

EXPLICIT KERNEL ELEMENTS FOR MULTIPLICATION BY A GENERAL LINEAR FORM

ON $\mathbb{k}[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2, e^2)$,
AND SPANNING SETS FOR THE KERNEL ON THE LINES $m = 3d - 3 - k$

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let $\mathbb{k}$ have characteristic zero, let $E = \mathbb{k}[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2)$, let $e = x_1 + \dots + x_m$, and let $A = E/(e^2)$, so that $A_j = E_j/e^2 E_{j-2}$. Crispin Quiñonez, Lundqvist and Nenashev conjectured that the Hilbert function of $\mathbb{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2, \ell_1^2, \ell_2^2)$, for general linear forms ℓ_1, ℓ_2 , is a lattice-path count $a(n, d)$; Booth, Singh and Vraciu translated the conjecture at (n, d) into the assertion $\dim \ker \varphi_{n+1, d} = \Delta(n, d)$, where $\varphi_{m, d}: A_{d-1} \rightarrow A_d$ is multiplication by a general linear form and Δ is the excess of a over its binomial approximation, and they exhibited exactly one element of that kernel. Boij and Lundqvist have since proved the conjecture for all n and all degrees, by a Hilbert-series computation that produces no element of the kernel at all. The dimension is therefore known; the elements are not.

We construct them. For every subset $L_0 \subseteq [m]$ of size k and every integer $s \geq 0$ we write down a pair $R_{L_0, s} \in E_{d-1}$, $C_{L_0, s} \in E_{d-2}$ with $L \cdot R_{L_0, s} = e^2 \cdot C_{L_0, s}$ identically in $\mathbb{Z}[a_1, \dots, a_m]$, so that $R_{L_0, s}$ lies in $\ker \varphi_{m, d}$ for a general linear form $L = \sum a_i x_i$; the member $k = 0$, $s = m - 2d + 2$ is the element of Booth, Singh and Vraciu, and the range of validity is sharp at its lower end. On the diagonal line $m = 3d - 3 - k$, where the parameter $s = d - 1 - k$ is forced, the resulting $\binom{m}{k}$ elements span the *whole* of $\ker \varphi_{m, d}$; we prove this at eight parameter sets on the lines $k = 0, 1, 2, 3$, and in particular the m elements attached to the singletons span the n -dimensional kernel on the line $n = 3d - 5$, for $d = 3, 4, 5, 6$ — the case Booth, Singh and Vraciu named as the next one. The identities are verified as polynomial identities over $\mathbb{Z}[a]$, and the independence at one integer specialisation — an asymmetry forced by the problem, since a rank at a point bounds the generic rank only from below. We record why no single numerical specialisation can decide the conjecture in either direction, and one transcription correction to the literature. Every computation reported below is machine-checked in Lean 4; the two places where such a computation is combined with a published upper bound to give an equality are marked as such.

1. INTRODUCTION

1.1. The conjecture, and the theorem that settles it. Let $\mathbb{k}$ be a field of characteristic zero and $S_n = \mathbb{k}[x_1, \dots, x_n]$. For general linear forms $\ell_1, \ell_2 \in S_n$, Crispin Quiñonez, Lundqvist and Nenashev [3, Conjecture 2] predicted the Hilbert series of the squarefree quotient

$$(1) \quad S_n/(x_1^2, \dots, x_n^2, \ell_1^2, \ell_2^2) = 1 + a(n, 1)t + a(n, 2)t^2 + \dots,$$

where, in the phrasing of their Conjecture 1, $a(n, s)$ is

the number of lattice paths inside the rectangle $(n + 2 - 2s) \times (n + 2)$ from the bottom left corner to the top right corner with moves of two types: $(x, y) \rightarrow (x + 1, y + 1)$ or $(x - 1, y + 1)$.

They proved the upper bound $\dim(S_n/(x_1^2, \dots, x_n^2, \ell_1^2, \ell_2^2))_s \leq a(n, s)$ [3, Theorem 8], and evaluated $a(n, s) = \binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}$ for $s \leq \lfloor n/3 \rfloor + 1$ [3, Proposition 7], which settles (1) on that range because the Fröberg-type lower bound meets the upper bound there. (That the Fröberg prediction can fail for $n + 2$ quadrics is not new with [3]: “the first published counterexamples were given in” Fröberg and Hollman [4], in the words of [3] itself, and (1) is the corrected prediction.) Everything past it stayed open for eight years.

It is open no longer. Boij and Lundqvist [1] prove the conjecture in full. Their Theorem 5.5, introduced with the sentence “We are now ready to prove [3, Conjecture 2]”, computes the Hilbert

series of $\mathbb{k}[x_1, \dots, x_n]/(\ell_1^2, \dots, \ell_{n+2}^2)$ — which after a change of coordinates is (1) — for every n and every degree, together with the closed form

$$(2) \quad a(n, s) = \sum_{k=0}^{\lceil s/3 \rceil} \left(\binom{n}{s-kD} - 2 \binom{n}{s-2-kD} + \binom{n}{s-4-kD} \right), \quad D = n - 2s + 4,$$

for $s \leq \lceil n/2 \rceil$. Their route is a Hilbert-series computation matched against a reflection count for the lattice paths.

1.2. The constructive question that remains. Between [3] and [1] sits the paper of Booth, Singh and Vraciu [2], which turns the conjecture into a statement about one kernel. Write

$$(3) \quad E = \mathbb{k}[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2), \quad e = x_1 + \dots + x_m, \quad A = E/(e^2),$$

let $a = (a_1, \dots, a_m)$ be indeterminates, $L = \sum_i a_i x_i$, and let $\varphi_{m,d}: A_{d-1} \rightarrow A_d$ be multiplication by L . Then (1) at (n, d) is equivalent to

$$(4) \quad \dim \ker \varphi_{n+1,d} = \Delta(n, d) := a(n, d) - \left(\binom{n}{d} - 2 \binom{n}{d-2} + \binom{n}{d-4} \right),$$

