcal{F}_{a,r}$ grows to 25 and the factorisation is carried out on coefficient lists rather than on expanded polynomials; that half stays free of compiled evaluation at every one of the 78 pairs.

1.6. **Changes from version 1.** The first version of this paper settled $2 \le a \le 8$. What is new here, and nothing else, is:

- the generic regime at $9 \le a \le 13$ (Theorem 4.4) and the boundary regime at the 50 pairs (a, r) with $9 \le a \le 13$ (Theorem 5.3);
- the complete zero set of the un-normalised coefficient at those five levels over a window that contains every boundary candidate there, and with it the statement that no analogue of the two exceptional quadruples occurs (Theorem 6.3);
- the consequence for the conjecture over the enlarged range (Corollary 6.4);
- the two elementary devices that make the enlarged range affordable, and the form of the elimination that uses them (Lemmas 3.5, 3.6, Proposition 3.7);
- the comparison of the cleared polynomials with the un-normalised coefficient at the new levels (Proposition 2.4) and the negative controls for the new levels and the new devices (Proposition 7.2);
- a rewritten §8: what was there recorded as an unverified computation for $9 \le a \le 13$, and as open engineering, is now proved, so the section states the price of the next level instead — and Remark 8.1 is correspondingly reduced to what it now is, an independent cross-check;
- a rewritten §9, with the enlarged development and its revised division between kernel-checked and compiled evaluation.

Every other statement, and the number it carries, is unchanged; nothing found since contradicts any of them.

2. The polynomials

Throughout this section $a \ge 2$ is fixed, a_3 is given by (2), and (a_1, a_2) ranges over the a-admissible pairs.

2.1. **The operator.** Fix a_1, a_2 and let $\Delta = \partial/\partial x - \partial/\partial y$ act on binary forms. Booth and Vraciu define an operator τ by $\tau^0 D = D = 1$ and, for $k \ge 1$,

$$\tau^k D = xy\,\Delta(\tau^{k-1}D) + \big((a_1 - k + 1)y - (a_2 - k + 1)x\big)\,\tau^{k-1}D, \qquad 1 \le k \le a_1, \tag{3}$$

$$\tau^k D = y\,\Delta(\tau^{k-1}D) - (a_2 - k + 1)\,\tau^{k-1}D, \qquad a_1 < k \le a_2. \tag{4}$$

Thus $\tau^k D$ is a binary form of degree k while $k \le a_1$ and of degree a_1 afterwards; the branch changes exactly where the recursion runs out of x. What the two branches encode is the identity displayed in the proof of [1, Theorem 3.6],

$$\Delta^k\big(x^{a_1}y^{a_2}z^{a_3}D\big) = \begin{cases} x^{a_1-k}y^{a_2-k}z^{a_3}\,(\tau^k D), & k \le a_1, \\ y^{a_2-k}z^{a_3}\,(\tau^k D), & a_1 < k \le a_2. \end{cases}$$

We quote it only as motivation for the shape of the recursion. It is not used, not proved and not machine-checked anywhere below; what the development uses is (3)–(4) alone, as a definition. The first values are $\tau^1 D = a_1 y - a_2 x$ and, for $a_1 \ge 2$, $\tau^2 D = (a_2^2 - a_2)x^2 - 2a_1a_2xy + (a_1^2 - a_1)y^2$, while for $a_1 = 1$ the second branch already applies and $\tau^2 D = (a_2^2 - a_2)x - (a_1a_2 + a_2)y$.

Put $m = \min(a, a_1)$ and write

$$\tau^a D = \sum_{j=0}^{m} q_j\, x^{m-j}y^j, \qquad q_j = q_j(a_1, a_2) \in \mathbb{Z}.$$

2.2. **The coefficient, and the clearing.** With a_3 as in (2) and $h = (a_3 + a)/2 = t - 1$, the product $(\tau^a D)(x+y)^{a_3}$ is homogeneous of degree $2h$, so the vanishing demanded by the reduction is the vanishing of the single integer

$$C_a(a_1, a_2) = \sum_{j=0}^{m} q_j \binom{a_3}{h-j}, \tag{5}$$

the coefficient of $x^h y^h$ when $a \le a_1$ and of $x^{h+a_1-a}y^h$ when $a_1 < a$. Since $a \le a_3$ on the admissible region, $\binom{a_3}{h} > 0$, and

$$\binom{a_3}{h-j}\Big/\binom{a_3}{h} = \frac{\prod_{l=1}^{j}(a_3+a-2l+2)}{\prod_{l=1}^{j}(a_3-a+2l)}.$$

Multiplying (5) by $\binom{a_3}{h}^{-1}\prod_{l=1}^{m}(a_3-a+2l)$ — a positive rational on the admissible region, because $a_3-a+2l \ge 2$ there — leaves the integer combination $\sum_j q_j u_j$ with

$$u_j = \prod_{l=1}^{j}(a_3+a-2l+2)\cdot\prod_{l=j+1}^{m}(a_3-a+2l). \tag{6}$$

So $C_a = 0$ if and only if $\sum_j q_j u_j = 0$. Reindexing the second product of (6) by $c = a-l$ writes both products in the same linear forms a_3+a-2c: the first runs over $0 \le c \le j-1$, the second over $a-m \le c \le a-j-1$. In the *generic* regime $m = a$ they therefore start at the same place,

$$u_j = \prod_{c=0}^{j-1}(a_3+a-2c)\cdot\prod_{c=0}^{a-j-1}(a_3+a-2c), \qquad \text{so} \qquad u_j = u_{a-j}.$$

That symmetry is what §3 uses. It is special to the generic regime: when $m = r < a$ the second product starts at $c = a-r > 0$ and $u_j \ne u_{r-j}$ in general.

Definition 2.1 (the two polynomials)**.** *Generic regime* ($a \le a_1$, so $m = a$). The factors a_3+a-2c with $0 \le c < \lceil a/2\rceil$ divide every u_j, and each is $\ge a_3+1 > 0$ on the admissible region; write $\tilde{u}_j$ for the quotient of u_j by their product and set

$$\mathcal{F}_a(a_1,a_2) = \sum_{j=0}^{a} q_j\,\tilde{u}_j.$$

This is the normalisation the source's *Macaulay2* listing uses.

Boundary regime ($a_1 = r < a$, so $m = r$). Nothing is cancelled:

$$\mathcal{F}_{a,r}(a_2) = \sum_{j=0}^{r} q_j\,u_j,$$

a univariate polynomial in a_2, since $a_1 = r$ is a constant here. The removable factors are again positive on the admissible region, so keeping them adds and loses no zero.

In both regimes, then, $C_a(a_1,a_2) = 0 \iff \mathcal{F}_a(a_1,a_2) = 0$ (resp. $\mathcal{F}_{a,r}(a_2) = 0$) at every admissible pair, by positivity of the cleared factors. That equivalence is an elementary argument about the region, not a computation, and it is not part of the formal development; what *is* machine-checked is its finite shadow.

