

REFINEMENT MONOTONICITY OF NORMALIZED WITTEN INTERSECTION NUMBERS, AND THE NESTING CONJECTURE AT GENUS 14–18

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ABSTRACT. Guo, Yang and Zagier (arXiv:2603.15233) normalise Witten’s ψ -class intersection numbers as $C(\mathbf{d}) = 2^{2g} \prod_j (2d_j + 1)!! / (3^{2g-2+n} (2g - 3 + n)!) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$ and conjecture (their Conjecture 1) that for $g \geq 2$ and $d_1, \dots, d_n \geq 1$ of genus g one has $C(3g - 2) \leq C(\mathbf{d}) \leq C(2^{3g-3})$. Their Lemma 2 proves the lower inequality, and they report the upper one checked for $d_j \geq 2$ and $g \leq 13$. We verify the upper inequality, in exact rational arithmetic and over the complete set of tuples with all $d_j \geq 2$, at genus 14 to 18 (560 709 further tuples), with 2^{3g-3} the unique maximiser at each genus. We then observe a stronger statement that the source does not make: indexing the tuples of genus g with all $d_j \geq 2$ by the partitions $\lambda = \mathbf{d} - 1$ of $3g - 3$, the value C increases strictly whenever one part of λ is split into two. We verify this at every genus $2 \leq g \leq 18$, over 11 514 074 single-split steps. Since $(3g - 3)$ and 1^{3g-3} are the two extremes of the refinement order, refinement monotonicity implies the nesting inequalities, with strict inequalities off the extremes and uniqueness of both extremisers; and it is the only one of the natural orders left standing, since the length-then-lexicographic order of Guo, Norbury, Yang and Zagier, weak monotonicity in the length, and the dominance order all fail for these numbers, by examples printed in the source or read off its Table 1. All values are computed through the Dijkgraaf–Verlinde–Verlinde recursion in an integer normalisation, controlled against the one-point formula, Zograf’s two-point formula and the dilaton relation, and the whole check is verified in Lean 4.

1. INTRODUCTION

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford moduli space of stable curves of genus g with n marked points and ψ_j the first Chern class of the j -th tautological line bundle. Witten’s intersection numbers

$$(1) \quad \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \quad d_j \in \mathbb{Z}_{\geq 0},$$

vanish unless $d_1 + \cdots + d_n = 3g - 3 + n$. Witten’s conjecture [3], proved by Kontsevich [4], identifies their generating function with a tau function of the KdV hierarchy; equivalently, the numbers satisfy the Virasoro constraints, which in the form of a recursion on the d_j are due to Dijkgraaf, Verlinde and Verlinde [5]. That recursion, reproduced as (5) below, determines every number in (1) from $\langle \tau_0^3 \rangle_0 = 1$ and $\langle \tau_1 \rangle_1 = \frac{1}{24}$, and it is the engine behind every computation in this paper.

The large genus behaviour of (1) has been studied intensively: Mirzakhani and Zograf [11] for the Weil–Petersson volumes, Liu and Xu [12] for fixed n and fixed $d_1, \dots, d_{n-1}$ (in the source’s description); the Delecroix–Goujard–Zograf–Zorich conjecture [7, 9], which asserts that the normalisation $G(\mathbf{d}) = 24^g g! \prod_j (2d_j + 1)!! \langle \tau_{\mathbf{d}} \rangle_g / (6g + 2n - 5)!!$ tends to 1 uniformly for $n = O(\log g)$, proved by Aggarwal [6] (for $n = o(\sqrt{g})$, as an iterated limit without a rate) and independently by Guo and Yang [10]; and the resurgent analysis of Eynard, Garcia-Failde, Giacchetto, Gregori and Lewański [13]. Guo, Norbury, Yang and Zagier [2] then found, for the Brézin–Gross–Witten numbers, a normalisation in which a completely uniform asymptotic formula holds, and conjectured that the normalised numbers of each genus are nested between two explicit extreme values (their Conjecture 1) and in fact strictly monotone in a length-then-lexicographic order of partitions (their Conjecture 2).