and the upper bound [3, Theorem 8] reads $\dim \ker \varphi_{n+1,d} \leq \Delta(n, d)$. Neither of these is a numbered statement of [2]; both are established inside the proof of [2, Corollary 5.5], and we recall the argument because everything below is measured against (4). Write $A^{(n+1)} = S_{n+1}/(x_1^2, \dots, x_{n+1}^2, e^2)$ and $h_j = \dim A_j^{(n+1)}$. After a change of coordinates one may take $\ell_1 = x_1 + \dots + x_n$, and then the cokernel of $\varphi_{n+1,d}: A_{d-1}^{(n+1)} \rightarrow A_d^{(n+1)}$ is $(S_n/(x_1^2, \dots, x_n^2, \ell_1^2, \ell_2^2))_d$, so that the left-hand side of (1) in degree d equals $h_d - h_{d-1} + \dim \ker \varphi_{n+1,d}$. The proof of [3, Proposition 7] gives $a(n, d) = \binom{n}{d} - 2 \binom{n}{d-2} + \binom{n}{d-4} + T(n, d)$, where $T(n, d)$ counts the paths that cross $x = n + 2 - 2d$ and then $x = 0$. Finally, the weak Lefschetz property of the monomial complete intersection gives $h_j = \binom{n+2}{j} - \binom{n+2}{j-1}$ whenever that number is positive [2, Remark 1.3], and Pascal's rule collapses $h_d - h_{d-1}$ to exactly $\binom{n}{d} - 2 \binom{n}{d-2} + \binom{n}{d-4}$. Hence $\Delta(n, d) = T(n, d)$, the conjecture (1) at (n, d) is equivalent to (4), and [3, Theorem 8] — which bounds the left-hand side of (1) above by $a(n, d)$ — is exactly the inequality $\dim \ker \varphi_{n+1,d} \leq \Delta(n, d)$. By [3, Proposition 7] the excess Δ vanishes for $n \geq 3d - 3$; it equals 1 on the line $n = 3d - 4$, where [2, Corollary 5.5] settles the conjecture using the single explicit kernel element Q produced in the proof of [2, Theorem 5.1]. Their Remark 5.6 then reads, verbatim — their Conjecture (5.4) being [3, Conjecture 2] and their ϕ our φ :

The next case of interest for Conjecture (5.4) is $n = 3d - 5$ we have $T(n, d) = n$. In order to establish that Conjecture (5.4) holds for this case, one would need to be able to produce n linearly independent elements in the kernel of $\phi_{n+1,d}$. A potential way to accomplish this is by choosing various specializations of the one element that is known to exist in the kernel of $\phi_{n+2,d}$.

Boij and Lundqvist's theorem supersedes the *conclusion* that was wanted here — the dimension is now known on every line — but not the request. Their proof exhibits no element of $\ker \varphi_{n+1,d}$ whatsoever, and no n independent kernel elements appear anywhere in the literature. That is the gap this paper closes, at eight parameter sets on the first four diagonal lines, with a fifth line measured.

1.3. Results. Fix $d \geq 3$. For $L_0 \subseteq [m]$ with $|L_0| = k$ and $s \geq 0$ we define in §4 an element $R_{L_0,s} \in E_{d-1} \otimes \mathbb{Q}[a]$ and a certificate $C_{L_0,s} \in E_{d-2} \otimes \mathbb{Q}[a]$, and prove:

- $L \cdot R_{L_0,s} = e^2 \cdot C_{L_0,s}$ as an identity of polynomials in $a_1, \dots, a_m$, so $R_{L_0,s}$ lies in $\ker \varphi_{m,d}$ for the general linear form L (Theorems 4.2 and 4.3). The member $k = 0$, $s = m - 2d + 2$ is exactly the element Q of the proof of [2, Theorem 5.1]; the member $k = 1$, $s = d - 2$ is the one-index deformation from which the whole family was found.
- Two inequalities delimit the family, $m - 2d + 2 \leq s$ and $s + k \leq d - 1$, and each is used exactly once in the proof. The first is sharp: lowering s by one destroys the identity (Theorem 4.2 (ii)), and so does the passage from $m = 3d - 4$ to $m = 3d - 3$ at $k = 1$ (Theorem 4.3). For the second we record no negative control.

- On the diagonal line $m = 3d - 3 - k$ — where the two constraints meet and force $s = d - 1 - k$ — the $\binom{m}{k}$ elements $R_{L_0, d-1-k}$, $|L_0| = k$, span the whole of $\ker \varphi_{m,d}$; we prove this at four parameter sets on the line $k = 1$ (Theorem 5.1) and at four more on the lines $k = 0, 2, 3$ (Theorem 5.2), and report the line $k = 4$ as a measurement (Remark 5.3). The line $k = 1$ is $n = 3d - 5$, the case named in [2, Remark 5.6], and there the spanning set consists of m elements of which exactly $n = m - 1$ are independent.

In both of those theorems the lower bound on the span is the machine-checked computation and the matching upper bound is [3, Theorem 8]; the resulting equality is therefore a theorem of this paper only in the sense that it combines a formal computation with a published inequality.

The certificates are of two different kinds, and the difference is forced by the problem rather than chosen. The identity $LR_{L_0, s} = e^2 C_{L_0, s}$ is verified as an identity in $\mathbb{Z}[a_1, \dots, a_m]$, hence at the generic point. The independence, by contrast, is verified at a single integer specialisation $a_i = i + 1$, because a rank at a point is a *lower* bound for the rank over $\mathbb{k}(a)$ — and a lower bound on $\dim \ker \varphi$ is precisely the half of (4) that the literature did not have. We say why in §6: the function $\dim(S_n/(x_1^2, \dots, x_n^2, \ell_1^2, \ell_2^2))_d$ is upper semicontinuous in the coefficients of ℓ_1, ℓ_2 , so *no* numerical specialisation can prove or disprove (1). This explains why a statement about 7-variable algebras could stay open while every individual case is a rank computation that finishes in seconds.

Sections 2 and 3 set up notation and reproduce, from the definitions, the lattice-path data of [3, 2, 1] that the later sections consume. §7 reports the machine verification.

2. PRELIMINARIES

2.1. The algebra. Throughout, $\mathbb{k}$ has characteristic zero, $[m] = \{1, \dots, m\}$, and for $S \subseteq [m]$ we write $x_S = \prod_{i \in S} x_i$ and $a_S = \prod_{i \in S} a_i$. In the exterior-like quotient E of (3) the monomials x_S form a basis, E_j has the x_S with $|S| = j$ as a basis, and

$$e^2 = 2e_2, \quad e_2 = \sum_{i < j} x_i x_j,$$

so the image of $(x_1^2, \dots, x_m^2, e^2)$ in E is the principal ideal $(e^2) = (e_2)$ and

$$A = \mathbb{k}[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2, e^2) = E/(e^2), \quad A_j = E_j/e^2 E_{j-2}.$$

Multiplication by e^2 sends $x_K \mapsto 2 \sum_{K \subset I, |I \setminus K|=2} x_I$; that is, it is twice the inclusion matrix of $(j-2)$ -sets in j -sets. With $a = (a_1, \dots, a_m)$ indeterminate we put $K_a = \mathbb{k}(a)$, $L = \sum_i a_i x_i$, and

$$\varphi_{m,d}: A_{d-1} \longrightarrow A_d, \quad u \longmapsto Lu,$$

a K_a -linear map; “a general linear form” means this one. We pair E_j with itself by declaring the monomial basis orthonormal, $\langle x_S, x_T \rangle = \delta_{S,T}$. Under that pairing the adjoint of multiplication by e_2 is the operator $\partial^{(2)} = \sum_{u < v} \partial_u \partial_v$, and since $\partial_i^2 = 0$ on E we have $D^2 = 2\partial^{(2)}$ for the total derivative $D = \sum_i \partial_i$. Hence

$$(5) \quad \xi \in E_{d-1}, \quad D\xi = 0 \implies \langle \xi, e^2 E_{d-3} \rangle = 0,$$

so such a ξ is a well-defined functional on A_{d-1} . This is how independence is certified in §5.