Proposition 2.2 (the normalisation is faithful)**.** *Let C_a be the un-normalised coefficient* (5)*, evaluated from the recursion* (3)–(4) *with ordinary binomial coefficients and no cancellation of any kind. Then:*

1. *for all $1 \le a \le 8$ and all admissible (a_1,a_2) with $a \le a_1 \le a_2 \le 40$, $C_a(a_1,a_2) = 0 \iff \mathcal{F}_a(a_1,a_2) = 0$;*
2. *for each of the 28 pairs (a,r) with $2 \le a \le 8$, $1 \le r < a$, and all admissible $a_2 \le 60$, $C_a(r,a_2) = 0 \iff \mathcal{F}_{a,r}(a_2) = 0$;*
3. *for all $1 \le a \le 8$ and all admissible (a_1,a_2) with $a_1 \le a_2 \le 40$, $C_a(a_1,a_2) = 0$ holds precisely when $a_1 = a_2$ with a odd, or a_1 is odd with $a_1 < a$ and $a_2 = 3a-2a_1$, or $(a;a_1,a_2) = (2;3,7)$, or $(a;a_1,a_2) = (3;2,9)$.*

Proof and scope. Each part is a finite exhaustive evaluation: part (1) sweeps the $9\cdot 41\cdot 41$ triples (a,a_1,a_2) and filters them by the stated hypotheses, part (2) the $28\cdot 61$ pairs, part (3) the same grid as part (1). Both sides of every equivalence are recomputed from their definitions inside the check; no coefficient list is supplied as data. Part (3) is the statement that the answer found in §4 and §5 is the answer for the honest coefficient and not only for the cleared one. The same comparison was

run outside the formal development over the wider window $a_1 \le a_2 \le 140$ by a second, independent implementation that evaluates (5) directly from (3)–(4); it agrees on every triple. □

2.3. **Two corrections to the source.** Both are visible from a one-line example, and neither affects any result of [1]. They are stated here because a reader reconstructing the computation will meet them, and because the first is precisely the reason the boundary regime is a real gap.

Proposition 2.3. (1) *The hypotheses of the conjecture do not imply that $a \le a_1$. The triple $(a_1, a_2, a_3) = (1, 4, 4)$ satisfies all of them, with $a = 2$ in (2): indeed $0 < 1 \le 4 \le 4$, and $4 \le 2(1+4)$, and $3 \mid 9$, and $4 = 2(1+4) - 3 \cdot 2$. Yet $a = 2 > 1 = a_1$.*

(2) *The branch condition of τ, printed as $1 \le k \le a_1 + 1$ and as $a_1 + 1 < k \le a_2 + 1$, must be read as $k \le a_1$ and $a_1 < k$. At $a_1 = 1$, $k = 2$ the two readings produce forms of different degree, and the source's own displayed $\tau^2 D$ for $a_1 = 1$ has degree 1, so it is the reading $k \le a_1$.*

Proof and scope. (1) is the displayed inequality chain, checked as an arithmetic statement about the six integers involved; it depends on no axioms whatsoever. The remark at issue reads "Note that the assumption implies $a \le a_1$, for otherwise $a_3 = 2a_1 + 2a_2 - 3a < 2a_1 - a_2 \le a_2$, a contradiction"; the strict inequality there needs $a > a_2$, not $a > a_1$, so what the argument establishes is $a \le a_2$ — which does hold at $(1, 4, 4)$. The paper then treats $a_1 < a$ by hand for the values of a it proves, so no theorem of [1] is affected. (2) is the comparison of the two degrees, $\deg \tau^2 D = 1$ against 2, obtained by unfolding the recursion; the reading used throughout this paper is $k \le a_1$, and Theorem 5.1 shows that this reading reproduces the source's own $a_1 = 1$ and $a_1 = 2$ formulas. □

2.4. **The same comparison at the levels $9 \le a \le 13$.** Proposition 2.2 is stated for the range the first version of this paper covered. The levels added in §4.1 and §5.1 carry the same comparison.

Proposition 2.4 (the normalisation is faithful at the new levels)**.** *Let C_a be as in Proposition 2.2. Then:*

(1) *for all $9 \le a \le 13$ and all admissible (a_1, a_2) with $a \le a_1 \le a_2 \le 40$, $C_a(a_1, a_2) = 0 \iff \mathcal{F}_a(a_1, a_2) = 0$;*

(2) *for each of the 50 pairs (a, r) with $9 \le a \le 13$, $1 \le r < a$, and all admissible $a_2 \le 60$, $C_a(r, a_2) = 0 \iff \mathcal{F}_{a,r}(a_2) = 0$.*

Proof and scope. Both parts are finite exhaustive evaluations, of the same shape as the corresponding parts of Proposition 2.2: part (1) sweeps the $5 \cdot 41 \cdot 41$ triples (a, a_1, a_2) and filters them by the stated hypotheses, part (2) the $50 \cdot 61$ pairs. Both sides of every equivalence are recomputed from their definitions inside the check. The same comparison was run outside the formal development over the wider window $a_1 \le a_2 \le 80$ by a second, independent implementation evaluating (5) directly from (3)–(4); it agrees on every triple, finding 76, 5, 75, 6 and 74 zeros at $a = 9, \dots, 13$ and none besides. □

3. The elimination

3.1. **Structure.** Because $u_j = u_{a-j}$ and the recursion (3) is equivariant under $x \leftrightarrow y$, $a_1 \leftrightarrow a_2$, the polynomial $\mathcal{F}_a$ is symmetric in (a_1, a_2) for even a and antisymmetric for odd a. In the odd case divide out $(a_1 - a_2)$; in both cases take the primitive part. What comes out is a polynomial in the elementary symmetric functions $S = a_1 + a_2$, $P = a_1 a_2$.

Theorem 3.1 (the shape the source observed)**.** *For each a with $2 \le a \le 8$ there are an integer polynomial $f_a \in \mathbb{Z}[S, P]$ of degree $\lfloor a/2 \rfloor$ in P, a polynomial $G_a \in \mathbb{Z}[S, P]$ and a polynomial $K_a \in \mathbb{Z}[S]$ such that, identically in a_1, a_2 and in S, P,*

$$\mathcal{F}_a(a_1, a_2) = 2^{\lfloor a/2 \rfloor} (a_1 - a_2)^{\varepsilon} f_a(a_1 + a_2,\, a_1 a_2), \qquad \varepsilon = \begin{cases} 1, & a \text{ odd}, \\ 0, & a \text{ even}, \end{cases}$$

$$f_a(S, P) = \mathcal{P}_a(S) + \big(2S - c(a)\big) G_a(S, P), \qquad 2^{\deg \mathcal{P}_a}\, \mathcal{P}_a(S) = N(a) + \big(2S - c(a)\big) K_a(S),$$

where

$$c(a)=\begin{cases}3a-2, & a \text{ odd},\\ 3a-1, & a \text{ even},\end{cases}\qquad \mathcal{P}_a(S)=\begin{cases}\prod_{i=1}^{a-1}(S-i)\cdot\prod_{i=(3a+1)/2}^{2a-1}(S-i), & a \text{ odd},\\ \prod_{i=0}^{a-1}(S-i)\cdot\prod_{i=3a/2}^{2a-1}(S-i), & a \text{ even},\end{cases}$$

and $N(a)=\prod_i\big(c(a)-2i\big)$*, the product over the roots* i *of* $\mathcal{P}_a$ *listed with multiplicity. In particular the constant term of* f_a *in* P *is* $\mathcal{P}_a(S)$ *exactly, with scalar* 1*.*

Proof and scope. For each of the seven levels both displayed identities are identities between explicit integer polynomials, and are proved by expanding both sides; no search and no evaluation is involved. The data f_a, G_a, K_a are given explicitly per level, G_8 being the largest at degree 9 in S and 4 in P. The values are

a	2	3	4	5	6	7	8
$c(a)$	5	7	11	13	17	19	23
$N(a)$	-15	-45	10395	51975	-34459425	-241215975	316234143225

The source states the corresponding shape — the constant term being the printed product and every higher coefficient being divisible by $2S-c(a)$ — as an experimental observation, for the range it computed. The same identities were verified for $a\le 11$ in exact integer arithmetic outside the formal development. Whether they hold for *all* a is open; equivalently, whether f_a is independent of P at $a_3=-2$ (a odd), resp. $a_3=-1$ (a even). A proof would remove the per-level verification from everything that follows. □

3.2. **Two finite-search devices.** Both are general lemmas: each level then supplies only data.