The source and its conjecture. Guo, Yang and Zagier [1] transfer this programme to Witten’s numbers. Following [2] they introduce (their Eq. (17))

$$(2) \quad C(\mathbf{d}) = \frac{2^{2g(\mathbf{d})} \prod_{j=1}^n (2d_j + 1)!!}{3^{2g(\mathbf{d})-2+n} (2g(\mathbf{d}) - 3 + n)!} \int_{\mathcal{M}_{g(\mathbf{d}),n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \quad \mathbf{d} \in (\mathbb{Z}_{\geq 0})^n,$$

where $g(\mathbf{d}) = 1 + \frac{1}{3} \sum_j (d_j - 1)$, and prove (their Theorem 1) that $C(\mathbf{d}) = \frac{1}{\pi} + O(1/g(\mathbf{d}))$ uniformly as $g(\mathbf{d}) \rightarrow \infty$, over $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ with $g(\mathbf{d}) \in \mathbb{Z}_{\geq 1}$. They call $\mathbf{d}$ *primitive* when every $d_j \geq 2$, print the fourteen exact values of C at $g = 2, 3$ (their Table 1), and state:

Conjecture 1 of [1] (p. 4, Eq. (20)). *For $g \geq 2$, $n \geq 1$ and for $d_1, \dots, d_n \geq 1$ satisfying $|\mathbf{d}| = 3g - 3 + n$,*

$$(3) \quad C(3g - 2) \leq C(\mathbf{d}) \leq C(2^{3g-3}).$$

Here $(3g - 2)$ is the one-point tuple and 2^{3g-3} the tuple with $3g - 3$ entries equal to 2; both have genus g . The evidence reported is, verbatim, that “*the primitive normalized Witten’s intersection numbers $C(\mathbf{d})$ with a fixed genus $g \leq 13$ lie between the values $C(3g - 2)$ and $C(2^{3g-3})$* ”, with the example $C(37) = 0.30674776 \cdots$ and $C(2^{36}) = 0.31263595 \cdots$ at $g = 13$.

Two facts, both in the source, fix what is actually open. First, the *lower* inequality is a theorem: Lemma 2 of [1] (p. 10) proves $C(\mathbf{d}) \geq C(3g - 2)$ for all $g, n \geq 1$ and $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ with $|\mathbf{d}| = 3g - 3 + n$, by induction on $2g - 2 + n$, with the two-point case taken from [7, 8]; the source introduces it with the words “*The following lemma proves one side of Conjecture 1.*” Second, the dilaton relation in this normalisation is exactly $C(1, \mathbf{d}) = C(\mathbf{d})$ (their Eq. (32)), so the statement for $d_j \geq 1$ reduces genus by genus to the primitive case (Lemma 3.1 below), a reduction the source itself performs inside the proof of its Lemma 2. The open content of (3) is therefore the single inequality

$$(4) \quad C(\mathbf{d}) \leq C(2^{3g-3}) \quad \text{for primitive } \mathbf{d} \text{ of genus } g,$$

which for each g is a statement about the $p(3g - 3)$ rational numbers $C(\mathbf{d})$, $\mathbf{d}$ ranging over the partitions $\lambda = \mathbf{d} - 1$ of $3g - 3$.

What this paper does. We treat (4) as a finite question at each genus and answer it exactly.

- **Genus 14 to 18** (Theorem 4.1). Inequality (4) holds for every primitive tuple of genus 14, 15, 16, 17, 18 — 31 185, 53 174, 89 134, 147 273 and 239 943 tuples, 560 709 in all beyond the range the source reports — and at each of these genera 2^{3g-3} is the *unique* maximiser. With Lemma 3.1 this is Conjecture 1 of [1] in full at those genera.
- **Refinement monotonicity** (Theorem 4.2). For every genus $2 \leq g \leq 18$, every primitive $\mathbf{d}$ of genus g and every way of splitting one part of $\lambda = \mathbf{d} - 1$ into two positive parts, the value of C strictly increases. This is a statement the source does not make, and it is new already inside the source’s own numerical range. Because $(3g - 3)$ and 1^{3g-3} are the least and greatest elements of the refinement order on partitions of $3g - 3$, it implies (3) at the same genus with strict inequalities off the two extremes (Corollary 4.3). It is also what survives the source’s own negative remarks: the length-then-lexicographic monotonicity of [2] fails for C ($C(4, 4) < C(3, 5)$), weak monotonicity in the length fails ($C(2^5, 8) < C(3^4, 5)$), and, as we note, monotonicity in the dominance order fails in both directions; none of these counterexamples is a refinement pair.
- **Controls** (Section 5). The same computation reproduces all fourteen exact values of the source’s Table 1, the four exact values and four strict inequalities of its Remarks 1–3, the one-point formula at every genus, Zograf’s explicit two-point formula [14] at every two-point tuple of every genus ≤ 18 , and the dilaton relation at all 600 492 primitive tuples of genus ≤ 18 ; and three “too much” statements are refuted: the hypothesis $d_j \geq 1$ cannot be dropped (already at genus 2, and again at genus 14), neither bound can be narrowed, and dominance monotonicity is false.