2.2. The scalars ε and ψ . Two scalar sequences from [2, §5] drive everything. For a fixed $d \geq 3$ set

$$(6) \quad \varepsilon(t) = \frac{(-1)^t}{\binom{d-1}{t}} \quad (0 \leq t \leq d-1), \quad \psi(t) = \frac{(-1)^t(t+1)}{\binom{d-2}{t}} \quad (0 \leq t \leq d-2).$$

These are the closed forms of the two recursively defined sequences of [2]: ε is determined by $\varepsilon(0) = 1$ and $\varepsilon(t) = -t\varepsilon(t-1)/(d-t)$, and ψ by the three-term recursion of Lemma 2.1 (iii) below. It is convenient to clear denominators:

$$(7) \quad \hat{\varepsilon}(t) := (d-1)! \varepsilon(t) = (-1)^t t! (d-1-t)!, \quad \hat{\psi}(t) := (d-2)! \psi(t) = (-1)^t (t+1)! (d-2-t)!,$$

both integers. The three identities we use are the following.

Lemma 2.1 (cf. [2, Observation 5.3]). *For $3 \leq d \leq 12$ and integers t in the indicated ranges,*

- (i) $t\hat{\varepsilon}(t-1) + (d-t)\hat{\varepsilon}(t) = 0$ for $1 \leq t \leq d-1$;
- (ii) $(t+1)\hat{\varepsilon}(t) + (d-1-t)\hat{\varepsilon}(t+1) = 0$ for $0 \leq t \leq d-2$;
- (iii) $\binom{d-t}{2}\hat{\psi}(t) + t(d-t)\hat{\psi}(t-1) + \binom{t}{2}\hat{\psi}(t-2) = 0$ for $1 \leq t \leq d-2$.

Identity (i) is the recursion defining ε and (iii) the one defining ψ ; both appear in [2, §5]. Identity (ii) is the one that permits a distinguished index to be inserted into the argument of ε , and it is the shape in which the $k = 1$ member of our family was first found; it is a formal consequence of (i), and we record it because the derivation of §4 passes through it. All three are verified over the stated range of d by direct evaluation of (7); the verification is kernel-checked, that is, it uses no compiled evaluation and no axioms at all (§7).

Remark 2.2. Since $\hat{\varepsilon} = (d-1)!\varepsilon$ and $\hat{\psi} = (d-2)!\psi$, parts (i) and (iii) hold verbatim for ε and ψ themselves, and summed over a d -set they become the two observations of [2, Observation 5.3]. For $|I| = d$, a set $G \subseteq [m]$ with $|G| \leq d-1$, and $t = |I \cap G|$,

$$(8) \quad \sum_{i \in I} \varepsilon(|(I \setminus i) \cap G|) = t\varepsilon(t-1) + (d-t)\varepsilon(t) = d[I \cap G = \emptyset]$$

(the t indices $i \in I \cap G$ contribute $\varepsilon(t-1)$ and the remaining $d-t$ contribute $\varepsilon(t)$; the middle expression vanishes for $1 \leq t \leq d-1$ by Lemma 2.1 (i) and equals $d\varepsilon(0) = d$ for $t = 0$), and, for $|I| = d$ and $|T| \leq d-2$,

$$(9) \quad \sum_{\{u,v\} \subseteq I} \psi(|(I \setminus \{u,v\}) \cap T|) = \binom{d}{2} [I \cap T = \emptyset].$$

These are parts (a) and (b) of [2, Observation 5.3], which are stated there for the exact cardinalities $|G| = m - 2d + 2$ and $|T| = d - 2$; the proof given there uses only $|G| \leq d - 1$, respectively $|T| \leq d - 2$, and it is in that form — the form the family of §4 needs, since there $|G| = s + k$ and $|T| = m - d - s$ vary — that we use them. Those two hypotheses are exactly where ε and ψ run out of definition, and they are what the sharpness statements of §4 turn on.

2.3. The path count and its excess. Let $a(n, d)$ be the lattice-path count quoted in §1, computed from the definition: the number of paths from $(0, 0)$ to $(w, n + 2)$, $w = n + 2 - 2d$, with steps $(x, y) \mapsto (x \pm 1, y + 1)$, staying inside the strip $0 \leq x \leq w$ (and $a(n, d) = 0$ when $2d > n + 2$). Put

$$(10) \quad F(n, j) = \binom{n}{j} - 2\binom{n}{j-2} + \binom{n}{j-4}, \quad \Delta(n, d) = a(n, d) - F(n, d),$$

with the convention $\binom{n}{j} = 0$ for $j < 0$ or $j > n$. So $F(n, d)$ is the value predicted by [3, Proposition 7] and Δ is the excess; by (4) it is the conjectural, and by [1] the actual, dimension of $\ker \varphi_{n+1, d}$. In [2] this quantity is written $T(n, d)$.

3. THE LATTICE-PATH DATA, REPRODUCED

The results of §4 and §5 are compared against the numbers $\Delta(n, d)$, so those numbers are recomputed here from the path definition rather than taken from the sources. Nothing in this section is new; the point is that each value the later sections consume has been produced independently of the paper that first printed it.

Proposition 3.1. *The four odd- n rows printed in [3] are reproduced from the path definition:*

$$\begin{aligned} (a(3, d))_{d \geq 0} &= 1, 3, 1; & (a(5, d))_{d \geq 0} &= 1, 5, 8, 1; \\ (a(7, d))_{d \geq 0} &= 1, 7, 19, 21, 1; & (a(9, d))_{d \geq 0} &= 1, 9, 34, 66, 55, 1, \end{aligned}$$

all further entries being 0. Moreover $\Delta(n, d) = 0$ whenever $3d \leq n + 3$ — the range $d \leq \lfloor n/3 \rfloor + 1$ of [3, Proposition 7] — for all $n \leq 25$ and $d \leq 11$; $\Delta(3d - 3, d) = 0$ and $\Delta(3d - 4, d) = 1$ for $3 \leq d \leq 12$.

The line $\Delta(3d - 4, d) = 1$ is the case settled in [2, Corollary 5.5]. The next line is the one [2, Remark 5.6] names.

Proposition 3.2. $\Delta(3d-5, d) = 3d-5$ for $3 \leq d \leq 12$; that is, $\Delta(n, d) = n$ on the line $n = 3d-5$. The corresponding assertions with $n+1$ and with $n-1$ in place of n both fail on $4 \leq d \leq 8$.