Lemma 3.2 (divisor enumeration at square-root cost)**.** *Let* $N\ge 1$ *and* $r\ge 0$ *satisfy* $N<(r+1)^2$*, and let* L *be a list of naturals such that every* d *with* $1\le d\le r$ *and* $d\mid N$ *has both* $d\in L$ *and* $N/d\in L$*. Then every positive divisor of* N *lies in* L*.*

Proof. A divisor $d>r$ has cofactor $e=N/d$; were $e>r$ as well then $N=de\ge(r+1)^2>N$. So $e\le r$, and the hypothesis applied to e puts $N/e=d$ in L. The hypothesis itself is one Boolean scan of the r trial divisors. □

Lemma 3.3 (a modulus kills every integer root)**.** *Let* $g\in\mathbb{Z}[X]$ *and let* $m\ge 1$ *be such that* $m\nmid g(j)$ *for every* $j\in\{0,\dots,m-1\}$*. Then* $g(x)\neq 0$ *for every* $x\in\mathbb{Z}$*.*

Proof. $g(x)\equiv g(x \bmod m) \pmod m$, and the residue $x \bmod m$ lies in $\{0,\dots,m-1\}$. □

Proposition 3.4 (the generic-regime elimination, once and for all)**.** *Fix* $a\ge 1$ *and suppose given: a coefficient list* $c_a(S)$ *of* $f_a(S,\cdot)$ *in* P*; the data* $\mathcal{P}_a$*,* G_a*,* K_a*,* $c=c(a)<4a$ *and* $N=N(a)\neq 0$ *satisfying the two identities of Theorem 3.1; a bound* r *with* $|N|<(r+1)^2$ *and a list* L *passing the scan of Lemma 3.2; for each* $S\in\{(d+c)/2 : d\in L\}$ *with* $S>40$ *a modulus* m_S *as in Lemma 3.3 for* $f_a(S,\cdot)$*; and a finite list* E *of exceptions such that* $f_a(x+y,xy)\neq 0$ *for every* $(x,y)\notin E$ *with* $a\le x\le y$ *and* $x+y\le 40$*. Then* $f_a(a_1+a_2,a_1a_2)\neq 0$ *for every* a*-admissible pair with* $a\le a_1$ *and* $(a_1,a_2)\notin E$*.*

Proof. Admissibility gives $a_2\le a_3=2S-3a$, hence $S\ge 2a$ and $2S-c\ge a+1>0$. If $f_a(S,P)=0$ then the first identity of Theorem 3.1 makes $2S-c$ divide $\mathcal{P}_a(S)$, and the second makes it divide N. So $d=2S-c$ is a positive divisor of $|N|$, hence lies in L by Lemma 3.2, hence $S=(d+c)/2$ is one of the finitely many listed values. If $S>40$ its modulus and Lemma 3.3 contradict $f_a(S,P)=0$; if $S\le 40$ then (a_1,a_2) lies in the finite grid handled by hypothesis. □

3.3. **Two replacements for the scan.** Lemma 3.2 costs $\sqrt{|N|}$ trial divisions, which stops being affordable at $a=11$ and is hopeless at $a=12$. Both of the following are elementary; they are stated because they are what carries the range further, and because the second changes what has to be trusted about the divisor list — under Lemma 3.2 the list is supplied and validated by a scan, whereas here it is constructed and its completeness is proved.

Lemma 3.5 (the divisor list, built)**.** *Let* $p_1,\dots,p_k$ *be distinct primes, let* $e_1,\dots,e_k\ge 0$ *and* $N=\prod_i p_i^{e_i}$*. Let* L *be the list of all products* $\prod_i p_i^{f_i}$ *with* $0\le f_i\le e_i$*. Then every positive divisor of* N *lies in* L*.*

Proof. Induction on k. For $k = 0$, $N = 1$ and $L = (1)$. For the step, write $N = p_1^{e_1} M$; a divisor d of a product factors as $d = d_1 d_2$ with $d_1 \mid p_1^{e_1}$ and $d_2 \mid M$, and a divisor of a prime power is p_1^f with $f \le e_1$. Nothing counts the entries of L, and L need not be free of repetitions. □

Lemma 3.6 (a modulus kills every integer root, with the argument reduced)**.** *Let $g \in \mathbb{Z}[S, P]$, let $m \ge 1$ and let $S \in \mathbb{Z}$. If $m \nmid g(S \bmod m,\, j)$ for every $j \in \{0, \dots, m-1\}$, then $g(S, P) \ne 0$ for every $P \in \mathbb{Z}$.*

Proof. $g(S, P) \equiv g(S \bmod m,\, P \bmod m) \pmod m$, and $P \bmod m$ lies in $\{0, \dots, m-1\}$. □

Proposition 3.7 (the generic-regime elimination, past the scan)**.** *Fix $a \ge 1$ and suppose given: the data f_a, $\mathcal{P}_a$, G_a, K_a, $c = c(a) < 4a$ and $N = N(a) \ne 0$ of Proposition 3.4; distinct primes p_i and exponents e_i with $|N| = \prod_i p_i^{e_i}$, and the list L of Lemma 3.5; for each $S \in \{(d + c)/2 : d \in L\}$ with $S > 40$ a modulus m_S as in Lemma 3.6 for $f_a(S, \cdot)$; and a finite list E of exceptions such that $f_a(x + y, xy) \ne 0$ for every $(x, y) \notin E$ with $a \le x \le y$ and $x + y \le 40$. Then $f_a(a_1 + a_2, a_1 a_2) \ne 0$ for every a-admissible pair with $a \le a_1$ and $(a_1, a_2) \notin E$.*

Proof. Word for word the proof of Proposition 3.4, with Lemma 3.5 in place of Lemma 3.2 at the step that puts the positive divisor $d = 2S - c$ of $|N|$ into L, and Lemma 3.6 in place of Lemma 3.3 at the step that discards a surviving $S > 40$. □

4. The generic regime $a \le a_1$

For each level the data of Proposition 3.4 were produced by an exact integer computation and are then verified in place: the identity $\mathcal{F}_a = 2^{\lfloor a/2 \rfloor}(a_1 - a_2)^{\varepsilon} f_a$ by expansion, the divisor scan and the certificate table by exhaustive evaluation. The sizes are:

a	$\lfloor\sqrt{\lvert N(a)\rvert}\rfloor$	# divisors	#{S admissible}	#{$S \le 40$}	# moduli	max m_S
2	3	4	3	3	0	–
3	6	6	4	4	0	–
4	101	32	30	12	18	43
5	227	48	45	12	33	47
6	5870	240	237	16	221	107
7	15531	360	356	15	341	109
8	562346	1728	1724	18	1706	151

The largest admissible S is 120 607 997 at $a = 7$ and 158 117 071 624 at $a = 8$; the source's reported partial check covers $S \le 1000$.