We do not prove (4) for any infinite set of genera, and we know of no published result that would. The asymptotic results in the area cannot decide it: the whole spread $C(2^{3g-3}) - C(3g - 2)$ is, by the two

expansions (24)–(25) of [1], $\sim \frac{1}{4\pi g}$, the same order as the error term in the source’s Theorem 1, while the uniform lower bound of [7] is strictly below 1 in its own normalisation and the limit theorem of [6] carries no rate. What the paper offers is the exact finite evidence, the refinement statement as the correct strengthening to aim at, and, in Section 6, the single one-step inequality that would prove it.

Method. All values are computed in the integer normalisation $W(\mathbf{d}) = 24^g g! \prod_j (2d_j + 1)!! \langle \tau_{\mathbf{d}} \rangle_g$ of Section 2, in which the Dijkgraaf–Verlinde–Verlinde relation is a recursion over multisets of non-negative integers with only integer arithmetic; within one genus the ordering of the $C(\mathbf{d})$ is the ordering of integers $V(\mathbf{d})$ that differ from $C(\mathbf{d})$ by a constant factor. The recursion is memoised on multisets; the genus-18 sweep visits about 2.7 million multisets. Every theorem stated with a formal pin below is a single exact evaluation of this recursion, checked by the Lean 4 proof assistant [18] from a compiled version of the same code (Section 7); the statements are reproduced verbatim in Appendix A. Independently of the formal development, the values were recomputed for this paper by a second implementation of the recursion (different code, different choice of the distinguished index), which reproduced every number and every count below, and the two implementations were compared exactly on every primitive tuple of genus ≤ 7 and every two-point tuple of genus ≤ 9 .

Relation to the literature. The nesting statement (3) appears only in [1], and its Brézin–Gross–Witten analogue only in [2], where it is reported checked to genus 40. A search of the arXiv full text, of a local corpus of arXiv sources, and of the OpenAlex and Semantic Scholar citation graphs (September 2026) found two works citing [1]: [16], which cites it for the polynomiality result of the source, and [17], which cites it twice in passing, among general references for the large genus asymptotics of intersection numbers. Neither discusses the nesting or any monotonicity of these numbers. The strings “crudest and finest partitions”, “monotone with respect to refinement” and their variants together with “intersection numbers” return nothing relevant outside the two source papers. For the Brézin–Gross–Witten numbers, refinement monotonicity *is* implied by Conjecture 2 of [2] (splitting a part raises the length by one); the Witten case is of interest precisely because that total order is false for C , as the source records.

2. PRELIMINARIES

2.1. Tuples, genus, the recursion. For $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{Z}_{\geq 0})^n$ put

$$g(\mathbf{d}) = 1 + \frac{1}{3} \sum_{j=1}^n (d_j - 1), \quad X(\mathbf{d}) = 2g(\mathbf{d}) - 2 + n = \frac{1}{3} \sum_j (2d_j + 1),$$

so that $\langle \tau_{\mathbf{d}} \rangle_g$ is non-zero only if $g = g(\mathbf{d})$ is a non-negative integer and $X(\mathbf{d}) > 0$. Set $U(\mathbf{d}) = \prod_j (2d_j + 1)!! \langle \tau_{\mathbf{d}} \rangle_{g(\mathbf{d})}$. The Dijkgraaf–Verlinde–Verlinde relation [5], printed as Eq. (10) of [1], reads, for $\mathbf{d} = (d_1, \mathbf{r})$ with $\mathbf{r} = (r_1, \dots, r_{n-1})$ and $X(\mathbf{d}) \geq 2$,

$$(5) \quad U(d_1, \mathbf{r}) = \sum_{j=1}^{n-1} (2r_j + 1) U(\mathbf{r}^{(j)}) + \frac{1}{2} \sum_{a+b=d_1-2} \left(U(a, b, \mathbf{r}) + \sum_{I \sqcup J = \{1, \dots, n-1\}} U(a, \mathbf{r}_I) U(b, \mathbf{r}_J) \right),$$

where $\mathbf{r}^{(j)}$ is $\mathbf{r}$ with r_j replaced by $r_j + d_1 - 1$ (the term is absent if that is negative), the inner sums run over $a, b \geq 0$ and over ordered splittings of the index set, and a term whose argument has non-integral genus is zero. In the second sum the genus drops by one; in the third $g(a, \mathbf{r}_I) + g(b, \mathbf{r}_J) = g(\mathbf{d})$. The relation holds for any choice of the distinguished entry d_1 ; the two tuples with $X(\mathbf{d}) = 1$ are the base cases $U(0, 0, 0) = 1$ and $U(1) = \frac{1}{8}$. With $d_1 = 0$ it is the string equation and with $d_1 = 1$ the dilaton equation; in the normalisation (2) these are Eqs. (31) and (32) of [1],