This is the assertion $T(n, d) = n$ of [2, Remark 5.6], whose proof there is a three-segment crossing argument; the value is obtained here from the path definition alone, so the combinatorial half of the case $n = 3d-5$ is certified independently of that argument. The two negative controls are not decoration: an early implementation of the count, started at the wrong corner, returned $n+1$ on this line, and the discrepancy with [2] was what exposed the error. See Remark 6.3.

Proposition 3.3. Write $n = 3d-3-j$. Then, on the ranges indicated,

$$\Delta(n, d) = \binom{n}{2} - 2 \quad (j = 3, 5 \leq d \leq 13), \quad \Delta(n, d) = \binom{n}{3} - 2n \quad (j = 4, 7 \leq d \leq 13),$$

$$\Delta(n, d) = \binom{n}{4} - 2\binom{n}{2} + 1 \quad (j = 5, 9 \leq d \leq 13),$$

and, uniformly for $0 \leq j \leq 5$, $3 \leq d \leq 13$ and $2j \leq d+1$,

$$(11) \quad \Delta(n, d) = F(n, 3d-4-n), \quad \text{equivalently} \quad a(n, d) = F(n, d) + F(n, 3d-4-n).$$

The restriction in (11) is necessary: $a(4, 4) \neq F(4, 4) + F(4, 4)$ and $a(5, 5) \neq F(5, 5) + F(5, 6)$. Finally $\Delta(3d-6, d) > 3d-6$ for $5 \leq d \leq 9$, so the line $j = 3$ does not repeat the value $\Delta = n$ that Proposition 3.2 gives on the line $j = 2$.

Formula (11) is the truncation of (2) to its terms $k = 0$ and $k = 1$: with $s = d$ and $D = n - 2d + 4$ the term $k = 1$ of (2) is $F(n, d-D) = F(n, 3d-4-n)$. It is therefore a special case of [1, Lemma 5.4], and is recorded here because it puts the excess in closed form on exactly the lines $m = 3d-3-k$ that §5 works on. The lower endpoints of the three ranges above are also necessary and not artefacts of the search: $\Delta(6, 4) = 14$ while $\binom{6}{2} - 2 = 13$, and the value 14 is the one used in Theorem 5.2.

4. AN EXPLICIT FAMILY OF KERNEL ELEMENTS

4.1. The elements. Fix $d \geq 3$ and m . For $L_0 \subseteq [m]$ with $|L_0| = k$ and an integer $s \geq 0$ put

$$(12) \quad R_{L_0, s} = \sum_{|I|=d-1} a_I \left(\sum_{\substack{J \subseteq [m] \setminus L_0 \\ |J|=s}} \varepsilon(|I \cap (J \cup L_0)|) a_J \right) x_I \in E_{d-1} \otimes \mathbb{Q}[a],$$

$$(13) \quad C_{L_0, s} = \frac{1}{d-1} \sum_{|K|=d-2} \left(\sum_{\substack{|T|=m-d-s \\ L_0 \subseteq T}} \psi(|K \cap T|) a_{[m] \setminus T} \right) x_K \in E_{d-2} \otimes \mathbb{Q}[a].$$

Both have denominators only through ε, ψ and the factor $1/(d-1)$; the rescalings

$$(14) \quad \hat{R}_{L_0, s} = (d-1)! R_{L_0, s}, \quad \hat{C}_{L_0, s} = (d-1)! C_{L_0, s}$$

have coefficients in $\mathbb{Z}[a_1, \dots, a_m]$, by (7), and it is in that integral form that the identities below are checked.

Two members deserve names. For $k = 0$ and $s = m-2d+2$, (12) is precisely the element Q displayed in the proof of [2, Theorem 5.1]. For $k = 1$, say $L_0 = \{l\}$, and $s = d-2$ we write $R_l := R_{\{l\}, d-2}$ and $C_l := C_{\{l\}, d-2}$; since $l \notin J$ one may expand $\varepsilon(|I \cap (J \cup \{l\})|) = \varepsilon(|I \cap J| + [l \in I])$, so that

$$(15) \quad R_l = \sum_{|I|=d-1} a_I \left(\sum_{\substack{J \subseteq [m] \setminus \{l\} \\ |J|=d-2}} \varepsilon(|I \cap J| + [l \in I]) a_J \right) x_I,$$

a one-index deformation of Q in which a distinguished index shifts the argument of ε . It is this shift, together with Lemma 2.1 (ii), that is not present in [2] or [1].

Remark 4.1 (why the hint of [2] does not by itself suffice; not part of the formal development). [2, Remark 5.6] suggests specialising the known element of $\ker \varphi_{m+1,d}$. Made precise, a substitution $\sigma: \mathbb{K}[x_1, \dots, x_{m+1}] \rightarrow \mathbb{K}[x_1, \dots, x_m]$ is usable exactly when $\sigma(I_{m+1}) \subseteq I_m$ and σ carries the general linear form to L . For $m \geq 3$ the only squares of linear forms lying in $(I_m)_2 = \langle x_1^2, \dots, x_m^2, e^2 \rangle$ are the scalar multiples of the x_k^2 and of e^2 : if $\lambda = \sum \lambda_i x_i$ has λ^2 in that span, then $\lambda_i \lambda_j$ is constant over all $i < j$, which forces $\lambda \propto x_k$ or $\lambda \propto e$. So every $\sigma(x_i)$ is a scalar multiple of some x_k or of e , or is 0. Enumerating those, the substitutions $x_{m+1} \mapsto 0$ return only the two symmetric elements Q with $|J| = d - 2$ and $|J| = d - 1$, because Q is equivariant for the symmetric group and its whole orbit collapses; the substitutions $x_{m+1} \mapsto \rho e$ and $x_l \mapsto vx_l$, $x_{m+1} \mapsto -e$, give genuinely new one- and two-parameter families, and (15) is what the linear part of the second simplifies to. This paragraph is a derivation, not a claim verified in the formal development; (12) and (13) are self-contained and nothing below refers back to it.

4.2. The identity.

Theorem 4.2. *Let (m, d, k, s) be one of*

$$(9, 4, 0, 3), \quad (7, 4, 2, 1), \quad (10, 5, 2, 2), \quad (9, 5, 3, 1), \quad (11, 6, 4, 1).$$

Then, for every subset $L_0 \subseteq [m]$ with $|L_0| = k$,

$$(16) \quad L \cdot \hat{R}_{L_0,s} = e^2 \cdot \hat{C}_{L_0,s}$$

holds in $E \otimes \mathbb{Z}[a_1, \dots, a_m]$. Consequently $R_{L_0,s} \in \ker \varphi_{m,d}$ for each such L_0 . Moreover:

- (i) *the first of these, $(m, d, k, s) = (9, 4, 0, 3)$, is the element Q of the proof of [2, Theorem 5.1], so that element is the member $k = 0$ of the family;*
- (ii) *the constraint $s \geq m - 2d + 2$ is sharp: for $(m, d, k, s) = (7, 4, 2, 0)$, $(10, 5, 2, 1)$ and $(9, 5, 3, 0)$ — each obtained from a case above by lowering s by one — identity (16) fails.*

Each of the five parameter sets satisfies $m = 3d - 3 - k$ and $s = d - 1 - k$; that is, they lie on the diagonal lines of §5, where the two constraints $m - 2d + 2 \leq s$ and $s + k \leq d - 1$ are both tight.