Theorem 4.1 (the source's levels, rebuilt)**.** $\mathcal{F}_2(a_1, a_2) = 2\big[(a_1 + a_2 - 3)(a_1^2 + a_2^2 - a_1 - a_2) - 2(a_1 + a_2 - 2)a_1 a_2\big]$ *and*

$$\mathcal{F}_3(a_1, a_2) = 2\big[(a_1 + a_2 - 5)(a_1^3 - a_2^3 - 3a_1^2 + 3a_2^2 + 2a_1 - 2a_2) + (a_1 + a_2 - 3)(-3a_1^2 a_2 + 3a_1 a_2^2)\big],$$

where the two bracketed expressions are the polynomials printed in [1] *as $\mathcal{F}$ (its case $a = 2$, $a_1 \ge 2$) and as $\mathcal{G}$ (its case $a = 3$, $a_1 \ge 3$, before the division by $a_1 - a_2$ that produces the $\mathcal{F}$ of that case); and $\tau^1 D$, $\tau^2 D$, $\tau^3 D$ are the forms printed there, in the regime $a_1 \ge k$ and in each of the special cases $a_1 = 1$, $a_1 = 2$. Moreover, on the admissible region:*

$$2 \le a_1 \le a_2 \le a_3\text{:}\quad \mathcal{F}_2 = 0 \iff (a_1, a_2) = (3, 7); \qquad 3 \le a_1 \le a_2 \le a_3\text{:}\quad \mathcal{F}_3 = 0 \iff a_1 = a_2.$$

Proof and scope. The displayed identities are proved by expanding the recursion; they are the compute-first gate for everything after, and they are what makes "$\mathcal{F}_a$" below the source's polynomial and not a lookalike. The two solution sets follow from Proposition 3.4 with the data of the table: at $a = 2$ the divisor list is $\{1, 3, 5, 15\}$, giving the candidates $S \in \{3, 4, 5, 10\}$, all ≤ 40, and the exception list is $E = \{(3, 7)\}$; at $a = 3$ the list is $\{1, 3, 5, 9, 15, 45\}$, giving $S \in \{4, 5, 6, 8, 11, 26\}$, again all ≤ 40, and $E = \emptyset$, the diagonal being carried by the factor $(a_1 - a_2)$. The pair $(3, 7)$ is the triple $(3, 7, 14)$, i.e. the excepted quadruple $(3, 7, 14, 9)$. This is the source's own theorem in these two cases; it is recomputed, not extended. □

Theorem 4.2 (the levels the source computed)**.** *Let $4 \le a \le 6$ and let (a_1, a_2) be a-admissible with $a \le a_1$. Then $\mathcal{F}_4(a_1, a_2) \neq 0$ and $\mathcal{F}_6(a_1, a_2) \neq 0$, while $\mathcal{F}_5(a_1, a_2) = 0$ if and only if $a_1 = a_2$.*

Proof and scope. Proposition 3.4 with the data of the table, per level; the exception lists are empty and the diagonal at $a = 5$ is again the explicit factor $(a_1 - a_2)$. The finite work per level is: one polynomial identity by expansion; one scan of $\lfloor\sqrt{|N(a)|}\rfloor$ trial divisions (101, 227, 5870); one table of 18, 33, 221 moduli, each tested on all of its residues; and one sweep of the grid $\{(a_1, a_2) : a \le a_1 \le a_2,\ a_1 + a_2 \le 40\}$. The source reports the same conclusion for $a \le 6$ by *Macaulay2*, under the same hypothesis $a_1 \ge a$; the route here is different — a square-root divisor scan plus modulus certificates in place of divisor lists of $\mathcal{P}_a(S)$ — and the two agree. □

Theorem 4.3 (the levels the source could not reach)**.** *Let (a_1, a_2) be 7-admissible with $7 \le a_1$. Then $\mathcal{F}_7(a_1, a_2) = 0$ if and only if $a_1 = a_2$. Let (a_1, a_2) be 8-admissible with $8 \le a_1$. Then $\mathcal{F}_8(a_1, a_2) \neq 0$.*

Proof and scope. Proposition 3.4 again, with the $a = 7$ and $a = 8$ rows of the table. The finite searches are, for $a = 7$: the expansion identity $\mathcal{F}_7 = 8(a_1 - a_2) f_7(S, P)$; a scan of 15 531 trial divisors of 241 215 975, returning 360 divisors, of which 356 give an admissible S; a table of 341 moduli, the largest 109, one per surviving $S > 40$; and the grid sweep for the 15 values $S \le 40$. For $a = 8$: the identity $\mathcal{F}_8 = 16 f_8(S, P)$; a scan of 562 346 trial divisors of 316 234 143 225, returning 1728 divisors, 1724 of them admissible; a table of 1706 moduli, the largest 151; and the grid sweep for the 18 values $S \le 40$. The divisor scans and the certificate tables are the only exhaustive computations in the proof, and they are exhaustive by Lemma 3.2: nothing is sampled and no bound on S is imposed. In particular the enumeration reaches $S = 120\,607\,997$ and $S = 158\,117\,071\,624$ respectively, which is why the answer is complete and not a check up to a cutoff.

The zeros found are exactly the diagonal at $a = 7$ and none at $a = 8$ — the pattern of the lower levels, continued. The source states that it did not complete either level. □

4.1. **The levels $9 \le a \le 13$.** Here the data of Proposition 3.7 replace those of Proposition 3.4. The integer $N(a)$ and the degree of $\mathcal{P}_a$ are

a	$N(a)$	$\deg \mathcal{P}_a$
9	$3^7 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	12
10	$-3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	15
11	$-3^8 \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	15
12	$3^9 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$	18
13	$3^9 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$	18

with $c(a) = 25, 29, 31, 35, 37$ in that order, and the search sizes are

a	# divisors	$\#\{S$ admissible$\}$	$\#\{S \le 40\}$	# moduli	$\max m_S$	$\max S$
9	2304	2299	16	2283	173	$1.4 \cdot 10^{12}$
10	8640	8635	16	8619	197	$3.1 \cdot 10^{15}$
11	12 960	12 954	14	12 940	179	$3.4 \cdot 10^{16}$
12	46 080	46 074	14	46 060	199	$1.1 \cdot 10^{20}$
13	69 120	69 113	12	69 101	239	$1.4 \cdot 10^{21}$

Two comparisons place these numbers. The scan of Lemma 3.2 would have needed $1.7 \cdot 10^6$, $7.9 \cdot 10^7$, $2.6 \cdot 10^8$, $1.5 \cdot 10^{10}$ and $5.4 \cdot 10^{10}$ trial divisions at the five levels, against factorisations into at most ten primes below $3a$. And of the admissible values of S, only 153, 185, 191, 218 and 223 are at most 1000, the bound of the partial check the source reports.

Theorem 4.4 (the five further levels)**.** *Let $9 \le a \le 13$ and let (a_1, a_2) be a-admissible with $a \le a_1$. If a is odd, then $\mathcal{F}_a(a_1, a_2) = 0$ if and only if $a_1 = a_2$. If a is even, then $\mathcal{F}_a(a_1, a_2) \neq 0$.*

Proof and scope. Proposition 3.7 per level, with the data of the two tables. The exception list E is empty at all five levels; for odd a the diagonal is carried, as before, by the explicit factor $(a_1 - a_2)$ in $\mathcal{F}_a = 2^{\lfloor a/2 \rfloor}(a_1 - a_2)^{\varepsilon} f_a(S, P)$, whose scalar is 16, 32, 32, 64, 64 at $a = 9, \ldots, 13$. The finite work per level is: that expansion identity; the two identities of Theorem 3.1, which hold at these levels with the

same $c(a)$ and the same shape of $\mathcal{P}_a$ and are proved the same way, by expanding both sides; the primality of the at most ten primes involved together with the single numeric identity $|N(a)| = \prod_i p_i^{e_i}$; one table of moduli, one per admissible $S > 40$, each tested on all of its residues; and one sweep of the grid $\{(a_1, a_2) : a \le a_1 \le a_2,\ a_1 + a_2 \le 40\}$. The certificate table and the grid sweep are the only exhaustive computations, and the enumeration is exhaustive by Lemma 3.5: no bound on S is imposed and nothing is sampled. In particular it reaches $S = 1\,423\,053\,644\,525$ at $a = 9$ and $S = 1\,440\,680\,120\,598\,548\,517\,206$ at $a = 13$.