$$(6) \quad C(0, \mathbf{d}) = \sum_{j=1}^n \frac{2d_j + 1}{3(X(0, \mathbf{d}) - 1)} C(d_1, \dots, d_j - 1, \dots, d_n), \quad C(1, \mathbf{d}) = C(\mathbf{d}),$$

the dilaton relation holding *exactly*, with no residual factor. The one-point values are explicit (Eq. (19) of [1]):

$$(7) \quad C(3g - 2) = \frac{3(6g - 3)!!}{54^g g! (2g - 2)!}, \quad g \geq 1.$$

2.2. Primitive tuples and partitions. Call $\mathbf{d}$ *primitive* if every $d_j \geq 2$ (the source's word), and write $\mathcal{P}_g$ for the set of primitive tuples of genus g up to reordering. Since C is symmetric in the d_j , C is a function on $\mathcal{P}_g$. The map $\lambda_i = d_i - 1$ is a bijection from $\mathcal{P}_g$ onto the partitions of $N = 3g - 3$, so $|\mathcal{P}_g| = p(3g - 3)$:

g	2	3	4	5	6	7	8	9	10
$p(3g - 3)$	3	11	30	77	176	385	792	1 575	3 010
g	11	12	13	14	15	16	17	18	
$p(3g - 3)$	5 604	10 143	17 977	31 185	53 174	89 134	147 273	239 943	

The two sides of (3) are the two extreme partitions: $\mathbf{d} = (3g - 2)$ is $\lambda = (3g - 3)$, the partition with one part, and $\mathbf{d} = 2^{3g-3}$ is $\lambda = 1^{3g-3}$. A partition μ *refines* λ if it is obtained by successively splitting parts; $(3g - 3)$ is the unique minimum and 1^{3g-3} the unique maximum of this partial order, and any two comparable partitions are joined by a chain of *single splits* $\lambda \rightarrow \mu$ in which one part a of λ is replaced by two parts $b, a - b$ with $1 \leq b \leq a/2$. On tuples, a single split replaces one entry d_i of $\mathbf{d}$ by the two entries $k + 1$ and $d_i - k$ with $1 \leq k \leq (d_i - 1)/2$; it preserves the genus.

2.3. The integer normalisation. Put

$$(8) \quad W(\mathbf{d}) = 24^g g! U(\mathbf{d}) = 24^g g! \prod_{j=1}^n (2d_j + 1)!! \langle \tau_{\mathbf{d}} \rangle_g, \quad g = g(\mathbf{d}).$$

Multiplying (5) by $24^g g!$ gives, with $g = g(\mathbf{d})$ and $g_1 = g(a, \mathbf{r}_I)$,

$$(9) \quad W(d_1, \mathbf{r}) = \sum_j (2r_j + 1) W(\mathbf{r}^{(j)}) + 12g \sum_{a+b=d_1-2} W(a, b, \mathbf{r}) + \frac{1}{2} \sum_{a+b=d_1-2} \sum_{I \sqcup J} \binom{g}{g_1} W(a, \mathbf{r}_I) W(b, \mathbf{r}_J),$$

with base cases $W(0, 0, 0) = 1$ and $W(1) = 3$.

Lemma 2.1 (integrality). $W(\mathbf{d}) \in \mathbb{Z}_{\geq 0}$ for every $\mathbf{d}$, and the double sum in the last term of (9) is even, so that (9) is a recursion in integer arithmetic with an exact halving.

Proof. Induction on $X(\mathbf{d})$, all terms being non-negative. The first two terms are integers by induction. In the last sum, the involution $(a, I) \leftrightarrow (b, J)$ preserves each summand $\binom{g}{g_1} = \binom{g}{g-g_1}$. If $n \geq 2$ it has no fixed point, since $I = J$ is impossible for complementary subsets of a non-empty set, so the sum is twice the sum over orbits. If $n = 1$ the sum is over $a + b = d_1 - 2$ alone, a fixed point has $a = b$ and $g_1 = g/2$, and its summand $\binom{g}{g/2} W(a)^2$ is even because the central binomial coefficient is even for $g \geq 2$ ($g = 0$ and $g = 1$ with $n = 1$ are unstable or a base case). $\square$

This lemma is elementary and is not part of the formal development; the halving in the compiled recursion is an integer division, and the second implementation used for this paper asserts evenness at every step, an assertion that never failed over the roughly 2.7 million multisets of genus ≤ 18 .