Proof sketch. Each of the eight assertions is an exhaustive check, run to completion, of a polynomial identity over $\mathbb{Z}[a_1, \dots, a_m]$. A monomial $\prod_i a_i^{c_i}$ occurring on either side of (16) has all $c_i \leq 2$, so it is encoded by the integer $\sum_i c_i 3^i$, and both sides are accumulated into one scratch array of length 3^m ; the check loops over every k -subset L_0 of $[m]$, every d -subset I of $[m]$, every $i \in I$ and every admissible J on the left, and every pair $\{u, v\}$ in I and every admissible T on the right, and asserts that every accumulated coefficient is zero. The finite search is therefore over the $\binom{m}{k} \binom{m}{d}$ pairs (L_0, I) for each of the five parameter sets, and over the three parameter sets of (ii), where the same loop returns a nonzero coefficient. No specialisation of a is used anywhere. The uniform statement behind the five cases, and its proof, are Remark 4.4. $\square$

Theorem 4.3. *Let (m, d) be one of*

$$(4, 3), \quad (5, 3), \quad (6, 4), \quad (7, 4), \quad (8, 4), \quad (9, 5), \quad (10, 5), \quad (11, 5), \quad (14, 6).$$

Then $L \cdot \hat{R}_l = e^2 \cdot \hat{C}_l$ in $E \otimes \mathbb{Z}[a_1, \dots, a_m]$ for every $l \in [m]$, where R_l, C_l are as in (15), (13); hence $R_1, \dots, R_m$ all lie in $\ker \varphi_{m,d}$. The pairs listed include $m = 3d - 4$ for $d = 3, 4, 5, 6$. The range is sharp: for $(m, d) = (6, 3)$, $(9, 4)$ and $(12, 5)$ — that is, at $m = 3d - 3$, where the constraint $s \geq m - 2d + 2$ reads $d - 2 \geq d - 1$ and fails — the identity fails.

Proof sketch. Identical in method to Theorem 4.2: for each of the nine pairs the check runs over all $l \in [m]$ and all I with $|I| = d$, accumulating both sides of the identity into a scratch array of length 3^m indexed by the base-3 encoding of the exponent vector, and verifying that every coefficient cancels. The largest instance, $(m, d) = (14, 6)$, is a scratch array of 3^{14} integers and about $3.6 \cdot 10^8$ accumulation steps. For the three negative instances the same loop exhibits a surviving coefficient. Nothing is evaluated at a numerical a . $\square$

Remark 4.4 (the uniform statement; proved by hand, formalised only in the instances above). Let $d \geq 3$, let $L_0 \subseteq [m]$ with $|L_0| = k$, and let $s \geq 0$ satisfy

$$(17) \quad m - 2d + 2 \leq s \quad \text{and} \quad s + k \leq d - 1.$$

Then $L \cdot R_{L_0, s} = e^2 \cdot C_{L_0, s}$. Indeed, the coefficient of x_I , $|I| = d$, on the left is $a_I \sum_J a_J \sum_{i \in I} \varepsilon(|(I \setminus i) \cap (J \cup L_0)|)$; since $|J \cup L_0| = s + k \leq d - 1$, (8) applies with $G = J \cup L_0$ and the inner sum is $d[I \cap (J \cup L_0) = \emptyset]$. Only the J with J, L_0 both disjoint from I survive, and for those $a_I a_J = a_{I \sqcup J}$ with $|I \sqcup J| = d + s$; setting $T = [m] \setminus (I \sqcup J)$ turns the conditions $J \cap I = J \cap L_0 = I \cap L_0 = \emptyset$ into $T \cap I = \emptyset$ and $L_0 \subseteq T$, so

$$(L \cdot R_{L_0, s})_I = d \sum_{\substack{|T|=m-d-s, \ L_0 \subseteq T \\ T \cap I = \emptyset}} a_{[m] \setminus T}$$

(both sides being zero when $I \cap L_0 \neq \emptyset$). On the right,

$$(e^2 C_{L_0, s})_I = \frac{2}{d-1} \sum_T a_{[m] \setminus T} \sum_{\substack{K \subseteq I, \ |K|=d-2}} \psi(|K \cap T|),$$

and $|T| = m - d - s \leq d - 2$ by the first inequality of (17), so (9) turns this into $\frac{2}{d-1} \binom{d}{2} \sum a_{[m] \setminus T}$, the sum being over the same T as before, which is the same. Each hypothesis of (17) is used exactly once: the second makes (8) applicable, the first makes (9) applicable. The failures recorded in Theorem 4.2 (ii) and in Theorem 4.3 are both the first inequality failing, and there $|T| = d - 1$, where (9) would call for the undefined value $\psi(d - 1)$; no negative control is recorded for the second inequality. Summing (15) over l gives $\sum_{l \in [m]} R_l = (m - 2d + 2) R_{\emptyset, d-2}$, by Lemma 2.1 (ii); on the line $m = 3d - 4$, where $m - 2d + 2 = d - 2$, the right-hand side is a multiple of the element Q of [2, Theorem 5.1], so there the known element lies in the span of the new ones and the content of the family is the remaining $m - 1$ directions. What is formalised is the list of instances in Theorems 4.2 and 4.3: the reduction of the display above to those instances is a re-indexing of sums over subsets that has not been carried out formally, so the general statement is stated here as a remark and not as a theorem.

One point of (17) is worth stating flatly, because the shape of (8) invites the opposite guess. The first inequality carries no k : what (9) requires is $|T| = m - d - s \leq d - 2$, that is $s \geq m - 2d + 2$, and L_0 enters T only through the condition $L_0 \subseteq T$, not through its cardinality. The symmetric-looking reading $m - 2d + 2 \leq s + k$ is false, and Theorem 4.2 (ii) refutes it three times over: at $(m, d, k, s) = (7, 4, 2, 0)$, $(10, 5, 2, 1)$ and $(9, 5, 3, 0)$ one has $s + k \geq m - 2d + 2$ and $s + k \leq d - 1$, yet (16) fails — in each case because $|T| = m - d - s = d - 1$.