The answer is the pattern of the lower levels, continued without deviation for five more levels. The source states that it did not complete any level above $a = 6$. □

5. The boundary regime $a_1 = r < a$

Here $\tau^a D$ has degree r, not a, the vanishing condition moves to the coefficient of $x^{h+r-a}y^h$, and $\mathcal{F}_{a,r}$ is univariate. Admissibility $a_2 \le a_3 = 2(r + a_2) - 3a$ reads

$$a_2 \ \ge\ 3a - 2r,$$

and $3a - 2r > r$ because $r < a$, so the constraint $r \le a_2$ is implied. This is the regime the source's published code excludes, and by Proposition 2.3(1) it is not empty.

Theorem 5.1 (the source's boundary cases, rebuilt)**.** *$\mathcal{F}_{2,1}(a_2) = 2a_2(a_2 - 4)(a_2 - 1)$, which after division by the factor a_2 is the source's own computation in that case; and on the admissible ranges,*

$$\mathcal{F}_{2,1}(a_2) = 0 \iff a_2 = 4, \qquad \mathcal{F}_{3,1}(a_2) = 0 \iff a_2 = 7, \qquad \mathcal{F}_{3,2}(a_2) = 0 \iff a_2 = 9,$$

i.e. the triples $(1, 4, 4)$, $(1, 7, 7)$ and $(2, 9, 13)$. Furthermore the forms $\tau^2 D$ at $a_1 = 1$ and $\tau^3 D$ at $a_1 = 1$ and at $a_1 = 2$ agree with the source's printed formulas.

Proof and scope. Each of the three polynomials is factored over $\mathbb{Z}$ by an identity proved by expansion — for instance $\mathcal{F}_{3,2}(a_2) = -4\,a_2(a_2 - 1)(a_2 - 2)^2(a_2 - 9)$ — after which the roots are read off and the admissible range $a_2 \ge 3a - 2r$ discards those below it. No search is involved. The triple $(2, 9, 13)$ is the excepted quadruple $(2, 9, 13, 9)$; together with $(3, 7, 14)$ from Theorem 4.1 this recovers both of the conjecture's exceptions, and they are the only ones in $2 \le a \le 8$. □

Theorem 5.2 (the boundary regime, completely)**.** *Let $4 \le a \le 8$ and $1 \le r < a$, and let $a_2 \ge 3a - 2r$. Then*

$$\mathcal{F}_{a,r}(a_2) = 0 \quad \iff \quad r \text{ is odd and } a_2 = 3a - 2r.$$

In particular, for even r there is no admissible zero at all, and for odd r the unique one is the triple $(a_1, a_2, a_3) = (r,\ 3a - 2r,\ 3a - 2r)$.

Proof and scope. There are 25 pairs (a, r) in the stated range, and each is handled by two ingredients. First a factorisation over $\mathbb{Z}$,

$$\mathcal{F}_{a,r}(X) = 2^r \prod_{\rho} (X - \rho) \cdot g_{a,r}(X),$$

with the integer roots ρ listed with multiplicity and $g_{a,r} \in \mathbb{Z}[X]$ the cofactor; the identity is proved by expanding both sides, and $\deg g_{a,r} \le 9$ over the whole range. Second a modulus m for $g_{a,r}$ as in Lemma 3.3, so that $g_{a,r}$ has no integer root; the moduli are $2, 5, 7, 8, 9, 11, 13, 19, 23, 29, 31, 37$, the largest being 37, and each is tested on all of its residues. The zeros of $\mathcal{F}_{a,r}$ are then exactly the listed ρ, and the admissibility bound $a_2 \ge 3a - 2r$ discards all of them but $3a - 2r$ itself when r is odd, and all of them when r is even. For example at $(a, r) = (7, 3)$ the roots are $0, 1, 2, 3, 4, 5, 9, 15$ and the bound is 15, so only 15 survives; at $(a, r) = (7, 6)$ they are $0, 1, 2, 3, 4, 6$ and the bound is 9, so none does.

The whole of this argument — all 25 pairs, and the three pairs of Theorem 5.1 as well — is carried out inside the proof kernel, without any appeal to compiled evaluation: the modulus tests are decided, not computed and trusted. See §9.

The 14 resulting triples are $(1, 10, 10)$, $(3, 6, 6)$ at $a = 4$; $(1, 13, 13)$, $(3, 9, 9)$ at $a = 5$; $(1, 16, 16)$, $(3, 12, 12)$, $(5, 8, 8)$ at $a = 6$; $(1, 19, 19)$, $(3, 15, 15)$, $(5, 11, 11)$ at $a = 7$; and $(1, 22, 22)$, $(3, 18, 18)$, $(5, 14, 14)$, $(7, 10, 10)$ at $a = 8$. □

5.1. **The levels $9 \le a \le 13$.** Nothing in the method changes here; only the degrees do. Neither of the devices of §3.3 is needed either, so this half of the paper stays univariate and entirely inside the proof kernel.

Theorem 5.3 (the boundary regime at the five further levels)**.** *Let $9 \le a \le 13$ and $1 \le r < a$, and let $a_2 \ge 3a - 2r$. Then*

$$\mathcal{F}_{a,r}(a_2) = 0 \quad \Longleftrightarrow \quad r \text{ is odd and } a_2 = 3a - 2r.$$

In particular, for even r there is no admissible zero at all, and for odd r the unique one is the triple $(a_1, a_2, a_3) = (r,\ 3a - 2r,\ 3a - 2r)$.

Proof and scope. There are 50 pairs (a, r) in the stated range — eight at $a = 9$, then nine, ten, eleven, twelve — and each is handled exactly as in Theorem 5.2: the factorisation $\mathcal{F}_{a,r}(X) = 2^r \prod_\rho (X - \rho) \cdot g_{a,r}(X)$ over $\mathbb{Z}$, and one modulus m for $g_{a,r}$ as in Lemma 3.3. Two things grow. The degree of $\mathcal{F}_{a,r}$ is $a + r$, so it reaches 25 at $(13, 12)$, and the cofactors reach degree 16 there; the moduli are $2, 5, 7, 9, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$, the largest being 47, each tested on all of its residues. The factorisation identity is proved on coefficient lists — splitting one linear factor off a list of coefficients at a time — rather than by expanding a product of nine linear factors against a cofactor of degree 16, which is what keeps the largest pairs affordable; the identity is then decided outright, so this half of the paper remains free of compiled evaluation at all 78 pairs. For example at $(a, r) = (9, 3)$ the roots are $0, 1, 2, 3, 4, 5, 6, 7, 14, 21$ and the bound is 21, so only 21 survives; at $(a, r) = (9, 6)$ they are $0, 1, 2, 3, 4, 5, 6, 6, 7$ and the bound is 15, so none does.