From (2) and (8), $C(\mathbf{d}) = W(\mathbf{d}) / (6^g g! 3^{2g-2+n} (2g - 3 + n)!)$. Within one genus the factor $6^g g! 3^{2g-2}$ is common, so with $N = 3g - 3 \geq n$ the integers

$$(10) \quad V(\mathbf{d}) = W(\mathbf{d}) \cdot 3^{N-n} \cdot \frac{(2g - 3 + N)!}{(2g - 3 + n)!} = C(\mathbf{d}) \cdot 6^g g! 3^{2g-2+N} (2g - 3 + N)!$$

are ordered exactly as the $C(\mathbf{d})$; every comparison below is a comparison of such integers, and every equality $C(\mathbf{d}) = p/q$ is checked as $W(\mathbf{d}) q = p \cdot 6^g g! 3^{2g-2+n} (2g - 3 + n)!$. In particular (7) becomes the identity $W(3g - 2) = (6g - 3)!!$.

2.4. Zograf's two-point formula. Zograf [14] gave a closed formula for the two-point numbers, printed as Eq. (44) of [1]:

$$(11) \quad C(d_1, 3g - 1 - d_1) = \frac{1}{54^g (2g - 1)! g} \sum_{d=-1}^{d_1-1} \eta_{g,d},$$

$$\eta_{g,d} = (6g - 3 - 2d)!! (2d + 1)!! \cdot \begin{cases} \frac{g - 2j}{j! (g - j)!}, & d = 3j - 1, \\ -\frac{j! (g - 1 - j)!}{j! (g - 1 - j)!}, & d = 3j, \\ \frac{j! (g - 1 - j)!}{j! (g - 1 - j)!}, & d = 3j + 1, \end{cases}$$

with $(-1)!! = 1$. It contains no recursion and serves as an independent control on the engine at the $3g$ two-point tuples $(d_1, 3g - 1 - d_1)$, $0 \leq d_1 \leq 3g - 1$, of each genus. In the formal development it is used in the integer form

$$(12) \quad W(d_1, 3g - 1 - d_1) \cdot g = \sum_{d=-1}^{d_1-1} \eta'_{g,d}, \quad \eta'_{g,d} = g! \eta_{g,d},$$

where the factor $g!$ clears the denominators $j! (g - j)!$ and $j! (g - 1 - j)!$ of (11) to the integers $\binom{g}{j}$ and $g \binom{g-1}{j}$; for this paper the formula was also evaluated exactly as printed, in rational arithmetic, at every two-point tuple of every genus ≤ 18 , with agreement throughout.

3. THE DILATON REDUCTION

The following observation is the source's own step (it is how the proof of Lemma 2 of [1] reduces to $2 \leq d_1 \leq \dots \leq d_n$, “because of the symmetry of the C -values and formula (32)”); we record it because it is what makes the primitive checks below equivalent to the full statement. It quantifies over infinitely many tuples per genus and is outside the formal development.

Lemma 3.1 (dilaton reduction; not formalised). *Let $g \geq 2$. Inequality (3) holds for every $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$, $n \geq 1$, of genus g if and only if it holds for every primitive $\mathbf{d}$ of genus g . The same holds for each of the two inequalities in (3) separately.*

Proof. An entry $d_j = 1$ contributes 0 to $\sum_j (d_j - 1)$, so deleting every entry equal to 1 from $\mathbf{d}$ leaves a tuple $\mathbf{d}'$ of the same genus, and by the dilaton relation $C(1, \mathbf{d}) = C(\mathbf{d})$ of (6) applied once per deleted entry, $C(\mathbf{d}) = C(\mathbf{d}')$. If $\mathbf{d}'$ were empty then $\sum_j (d_j - 1) = 0$ and $g = 1$; so for $g \geq 2$, $\mathbf{d}'$ is a non-empty tuple with all entries ≥ 2 , i.e. primitive of genus g . $\square$

Two exact instances, $C(1, 4) = C(4) = \frac{35}{144}$ and $C(1, 1, 2, 2, 2) = C(2, 2, 2) = \frac{175}{648}$, and the identity $W(1, \mathbf{d}) = 3(2g - 2 + n)W(\mathbf{d})$ at every primitive tuple of genus ≤ 18 , are part of the formal controls (Propositions 5.1 and 5.2).

4. MAIN RESULTS

Throughout, “for every primitive $\mathbf{d}$ of genus g ” means for every element of $\mathcal{P}_g$, enumerated as the $p(3g - 3)$ partitions of $3g - 3$; the count is part of each formal statement.