Remark 4.5 (a transcription correction to [2]; measured outside the formal development). In the proof of [2, Theorem 5.1] the displayed element Q' satisfies

$$LQ = \frac{1}{d-1} Q' (x_1 + \cdots + x_n)^2 \quad \text{in } \mathbb{K}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2),$$

rather than $LQ = Q' (x_1 + \cdots + x_n)^2$, as displayed there. The two coefficient computations in that proof produce, for $|I| = d$, the value $d a_I \sum_J a_J$ on the left and $\binom{d}{2} a_I \sum_J a_J$ on the right — the sums being over the J of size $n - 2d + 2$ disjoint from I — and the right-hand computation is carried out for $Q' e_2$ while the display asks for $Q' (x_1 + \cdots + x_n)^2 \equiv 2Q' e_2$, so the two sides differ by $2\binom{d}{2}/d = d - 1$. We measured the ratio exactly, by symbolic expansion, for $d = 3, 4, 5, 6$ at both $m = 3d - 4$ and $m = 3d - 3$, and it is $1/(d - 1)$ in every case. Their theorem is unaffected — only $LQ \in I_n$ is claimed — but the constant matters to anyone re-deriving the identity, and it is the reason for the factor $1/(d - 1)$ in (13). The reading is that of the e-print v3 cited below; we were not able to consult the published text.

5. INDEPENDENCE, AND SPANNING SETS FOR THE KERNEL

5.1. The line $n = 3d - 5$. Take $m = 3d - 4$, so $n = m - 1 = 3d - 5$, and consider the m elements $R_1, \dots, R_m$ of Theorem 4.3. Independence is certified by pairing them with functionals that descend to

$A_{d-1} = E_{d-1}/e^2 E_{d-3}$. For $i \in [m]$ set

$$(18) \quad \xi_i = \prod_{j=0}^{d-2} (x_{(i+3j) \bmod m} - x_{(i+3j+1) \bmod m}) \in E_{d-1}.$$

The $2(d-1)$ indices occurring are pairwise distinct, since the largest offset is $3(d-2)+1 = 3d-5 < m$; so ξ_i is a product of $d-1$ differences over disjoint index pairs, each killed by the derivation $D = \sum_k \partial_k$, whence $D\xi_i = 0$ and, by (5), ξ_i annihilates $e^2 E_{d-3}$.

Theorem 5.1. *Let $d \in \{3, 4, 5, 6\}$ and $m = 3d - 4$, so $(m, d) \in \{(5, 3), (8, 4), (11, 5), (14, 6)\}$. Then:*

- (i) $\langle \xi_i, e^2 x_K \rangle = 0$ for every $i \in [m]$ and every K with $|K| = d - 3$, so each ξ_i is a functional on A_{d-1} ;
- (ii) the $m \times m$ integer matrix $[\langle \xi_i, \hat{R}_l \rangle]_{i,l}$, evaluated at $a_i = i + 1$, has a nonzero $(m-1) \times (m-1)$ minor, namely the one obtained by deleting the row $i = 1$ and the column $l = 1$; and its full determinant is 0.

Consequently — combining (ii) with the upper bound $\dim \ker \varphi_{m,d} \leq \Delta(m-1, d)$ of [3, Theorem 8], which is not part of the formal development — the images of $R_1, \dots, R_m$ in A_{d-1} span a subspace of dimension exactly $m-1 = n$ over $\mathbb{K}(a)$, and that subspace is the whole of $\ker \varphi_{m,d}$.

Proof sketch. Parts (i) and (ii) are exhaustive computations, run to completion for each of the four pairs (m, d) . For (i) the check evaluates, for every $i \in [m]$ and every one of the $\binom{m}{d-3}$ sets K , the sum of ± 1 coefficients of ξ_i over the $\binom{m-d+3}{2}$ monomials of $e^2 x_K$, and finds 0; this is a direct verification of (5) for these ξ_i rather than an appeal to it. For (ii) each entry $\langle \xi_i, \hat{R}_l \rangle$ is obtained by expanding ξ_i over its 2^{d-1} signed monomials x_I and reading off the coefficient of x_I in $\hat{R}_l$ at the point $a_i = i + 1$; the determinants are then computed exactly, in integer arithmetic, by the fraction-free Bareiss algorithm. For the last sentence: (ii) gives $\dim \text{span}\{R_l\} \geq m-1$ over $\mathbb{K}(a)$, because the rank of a matrix at a specialisation is a lower bound for its rank over the function field. In the other direction the span lies in $\ker \varphi_{m,d}$, whose dimension is at most $\Delta(m-1, d) = m-1$ by [3, Theorem 8] together with Proposition 3.2. Hence both inequalities are equalities. The vanishing of the full $m \times m$ determinant in (ii) is the matching control: it is what the equality $\dim \text{span} = m-1$ predicts, and a nonzero value there would have contradicted [3, Theorem 8]. $\square$

5.2. The diagonal lines $m = 3d - 3 - k$. On the line $m = 3d - 3 - k$ the two inequalities (17) read $d-1-k \leq s$ and $s \leq d-1-k$, so the parameter s is forced to $d-1-k$ and the family contributes exactly $\binom{m}{k}$ elements $R_{L_0, d-1-k}$. Passing to $n = m-1 = 3d-4-k$, these are the lines $n = 3d-4, 3d-5, 3d-6, \dots$, of which $k=0$ is the case of [2, Corollary 5.5] and $k=1$ the case of Theorem 5.1.

Theorem 5.2. *Let (m, d, k, s) be one of $(9, 4, 0, 3)$, $(7, 4, 2, 1)$, $(10, 5, 2, 2)$, $(9, 5, 3, 1)$, so that $m = 3d - 3 - k$ and $s = d - 1 - k$. Evaluate the $\binom{m}{k}$ elements $\hat{R}_{L_0, s}$ at $a_i = i + 1$ and reduce modulo the prime $p = 2^{31} - 1$. Then the rank of their images in A_{d-1} equals*

$$1, \quad 14, \quad 34, \quad 48$$

respectively, and in each case this number is exactly $\Delta(m-1, d)$: the four values $\Delta(8, 4) = 1$, $\Delta(6, 4) = 14$, $\Delta(9, 5) = 34$ and $\Delta(8, 5) = 48$ are recomputed from the path definition of §2 and compared with the ranks inside the same verified computation. Consequently — again using the upper bound of [3, Theorem 8], which is not part of the formal development — the $\binom{m}{k}$ elements $R_{L_0, d-1-k}$ span the whole of $\ker \varphi_{m,d}$, whose dimension is $\Delta(m-1, d)$.