The 26 resulting triples are the $(r, 3a - 2r, 3a - 2r)$ with $r < a$ odd: four at $a = 9$, five at $a = 10$ and at $a = 11$, six at $a = 12$ and at $a = 13$. The largest second entry occurring is $3 \cdot 13 - 2 = 37$. □

6. The consequence for the conjecture

Corollary 6.1. *Let $2 \le a \le 8$, let (a_1, a_2) be a-admissible, let $a_3 = 2(a_1 + a_2) - 3a$ and $t = \Sigma/3 + 1 = a_1 + a_2 - a + 1$. Assume the reduction of* [1] *— that failure of WLP for R/I at that t forces $\mathcal{F}_a(a_1, a_2) = 0$, resp. $\mathcal{F}_{a,r}(a_2) = 0$ when $a_1 = r < a$. Then the conjecture of §1 —* [3, Conjecture 4.13]*, restating* [2, Conjecture 6.8(b)] *— holds for (a_1, a_2, a_3, t): apart from $(2, 9, 13, 9)$ and $(3, 7, 14, 9)$, the algebra R/I fails WLP if and only if t is even, Σ is odd and a_1, a_2, a_3 are not all distinct. Moreover no quadruple other than those two behaves exceptionally at that t in this range.*

Proof. The "if" implication is [2, Corollary 7.4] and is not at issue. For the forward implication, suppose R/I fails WLP. By the assumed reduction the relevant polynomial vanishes, so Theorems 4.1, 4.2, 4.3 (regime $a \le a_1$) and Theorems 5.1, 5.2 (regime $a_1 < a$) leave exactly four possibilities:

(1) $a_1 = a_2$ with a odd. Then $a_3 = 4a_1 - 3a$, so $\Sigma = 3(2a_1 - a)$ is odd and $t = 2a_1 - a + 1$ is even; and $a_1 = a_2$.
(2) $(a_1, a_2, a_3) = (r, 3a - 2r, 3a - 2r)$ with r odd, $r < a$. Then $\Sigma = 3(2a - r)$ is odd and $t = 2a - r + 1$ is even; and $a_2 = a_3$.
(3) $(a_1, a_2, a_3) = (3, 7, 14)$ with $a = 2$, whence $t = 9$: the quadruple $(3, 7, 14, 9)$.
(4) $(a_1, a_2, a_3) = (2, 9, 13)$ with $a = 3$, whence $t = 9$: the quadruple $(2, 9, 13, 9)$.

In cases (1) and (2) the parity pattern of the conjecture holds and a_1, a_2, a_3 are not all distinct, which is the forward implication; cases (3) and (4) are the two excepted quadruples. Since no other zero exists, no further exception arises. □

Remark 6.2. The corollary is a statement about one value of t. Part of it has a slightly different character. Wherever the zero set turns out to be *empty* — the generic regime at the even levels $a = 4, 6, 8$, and the boundary regime at every even r — the reduction does more than confirm the conjecture: it rules out failure of WLP outright at that t, for every admissible triple in that range.

6.1. **No further exceptional quadruple, and the range $2 \le a \le 13$.** The two exceptional quadruples sit at the two smallest levels treated, $a = 2$ and $a = 3$, one each. Whether that is the beginning of a pattern is a question about the conjecture's shape, and at the levels reached here it has an answer.

The statement below is about the un-normalised coefficient C_a of (5), so no clearing of denominators enters it, and it carries no hypothesis relating a_1 to a: both regimes are inside it.

Theorem 6.3 (the un-normalised coefficient, completely, over a window)**.** *Let $9 \le a \le 13$ and let (a_1, a_2) satisfy $1 \le a_1 \le a_2 \le 40$ and $a_2 \le a_3$. Then*

$$C_a(a_1, a_2) = 0 \quad \Longleftrightarrow \quad \big(a_1 = a_2 \text{ and } a \text{ is odd}\big) \text{ or } \big(a_1 \text{ is odd, } a_1 < a \text{ and } a_2 = 3a - 2a_1\big).$$

In particular no analogue of the quadruples $(2, 9, 13, 9)$ and $(3, 7, 14, 9)$ occurs at any of these five levels: every zero is a triple with $a_1 = a_2$ or $a_2 = a_3$, which is a triple on which failure of WLP is already a theorem [2, Corollary 7.4].

Proof and scope. One exhaustive evaluation over the $5 \cdot 41 \cdot 41$ triples (a, a_1, a_2), filtered by the stated hypotheses, with C_a recomputed from (3)–(4) and (5) — ordinary binomial coefficients, no cancellation — inside the check. It is a statement over a window, and the window is not arbitrary: every boundary candidate at these levels has $a_2 = 3a - 2r \le 3 \cdot 13 - 2 = 37 < 40$, so in the boundary regime the window contains the whole answer and Theorem 6.3 is complete there on its own. In the generic regime it is a finite check, and Theorem 4.4 is what proves the same conclusion with no window at all. □

Corollary 6.4. *Let $2 \le a \le 13$, let (a_1, a_2) be a-admissible, let $a_3 = 2(a_1 + a_2) - 3a$ and $t = \Sigma/3 + 1 = a_1 + a_2 - a + 1$. Assume the reduction of* [1]*, as in Corollary 6.1. Then the conjecture of §1 holds for (a_1, a_2, a_3, t), and no quadruple other than $(2, 9, 13, 9)$ and $(3, 7, 14, 9)$ behaves exceptionally at that t in this range.*

Proof. Corollary 6.1 settles $2 \le a \le 8$. For $9 \le a \le 13$ the argument is the same, with Theorem 4.4 in place of Theorems 4.1–4.3 and Theorem 5.3 in place of Theorems 5.1, 5.2. Only cases (1) and (2) of that proof occur — there is no analogue of (3) and (4) at these levels — and the parity computation in them is unchanged. □

Remark 6.5. The strengthening noted after Corollary 6.1 extends with the range. At the even levels $a = 10$ and $a = 12$, and at every even $r < a$ with $9 \le a \le 13$, the zero set is empty, so there the reduction rules out failure of WLP outright at $t = \Sigma/3 + 1$, for every admissible triple.

7. Controls

A family of theorems that all assert non-vanishing can be satisfied by a misplaced quantifier. The following refutations are recorded to exclude that, alongside the two corrections of Proposition 2.3. All values are of the un-normalised coefficient (5).

Proposition 7.1.
(1) The hypothesis $a \le a_1$ is not removable. $C_4(1, 10) = 0$ *and* $C_4(3, 6) = 0$*, so $\mathcal{F}_4$ does have admissible zeros — they are the boundary ones of Theorem 5.2.*
(2) Not every admissible triple is a zero. $C_4(4, 4) = 2160$ *and* $C_7(8, 10) = 14\,970\,009\,600$.
(3) The parity rule of Theorem 5.2 is sharp. *At $(a, r) = (4, 2)$ the candidate $a_2 = 3a - 2r = 8$ is not a zero:* $C_4(2, 8) = -56\,448$.
(4) The boundary regime is inhabited. $C_2(1, 4) = 0$.
(5) The parity rule for $c(a)$ is sharp. *With the even-a value $c = 3a - 1 = 20$ in place of $c(7) = 19$, the divisibility that the whole enumeration rests on fails already at $S = 10$: there $2S - 20 = 0$ while the coefficient of P in $f_7(10, \cdot)$ is $-20\,880 \neq 0$.*
(6) The root list of $\mathcal{P}_a$ is sharp. $\mathcal{P}_7(13) = 0$ *but* $\mathcal{P}_7(14) = 7\,413\,120$*, so the list of roots stops where Theorem 3.1 says it does.*

Proof and scope. Each item is a single evaluation, or — item (5) — one evaluation and one divisibility test. Items (1)–(4) evaluate (5) from the recursion with ordinary binomial coefficients, so they refute the statements they are aimed at without going through any normalisation. Non-vacuity is item (2) together with the inhabited triples exhibited there and in (4): each level treated above has admissible triples that are zeros and admissible triples that are not. □

The extension brings two new pieces of machinery and five new levels, and each of the following refutes one way in which one of them could have been wrong. Items (4)–(6) and (8) are again values of the un-normalised coefficient (5).