Theorem 4.1 (the open inequality at genus 14 to 18). *Let $g \in \{14, 15, 16, 17, 18\}$. For every primitive $\mathbf{d}$ of genus g ,*

$$C(\mathbf{d}) \leq C(2^{3g-3}),$$

with equality only for $\mathbf{d} = 2^{3g-3}$. The numbers of tuples inspected are $p(39) = 31\,185$, $p(42) = 53\,174$, $p(45) = 89\,134$, $p(48) = 147\,273$ and $p(51) = 239\,943$.

Proof. Exhaustive exact computation. For each g the $p(3g-3)$ partitions of $3g-3$ are enumerated, $V(\mathbf{d})$ of (10) is computed for each through the memoised recursion (9), and it is checked that $V(\mathbf{d}) \leq V(2^{3g-3})$ for all $\mathbf{d}$ and that exactly one enumerated tuple attains the value $V(2^{3g-3})$. The five genera are five separate formal statements (Appendix A), each reading one line of a single report computed once for genus 2 to 18. $\square$

Together with Lemma 3.1 and Lemma 2 of [1], Theorem 4.1 gives Conjecture 1 of [1], for all $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$, at each genus $14 \leq g \leq 18$; the source reports $g \leq 13$. The extreme values at the new genera are

g	$C(3g-2)$	$C(2^{3g-3})$	$4\pi g (C(2^{3g-3}) - C(3g-2))$
14	0.30757362...	0.31305742...	0.9648...
15	0.30828936...	0.31342059...	0.9672...
16	0.30891563...	0.31373677...	0.9693...
17	0.30946823...	0.31401454...	0.9712...
18	0.30995942...	0.31426048...	0.9729...

(exact rationals; in lowest terms their numerators and denominators run from 31 digits at $g = 14$ to 81 digits at $g = 18$, and only the leading decimals are shown. The last column, computed here, is consistent with the spread $\sim 1/(4\pi g)$ that the expansions (24)–(25) of [1] predict.)

Theorem 4.2 (refinement monotonicity, genus 2 to 18). *Let $2 \leq g \leq 18$, let $\mathbf{d}$ be primitive of genus g , let d_i be an entry of $\mathbf{d}$ and let $1 \leq k \leq (d_i - 1)/2$. Let $\mathbf{d}'$ be $\mathbf{d}$ with the entry d_i replaced by the two entries $k+1$ and $d_i - k$. Then*

$$C(\mathbf{d}) < C(\mathbf{d}').$$

Equivalently: for every partition λ of $3g-3$ and every single split $\lambda \rightarrow \mu$, $C(\mathbf{d}(\lambda)) < C(\mathbf{d}(\mu))$ strictly.

Proof. Exhaustive exact computation, in the same sweep as Theorem 4.1: for each enumerated $\mathbf{d}$, each index i and each k in range, $V(\mathbf{d}) < V(\mathbf{d}')$ is checked. The number of triples $(\mathbf{d}, i, k)$ inspected per genus (indices counted with multiplicity, as in the formal statement) is

g	2	3	4	5	6	7	8	9	10
steps	2	21	90	328	959	2 588	6 311	14 579	31 703
g	11	12	13	14	15	16	17	18	
steps	66 315	133 182	259 637	491 295	908 008	1 639 978	2 905 593	5 053 485	

515 715 for $g \leq 13$ and 10 998 359 for $14 \leq g \leq 18$, 11 514 074 in all, with no violation. (Counting each split $\lambda \rightarrow \mu$ once rather than once per index gives 2, 17, 73, 253, 738, 1 948, 4 721, 10 767, 23 286, 48 297, 96 524, 187 019, 352 413, 648 245, 1 166 600, 2 059 004, 3 569 927 distinct splits at $g = 2, \dots, 18$, 8 169 834 in all.) The step counts are not part of the formal statements, which record only that the check passed at each genus together with $p(3g-3)$; they were recomputed for this paper by the second implementation. $\square$

The formal statement is split at genus 13, the boundary of the source's own range, so that the part of Theorem 4.2 lying inside that range and the part beyond it carry separate pins.

Corollary 4.3 (refinement monotonicity implies nesting; not formalised). *Let $2 \leq g \leq 18$ and let $\mathbf{d}$ be primitive of genus g . Then*

$$C(3g-2) \leq C(\mathbf{d}) \leq C(2^{3g-3}),$$

the left inequality being strict unless $\mathbf{d} = (3g-2)$ and the right one strict unless $\mathbf{d} = 2^{3g-3}$. In particular both extremisers are unique, and with Lemma 3.1 Conjecture 1 of [1] holds for all $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ at every genus $2 \leq g \leq 18$.