Proof sketch. The computation forms the matrix whose rows are the $\binom{m}{d-3}$ generators $e^2 x_K$ of $e^2 E_{d-3}$ followed by the $\binom{m}{k}$ vectors $\hat{R}_{L_0, s}$ specialised at $a_i = i + 1$, all expressed in the monomial basis of E_{d-1} , and computes its rank over the field with $p = 2^{31} - 1$ elements by Gaussian elimination; the quoted number is that rank minus $\binom{m}{d-3}$. The subtraction is exact because the rank of the $e^2 E_{d-3}$ block alone is verified, in the same computation, to be the full $\binom{m}{d-3}$ — namely 9, 7, 45 and 36 for the four cases

in the order listed. Both reductions used — from $\mathbb{k}(a)$ to the point $a_i = i + 1$, and from $\mathbb{Z}$ to $\mathbb{Z}/p$ — can only lower the rank, so the number produced is a lower bound for $\dim_{\mathbb{k}(a)} \text{span}\{R_{L_0, d-1-k}\}$, which is the direction needed. The comparison with $\Delta(m-1, d)$ is made by recomputing Δ from the path definition of §2 inside the same verified computation, not by transcribing the table of §3. For the final sentence, the span lies in $\ker \varphi_{m,d}$ and $\dim \ker \varphi_{m,d} \leq \Delta(m-1, d)$ by [3, Theorem 8], so lower and upper bound meet; the same argument also re-proves (1) at $(n, d) = (m-1, d)$, which [1] now gives for all (n, d) . $\square$

Remark 5.3 (the measured table; the entries outside Theorems 5.1 and 5.2 are computations outside the formal development). The same rank computation was run in a separate, unverified implementation over a wider range. In every case the $\binom{m}{k}$ elements span the whole kernel:

k	(m, d)	$\binom{m}{k}$	$\dim \text{span} = \Delta(m-1, d)$
0	(9, 4)	1	1
1	(5, 3), (8, 4), (11, 5), (14, 6), (17, 7), (20, 8)	m	$m-1$
2	(7, 4), (10, 5), (13, 6), (16, 7)	21, 45, 78, 120	14, 34, 64, 103
3	(9, 5), (12, 6), (15, 7)	84, 220, 455	48, 144, 336
4	(11, 6), (14, 7)	330, 1001	165, 573

Of these, the rows carried by the formal development are $k = 0$ and $k = 2, 3$ at the four parameter sets of Theorem 5.2, and $k = 1$ at $d = 3, 4, 5, 6$ by Theorem 5.1; the identity (16) itself is additionally verified for $k = 4$ at $(m, d) = (11, 6)$ in Theorem 4.2. The remaining entries are reported as measurements. The cost of the unverified sweep is under a minute in total; the $d = 7$ independence pairing inside the formal development was priced at about $7 \cdot 10^7$ interpreted steps and deferred.

6. WHAT THIS DOES, AND DOES NOT, SETTLE

Remark 6.1 (the slice $n = 3d-5$; a consequence of the above and of published results, not a new theorem). Combining Theorem 5.1 with [3, Theorem 8] and Proposition 3.2 gives $\dim \ker \varphi_{n+1,d} = n$, hence (1), for $(n, d) = (4, 3), (7, 4), (10, 5), (13, 6)$, where the path count takes the values $a(n, d) = 0, 1, 32, 377$. This conclusion is *not* new: [1, Theorem 5.5] gives (1) for all n and all degrees. What is different here is the route. Boij and Lundqvist compute a Hilbert series; the argument above produces the kernel elements themselves, and it is constructive at every step. We note also that the case $d = 3$ is degenerate — there $a(4, 3) = 0$, so the assertion is that the algebra vanishes in degree 3, and a single specialisation settles it; the cases $d = 4, 5, 6$ are not of that kind.

Remark 6.2 (why no numerical specialisation can decide (1); a measurement). For fixed n and d the function $\dim(S_n/(x_1^2, \dots, x_n^2, \ell_1^2, \ell_2^2))_d$ is upper semicontinuous in the coefficients of ℓ_1, ℓ_2 , so its generic value is the *minimum* over specialisations. Since [3, Theorem 8] already gives $\dim \leq a(n, d)$, a computation at any particular pair ℓ_1, ℓ_2 that returns $a(n, d)$ re-proves the known inequality and yields nothing else, while a computation returning a smaller value says nothing about the generic one. Equivalently, in the form (4): with M_L the matrix of $\times L: E_{d-1} \rightarrow E_d$ and U the inclusion matrix of $(d-2)$ -sets in d -sets,

$$\dim \ker \varphi_{m,d} = \binom{m}{d-1} + \binom{m}{d-2} - \binom{m}{d-3} - \text{rank}[M_L \mid U],$$

and $\text{rank}[M_L \mid U]$ is lower semicontinuous in a , so a numerical a gives a *lower* bound for the generic rank and hence only an upper bound for $\dim \ker \varphi_{m,d}$ — the side the literature already owns. This is why a conjecture about 7-variable algebras stayed open for eight years although every individual case is a rank computation that finishes in seconds — we measured it: over $5 \leq m \leq 14$ and $3 \leq d \leq 6$ the kernel dimension at a random L modulo a 31-bit prime reproduces $\Delta(m-1, d)$ in every case, the largest being a 3003×3003 elimination taking 8 seconds, and it carries no information the literature did not already have. It also fixes the shape of the certificates in this paper: generic on the side of the identity (16), specialised on the side of the rank, because a rank at a point bounds the generic rank from below.

Remark 6.3 (a process note). In the course of Proposition 3.2 our first implementation of the path count started the paths at the corner $(0, 0)$ over $n + 2$ steps and returned $n + 1$ on the line $n = 3d - 5$, contradicting [2, Remark 5.6]. The source is right and the counter was wrong: [2] count the n free steps from $(1, 1)$ to $(n + 1 - 2d, n + 1)$ after the two forced end moves. Rebuilt that way, the whole crossing decomposition of [2, Remark 5.6] is confirmed at $n = 3d - 5$: with LR the set of paths crossing $x = 0$ and then $x = n + 2 - 2d$, and RL the set crossing $x = n + 2 - 2d$ and then $x = 0$ — so that $|\text{RL}|$ is the quantity [2] call $T(n, d)$ —

$$a(n, d) = \binom{n}{d} - 2 \binom{n}{d-2} + |\text{LR}| + |\text{RL}|, \quad |\text{LR}| = \binom{n}{d-4}, \quad |\text{RL}| = n,$$

and for $d = 4, \dots, 9$ the two counts come out as 1, 10, 78, 560, 3876, 26334 and 7, 10, 13, 16, 19, 22 respectively. We record the episode because it is the reason Proposition 3.2 carries two negative controls.

Remark 6.4 (what remains). Two questions are left by the results above. First, the independence in Theorems 5.1 and 5.2 is established case by case; the identity (16) is uniform in (k, s, m, d) but the spanning statement is not. The $m \times m$ matrix $[\langle \xi_i, R_l \rangle]$ has a one-dimensional left kernel that is equivariant for the symmetric group, so the relation is a single rational function of a_l and of the symmetric functions of the remaining variables; computing it in closed form would give the spanning statement on the whole line $n = 3d - 5$ at once, and the analogous relation space of dimension $\binom{m}{k} - \Delta(m - 1, d)$ would do the same for every line. Second, [1] know $\dim \ker \varphi_{n+1, d}$ for every (n, d) ; a constructive basis for general (n, d) is open, as is — for the exterior-algebra conjecture that (1) is paired with — the case of an odd number of generators, [1, Corollary 5.6] covering the even case only.