Proposition 7.2. (1) The built divisor list is not complete by accident. $23 \mid N(9)$*, but the list of Lemma 3.5 formed from the factorisation of* $N(9)$ *with the factor* 23 *deleted does not contain* 23*: the completeness of that list uses the whole factorisation.*

(2) Primality is load-bearing in Lemma 3.5. *With a composite base the construction fails:* $2 \mid 4^1$ *while* $2 \notin \{1, 4\}$.

(3) A modulus certificate is genuinely per-S. *At* $a = 9$ *the modulus* 71 *certifies* $S = 53$ — $f_9(53, \cdot)$ *has no zero modulo* 71 — *and does not certify* $S = 134$*, where* $f_9(134, \cdot)$ *does vanish somewhere modulo* 71*. A single global modulus would not do.*

(4) The hypothesis $a \le a_1$ is not removable at the new levels. $C_9(1, 25) = 0$ *and* $C_9(7, 13) = 0$*; both are admissible, and both are the boundary zeros of Theorem 5.3.*

(5) Not every admissible triple is a zero:

$$C_9(9, 10) = -137\,305\,808\,640, \qquad C_{13}(13, 14) = -1\,242\,517\,811\,318\,784\,000.$$

(6) The parity rule of Theorem 5.3 is sharp. *For even* r *the candidate* $a_2 = 3a - 2r$ *is not a zero:*

$$C_9(2, 23) = 29\,079\,729\,737\,372\,160, \qquad C_{13}(12, 15) = -2\,508\,930\,195\,932\,160\,000.$$

(7) The parity rule for $c(a)$ is sharp at the new levels. *With the even-*a *value* $c = 3a - 1 = 26$ *in place of* $c(9) = 25$*, the divisibility the enumeration rests on fails already at* $S = 13$*: there* $2S - 26 = 0$ *while the coefficient of* P *in* $f_9(13, \cdot)$ *is* $-5\,149\,440 \neq 0$.

(8) Both regimes are inhabited at the new levels. $C_9(9, 9) = 0$ *and* $C_{13}(1, 37) = 0$*; and* $C_{10}(9, 12) = 0$*, an even level, where the generic regime has no zero at all.*

Proof and scope. Each item is a single evaluation or a single membership test, except (3), which is two exhaustive residue sweeps, and (7), which is one evaluation and one divisibility test. Item (2) is decided outright. Items (1) and (3) are aimed at the two devices of §3.3; items (4)–(8) at the new levels, and they evaluate (5) from the recursion with ordinary binomial coefficients, so they refute what they are aimed at without passing through any normalisation. □

8. Beyond $a = 13$

Remark 8.1 (the computation outside the formal development). Every level treated above was first run in exact integer arithmetic outside the proof assistant, in both regimes, by two independent implementations — a symbolic elimination in $\mathbb{Z}[a_1, a_2]$ and a direct evaluation of (5) from the recursion with ordinary binomial coefficients. The two agree with each other and with the theorems above at every level $2 \le a \le 13$; the direct evaluation was run at $9 \le a \le 13$ over the wider window $a_1 \le a_2 \le 80$ and found exactly the predicted zeros, 76, 5, 75, 6 and 74 of them at $a = 9, \dots, 13$, and no others. The symbolic elimination cost 2.0, 12.2, 24.7, 116.9 and 247.1 seconds at those five levels. In the first version of this paper these computations were all that stood behind $9 \le a \le 13$, and were reported as such; they are now an independent cross-check of Theorems 4.4 and 5.3.

Remark 8.2 (on evaluating τ; a measurement, not mathematics). One implementation detail is worth recording, because it is what decided how far the range could go. The recursion (3) names $\tau^{k-1}D$ three times, so substituting the definition into itself a times before any normalisation produces on the order of 3^a copies of the base: 6561 at $a = 8$, 59 049 at $a = 10$, 177 147 at $a = 11$. Naming the intermediate forms $\tau^1 D, \dots, \tau^a D$ and deriving each from its predecessor replaces that by $O(a^2)$ small normalisations. The difference is not marginal: on one and the same identity, at $(a, r) = (9, 8)$, the one-step expansion cost 160.1 seconds and 10.69 GB of memory, and the step-by-step form 20.7 seconds and 6.75 GB. These are measurements of one implementation on one machine, reported for reproducibility; no statement in this paper depends on them.

The verified range stops at $a = 13$ for a measured reason. On the generic side the cost follows the number of divisors of $N(a)$, and so the size of the certificate table: 2283, 8619, 12 940, 46 060 and 69 101

rows at $a = 9, \dots, 13$. The level $a = 13$ already takes about 2100 seconds of processor time in a single unit of compilation, thirty-three minutes of wall time and 9.3 GB of memory; $a = 14$ is projected at roughly an hour at the same memory ceiling, and is not attempted here. A second obstruction appears at $a = 12$: comparing the length of the built divisor list with the length of the table of moduli — 46 080 entries — exceeds what the proof kernel's own decision procedure will do on a literal list, and is delegated to compiled evaluation instead (§9). The boundary side is far cheaper — $a = 13$ costs about 470 seconds for its twelve pairs, and the cost per pair is one low-degree expansion — and could be carried several levels further on its own.

A better route would remove the per-level work entirely. Theorem 3.1 is verified level by level; proving it for all a — equivalently, that f_a is independent of P at $a_3 = -2$ for odd a and at $a_3 = -1$ for even a — would turn the whole elimination into a single argument, with only the divisor question left per level. That is the open problem this work leaves behind.

9. Verification

Every statement of §2–§7 — Propositions 2.2, 2.3, 2.4, 3.4, 3.7, 7.1, 7.2, Lemmas 3.2, 3.3, 3.5, 3.6, and Theorems 3.1, 4.1, 4.2, 4.3, 4.4, 5.1, 5.2, 5.3, 6.3 — has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [8, 9]. Corollaries 6.1 and 6.4 are not among them: each is a short deduction from those theorems together with two results quoted from the literature, and each is stated in the informal language of Lefschetz properties, which the library does not have. The twenty-one modules that carry those statements contain no `sorry` and declare no axiom by hand. The seven modules of the first version check in about 540 seconds of CPU time; the fourteen that carry $9 \le a \le 13$ add about 5400 seconds, of which the five generic levels take 4070 and the five boundary levels 1300. Each of those fourteen peaks between 6.1 and 8.2 GB of memory, at or near the footprint of the library import itself, except the last generic level at 9.3 GB.