Proof. Let $\lambda = \mathbf{d} - 1$. If $\lambda \neq 1^{3g-3}$ it has a part ≥ 2 , which can be split; iterating, there is a chain of single splits from λ to 1^{3g-3} , of positive length, along which C strictly increases by Theorem 4.2. Likewise λ is reached from $(3g-3)$ by a chain of single splits (merge the parts of λ one at a time and read the chain backwards), of positive length if $\lambda \neq (3g-3)$. $\square$

The two inequalities of Corollary 4.3 are also checked directly and separately in the formal development (Theorem 4.1, Propositions 4.4 and 4.5); the corollary records that the refinement statement alone accounts for them.

Proposition 4.4 (the source's range, exactly). *For every genus $2 \leq g \leq 13$ and every primitive $\mathbf{d}$ of genus g , $C(\mathbf{d}) \leq C(2^{3g-3})$, with equality only for $\mathbf{d} = 2^{3g-3}$; the numbers of tuples inspected are $p(3g-3) = 3, 11, 30, 77, 176, 385, 792, 1575, 3010, 5604, 10143, 17977, 39783$ in all.*

The inequality here is exactly what [1] reports having observed numerically; what is added is that the check is exact, covers the complete set $\mathcal{P}_g$, and records the uniqueness of the maximiser. At $g = 13$ the computed extremes are $C(37) = 0.3067477612\dots$ and $C(2^{36}) = 0.3126359483\dots$, in agreement with the source's $0.30674776\dots$ and $0.31263595\dots$.

Proposition 4.5 (the proved inequality, as a control). *For every genus $2 \leq g \leq 18$ and every primitive $\mathbf{d}$ of genus g , $C(3g-2) \leq C(\mathbf{d})$, with equality only for $\mathbf{d} = (3g-2)$; the numbers of tuples inspected are the $p(3g-3)$ of Section 2, 600 492 in all.*

The inequality is Lemma 2 of [1], proved there for all $\mathbf{d} \in (\mathbb{Z}_{\geq 1})^n$ and every genus; Proposition 4.5 is a control on the engine, not a new claim. The uniqueness of the minimiser is a strictness increment over the lemma, which gives only " $\geq$ " (its proof carries slack at every step); we do not claim it beyond genus 18.

5. CONTROLS

The value of an exhaustive computation is the value of its controls. Three kinds are reported: reproduction of everything the source prints, closed formulas checked inside the sweep, and refutations of natural over-statements.

Proposition 5.1 (Table 1 and the Remarks of the source). *The recursion (9) gives exactly the fourteen values of Table 1 of [1]:*

$$\begin{array}{lll} C(4) = \frac{35}{144}, & C(2, 3) = \frac{1015}{3888}, & C(2^3) = \frac{175}{648}, \\ C(7) = \frac{25025}{93312}, & C(2, 6) = \frac{77077}{279936}, & C(3, 5) = \frac{38731}{139968}, \\ C(4, 4) = \frac{4249}{15552}, & C(2, 2, 5) = \frac{6545}{23328}, & C(2, 3, 4) = \frac{39235}{139968}, \\ C(3^3) = \frac{714175}{2519424}, & C(2^3, 4) = \frac{6625}{23328}, & C(2^2, 3^2) = \frac{179375}{629856}, \\ C(2^4, 3) = \frac{120625}{419904}, & C(2^6) = \frac{546875}{1889568}, & \\ C(1, 4) = \frac{35}{144}, & C(1, 1, 2, 2, 2) = \frac{175}{648}, & \end{array}$$

the last two being instances of the dilaton relation; and it gives the four exact values and the four strict inequalities asserted in Remarks 1–3 of [1] (p. 5):

$$\begin{array}{ll} C(0^6, 10) = \frac{1616615}{6718464} < C(4), & C(2^3) < C(0, 0, 6) = \frac{5005}{15552}, \\ C(2^5, 8) = \frac{727759375}{2448880128} < \frac{419588015525}{1410554953728} = C(3^4, 5), & C(4, 4) < C(3, 5). \end{array}$$

The first pair (Remark 1) shows that (3) fails once an entry 0 is allowed; the third inequality (Remark 3) is the source's counterexample to weak monotonicity in the length (the source prints the decimals $0.29718 < 0.29746$); the fourth (Remark 2) is its counterexample to the length-then-lexicographic monotonicity of [2]. Outside the formal development the same engine reproduces the classical values $\langle \tau_1 \rangle_1 = \frac{1}{24}$, $\langle \tau_4 \rangle_2 = \frac{1}{1152}$, $\langle \tau_2 \tau_3 \rangle_2 = \frac{29}{5760}$, $\langle \tau_2^3 \rangle_2 = \frac{7}{240}$, $\langle \tau_7 \rangle_3 = \frac{1}{82944}$ and the integer column $D \cdot C(\mathbf{d})$ of the source's Table 1 ($D = 3888$ at $g = 2$, $D = 7558272$ at $g = 3$).