7. VERIFICATION

Every computation asserted above is machine-checked in Lean 4; the modules import no part of *Mathlib*, so the trusted base is the Lean kernel alone. Two things are *not* covered by that, and both are marked where they occur. The closing sentence of Theorem 5.1 and the closing sentence of Theorem 5.2 — that the span is the *whole* kernel — combine the machine-checked lower bound with the upper bound $\dim \ker \varphi_{m, d} \leq \Delta(m - 1, d)$ of [3, Theorem 8], which is a published theorem and not part of the formal development; and Remarks 4.1, 4.4, 4.5, 5.3, 6.1, 6.2, 6.3 and 6.4 are prose, measurements or consequences of published results, as their headings say. The split between what the kernel evaluates and what is delegated to compiled code is exactly the split between the two kinds of statement in this paper. Lemma 2.1 in all three parts is proved by kernel evaluation and depends on *no* axioms; Propositions 3.1, 3.2 and 3.3, being lattice-path arithmetic over small ranges, are proved by kernel evaluation and depend only on **propext**. The polynomial identities of Theorems 4.2 and 4.3, the orthogonality and determinant certificates of Theorem 5.1, and the ranks of Theorem 5.2 are Boolean tests over scratch arrays of length 3^m and over integer matrices with up to a few hundred rows, too large for kernel reduction; they are discharged by Lean’s **native_decide**, so each such theorem carries one compiled-evaluation axiom per Boolean it asserts and **propext**, and nothing else. There is no **sorry** and no other axiom anywhere in the development. All the arithmetic that matters is exact: the identities are checked over $\mathbb{Z}[a_1, \dots, a_m]$ with the base-3 monomial encoding of §4, the determinants of Theorem 5.1 in exact integers by the Bareiss algorithm, and the only modular arithmetic is in Theorem 5.2, where reduction modulo $p = 2^{31} - 1$ can only lower a rank and therefore preserves the direction of the bound. Timings, on one core: the lattice-path module 97 seconds; the kernel-element module about 11 minutes, of which the single instance $(m, d) = (14, 6)$ of Theorem 4.3 accounts for 538 seconds; the k -index module 45 seconds. Each numerical value quoted in this paper was first produced by an independent implementation in C or Python and the formal statements were written against it afterwards; the entries of Remark 5.3 outside Theorems 5.1 and 5.2, the measurements of Remark 6.2, and the constant of Remark 4.5 come from those implementations only and are labelled as such above. The source of the development is currently held in a private repository: repository reference to be added at submission.

PROVENANCE OF THE NOVELTY CLAIMS

Theorems 4.2, 4.3, 5.1 and 5.2 are offered as new, and the ground for that is a recorded search. As of 3 September 2026: the three primary sources [3, 2, 1] were read from their live e-prints, and their version histories checked ([3] has one version; [2] three, the latest of 20 May 2025; [1] one, of 24 August 2026). A full-text *arXiv* corpus contains the string “quadratic forms in the exterior algebra” in exactly three papers, namely those three. The citation graph of [3] (eight citing works) and of [2] (six) was examined; the on-topic citing works are [1] and [2] themselves and one further preprint, none of which constructs kernel elements. The closest prior art is [5], whose Propositions 3.5 and 3.8 use the same ansatz $a_T \sum_i \lambda_i \sum_{|S \cap T|=i}$ in the same ring, but produce kernel elements of multiplication by ℓ^p , respectively ℓ^{p-2} , out of degrees 1 and 2, not of $\times L: A_{d-1} \rightarrow A_d$; we read them and they are neither the element of [2, Theorem 5.1] nor any $R_{L_0, s}$. The surrounding weak-Lefschetz literature [6] argues by dimension counts and syzygy bundles and contains no kernel constructions. Searches of the OEIS for the path triangle, for the diagonal 1, 32, 377, 3280, 25213 and for the rows 1, 9, 34, 66, 55, 1 and 1, 11, 53, 143, 221, 144, 1 returned nothing. This is a negative search result and should be read as “not found”, not as “does not exist”. A referee could reasonably hold that, given [2, Theorem 5.1], the deformation (15) is a short step; the honest position is that it is a short step nobody had published, and that its principal corollary is now known anyway by [1].

REFERENCES

- [1] M. Boij and S. Lundqvist, *On the Hilbert series of ideals generated by general linear forms*, arXiv:2608.22823v1 [math.AC], 24 August 2026. The submission history shows exactly one version. Lemma 5.4, Theorem 5.5 and Corollary 5.6 are cited above by the numbering of that e-print; note that its title page reads *On the Hilbert series of ideals generated by powers of general linear forms*, while the *arXiv* metadata title is the one given here.
- [2] M. D. Booth, P. Singh and A. Vraciu, *On the weak Lefschetz property for ideals generated by powers of general linear forms*, J. Commut. Algebra **18** (2026), no. 2, 139–158, doi:10.1216/jca.2026.18.139; arXiv:2410.22542v3 [math.AC], revised 20 May 2025 (three versions; the *arXiv* record carries no journal reference). Remark 1.3, Theorem 5.1, Observation 5.3, Corollary 5.5 and Remark 5.6 are cited above by the numbering of the e-print v3, which is the text we read; we were not able to consult the published version, although the citations of it in [1] (Theorem 3.7, Corollary 4.11, Remark 4.12) agree with that numbering.
- [3] V. Crispin Quiñonez, S. Lundqvist and G. Nenashev, *On ideals generated by two generic quadratic forms in the exterior algebra*, J. Pure Appl. Algebra **223** (2019), no. 12, 5067–5082, doi:10.1016/j.jpaa.2019.03.010; arXiv:1803.08918 (one version only). Source of Conjecture 1, Conjecture 2, Proposition 7 and Theorem 8 as cited above.
- [4] R. Fröberg and J. Hollman, *Hilbert series for ideals generated by generic forms*, J. Symbolic Comput. **17** (1994), no. 2, 149–157, doi:10.1006/jSCO.1994.1008. The first published counterexamples to the Fröberg prediction for $n + 2$ quadrics, in the words of [3].
- [5] F. Jonsson Kling, S. Lundqvist, F. Mohammadi, M. Orth and E. Sáenz-de-Cabezón, *Gröbner bases, resolutions, and the Lefschetz properties for powers of a general linear form in the squarefree algebra*, J. Algebra **700** (2026), 146–184, doi:10.1016/j.jalgebra.2026.04.012; arXiv:2411.10209v2. The closest prior kernel constructions in the same ring; Propositions 3.5 and 3.8 are cited above by the numbering of that e-print.
- [6] J. C. Migliore, R. M. Miró-Roig and U. Nagel, *On the weak Lefschetz property for powers of linear forms*, Algebra Number Theory **6** (2012), no. 3, 487–526, doi:10.2140/ant.2012.6.487; arXiv:1008.2149. The surrounding weak-Lefschetz line, consulted for kernel constructions; none occurs there.
- [7] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.