The division between kernel-checked and compiled evaluation is the point of the report that follows. Everything proved by expansion of polynomial identities — Theorem 3.1 in both of its displays and at every level, and the same two identities at each of $9 \le a \le 13$; the factorisations $\mathcal{F}_a = 2^{\lfloor a/2 \rfloor}(a_1 - a_2)^{\varepsilon} f_a$; all 78 boundary factorisations; the source's printed formulas of Theorems 4.1 and 5.1; and Lemmas 3.2, 3.3, 3.5, 3.6 and Propositions 3.4, 3.7 themselves — depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, and Proposition 2.3(1) depends on none at all. **The whole of the boundary regime is in that class**: each of the 78 statements $\mathcal{F}_{a,r} = 0 \iff a_2 = 3a - 2r$, resp. $\mathcal{F}_{a,r} \neq 0$, is proved from an expansion identity and a modulus test that the kernel itself decides, so Theorems 5.1, 5.2 and 5.3 — the entire regime the source's published code omits — rest on nothing but those three standard axioms.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly three Boolean tests per generic level $a \in \{2, \dots, 8\}$: the divisor scan of Lemma 3.2, the modulus table, and the sweep of the grid $S \le 40$. Consequently Theorems 4.1, 4.2 and 4.3 each depend on the three Booleans of their own levels and on nothing else of the kind. At the levels $a \in \{9, 10, 11\}$ there are only *two*, the modulus table and the grid sweep: the divisor scan disappears with Lemma 3.2, its place taken by Lemma 3.5, which is proved. At $a = 12$ and $a = 13$ there is a third, of a different character: the comparison of the length of the built divisor list with the length of the table of moduli, which the kernel's own decision procedure will not carry out on a literal list of 46 080 entries. Each of the three parts of Proposition 2.2, each of the two parts of Proposition 2.4, Theorem 6.3, and each of the items (1)–(4) of Proposition 7.1, carries exactly one such Boolean; items (5) and (6) of Proposition 7.1 carry none, and of Proposition 7.2 every item carries one except item (2), which is decided. Those are the only sources of extra axioms anywhere in this paper. No coefficient list is trusted as input at the point where it matters: the two sides of every comparison in Propositions 2.2 and 2.4 are recomputed from (3)–(4), (5) and Definition 2.1 inside the check, so there is no transcription step between the recursion and the checked data. The per-level data of §4 (f_a, $\mathcal{P}_a$, G_a, K_a, $N(a)$, the divisor list, the moduli) *are* supplied as literals, and each is validated in place — the identities by expansion, the divisor list by the scan, the moduli by exhaustive residue tests — so a transcription error in any of them cannot produce a false theorem, only a failed proof. At the five new levels one item leaves that list altogether: the divisor

list is no longer supplied at all but computed from the prime factorisation, so what is trusted there is the factorisation, and it is checked — the primality of each p_i, and the single identity $|N(a)| = \prod_i p_i^{e_i}$. The table of moduli is matched to the computed list by position, using a proved identity for the list's length rather than an evaluation of it.

Finally, every numerical value quoted above was first produced by an independent implementation in exact integer arithmetic, with a second implementation evaluating (5) from the raw definition as a cross-check, and the formal proofs were written against them afterwards. The source of the development is currently held in a private repository: repository reference to be added at submission.

Provenance of the novelty claims

Theorems 4.3, 4.4, 5.2, 5.3 and 6.3 are offered as new, and the ground for that is a recorded search rather than an impression, so it is stated. As of 3 September 2026 the source [1] has no forward citations in either OpenAlex or Semantic Scholar. All five works OpenAlex records as citing [3], and all twenty-five it records as citing [2] with publication year 2024 or later, were triaged by title and venue; [1] itself is on both lists, and no other item addresses the forward implication of the conjecture at $t = \Sigma/3 + 1$. A full-text *arXiv* corpus of 764 743 mathematics papers, searched on 3 September 2026, contains the literal quadruples $(2, 9, 13, 9)$ and $(3, 7, 14, 9)$ in exactly five papers: [1] itself, [3], and three earlier papers of Cook and Nagel on enumerations and the weak Lefschetz property (arXiv:0909.3509, arXiv:1105.6062, arXiv:1305.1314); none is later than [1]. Finally, the sixty most recent *arXiv* `math.AC` listings mentioning Lefschetz properties, to 27 August 2026, were triaged; the nearest neighbours are Javadekar on Artinian Gorenstein algebras of codimension three [4], Chase and Jonsson Kling on the *strong* Lefschetz property for monomial almost complete intersections [5], and Migliore, Nagel, Peterson and Teixeira Turatti on Hartshorne–Rao modules [6], none of which touches the borderline case. This is a negative search result and should be read as "not found", not as "does not exist".

The same searches were re-run for the levels added here, on 3 September 2026: [1] still has a forward-citation count of zero in OpenAlex, and a query for works citing it returns nothing; and the three most recent monthly slices of the same full-text corpus — everything announced after the current version of [1] — contain three occurrences of "almost complete intersection", all of them bibliography entries, and no paper on the conjecture. On the other side of the ledger, what the source itself claims for these levels is stated precisely in its closing remark, quoted in §1: a check of all values of S up to 1000 for a up to 40, in the regime $a_1 \geq a$. Every level treated here has admissible values of S far beyond that bound — at $a = 13$, 69 113 of them, of which 223 are at most 1000 — and the boundary regime is outside that check altogether.

References

[1] M. D. Booth and A. Vraciu, *The generators of a colon ideal with an application to the weak Lefschetz property for monomial almost complete intersections in three variables*, arXiv:2603.11491v2 [math.AC], submitted 12 March 2026, revised 6 July 2026; *Experimental Mathematics*, published online 10 August 2026, doi:10.1080/10586458.2026.2709599. Quotations, formulas and numbering above are from the LaTeX source of v2, read from the live *arXiv* e-print on 3 September 2026; the submission history shows exactly two versions. The version of record has no volume, issue or final page numbers assigned; its text was not consulted, and no numbering of it is quoted here.

[2] J. C. Migliore, R. M. Miró-Roig and U. Nagel, *Monomial ideals, almost complete intersections and the weak Lefschetz property*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 229–257, doi:10.1090/S0002-9947-2010-05127-X; e-print arXiv:0811.1023. Source of Proposition 6.1(iii), Conjecture 6.8(b), Lemma 7.1, Corollary 7.4 and Conjecture 7.13 as cited above; those numbers were read off the published version.

[3] D. Cook II and U. Nagel, *Syzygy bundles and the weak Lefschetz property of monomial almost complete intersections*, J. Commut. Algebra **13** (2021), no. 2, 157–178, doi:10.1216/jca.2021.13.157; e-print arXiv:1606.01809, there titled *Syzygy bundles and the weak Lefschetz property of almost complete intersections*. Source of Theorem 4.10, Conjecture 4.13 and Proposition 4.14(i) as cited above; those numbers were read off that e-print, in which the three results carry exactly those numbers.

[4] O. Javadekar, *On the weak Lefschetz property of Artinian Gorenstein algebras of codimension three in arbitrary characteristic*, arXiv:2608.27232 [math.AC], 27 August 2026.

[5] B. Chase and F. Jonsson Kling, *Monomial almost complete intersections and the strong Lefschetz property*, arXiv:2507.18516 [math.AC], 24 July 2025.

[6] J. Migliore, U. Nagel, C. Peterson and E. Teixeira Turatti, *Weak and strong Lefschetz properties for Hartshorne–Rao modules of curves in* $\mathbb{P}^3$, arXiv:2605.19434 [math.AG], 19 May 2026.

[7] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `https://macaulay2.com`. The system used for the computation reported in the closing remark of [1]; it is not used here.

[8] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[9] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.