Proposition 5.2 (closed formulas inside the sweep). *For every genus $2 \leq g \leq 18$, inside the same computation as Theorems 4.1 and 4.2: (i) $W(3g-2) = (6g-3)!!$, the one-point formula (7); (ii) Zograf's formula (11), in its integer form, at all $3g$ two-point tuples $(d_1, 3g-1-d_1)$, $0 \leq d_1 \leq 3g-1$; (iii) the dilaton identity $W(1, \mathbf{d}) = 3(2g-2+n)W(\mathbf{d})$ at every primitive $\mathbf{d}$ of genus g ; (iv) every enumerated tuple has all entries ≥ 2 , degree $3g-3+n$ and genus g .*

Item (ii) is the independent second implementation of the two-point numbers: it agrees with the recursion at all $\sum_{g \leq 18} 3g = 510$ two-point tuples, 0 mismatches. (A transcription of the *other* two-point formula printed by the source, its Eq. (43) from [15], did not reproduce the numbers. Read as printed, with $(-1)!! = 1$, its right-hand side at $g = 1$, $d_1 = 0$ is $\xi_{-1,3}/(1!!5!!) = -\frac{1}{48}$ against $\langle \tau_0 \tau_2 \rangle_1 = \frac{1}{24}$, a factor -2 ; and the discrepancy is not a constant factor, the ratios at $g = 1$, $d_1 = 0, 1, 2$ being -2 , $-\frac{1}{4}$, $-\frac{2}{7}$. The first place the intended convention is in doubt is ξ_{k_1, k_2} at $k_1 = -1$, where the printed factor $(6g_1 + 1)/(6g_1 - 1)$ is evaluated at $g_1 = 0$ and contributes -1 . We did not resolve it and do not use the formula; this is recorded as an unresolved transcription, not as a disagreement with the source.) Item (iv) checks that the enumeration lands in the primitive locus of the right genus; that it covers all of it rests on the count $p(3g - 3)$, which is pinned in every formal statement and agrees with the partition numbers, together with the canonical (ascending) generation of the partitions; the distinctness of the generated lists is not separately checked in the formal development, and was checked for this paper by the second implementation.

Proposition 5.3 (the hypothesis $d_j \geq 1$ cannot be dropped). *Neither inequality in (3) survives entries equal to 0: at $g = 2$, $C(0^6, 10) < C(4)$ and $C(2^3) < C(0, 0, 6)$; at $g = 14$, $C(0^6, 46) < C(40)$ and $C(2^{39}) < C(0, 0, 42)$.*

The $g = 2$ pair is the source's Remark 1; the $g = 14$ pair shows that the phenomenon is not an artefact of small genus. (Corollary 2 of [1] (p. 11), which follows its Lemma 2, gives the lower bound with zeros allowed, namely $C(3g - 2)$ times $\prod_{j=1}^{p_0} (1 + \frac{j+2-p_0}{6g-6+3n-3j})$ with p_0 the multiplicity of 0 in $\mathbf{d}$. The bound is sharp: it is an equality at each of the four tuples above.)

Proposition 5.4 (neither bound can be narrowed). *Already at $g = 3$ one has $C(7) < C(2, 6)$ and $C(2, 2, 2, 2, 3) < C(2^6)$; so replacing $C(3g - 2)$ by the next value up, or $C(2^{3g-3})$ by the next value down, breaks (3).*

Both witnesses are entries of the source's Table 1; the uniqueness of the two extremisers in Theorem 4.1 and Propositions 4.4, 4.5 is the same statement at every genus ≤ 18 .

Proposition 5.5 (dominance is not the order). *$C(4, 4) < C(3, 5)$ and $C(7) < C(2^6)$.*

In terms of $\lambda = \mathbf{d} - 1$: $(4, 2)$ dominates $(3, 3)$ and C is larger at $(4, 2)$, while the one-part partition $(3g - 3)$ dominates everything and C is smallest there. So C is monotone in neither direction of the dominance order. Refinement is a proper suborder of reverse dominance (μ refining λ implies $\lambda \supseteq \mu$), and the two counterexamples to the orders discussed in the source are not refinement pairs: $(3, 3)$ and $(4, 2)$ are incomparable, and so are $(1^5, 7)$ and $(2^4, 4)$ (a part 7 cannot arise by splitting from a partition whose largest part is 4). Of the natural candidates, refinement is the one left standing, and Theorem 4.2 is the evidence for it.

6. DISCUSSION

The one-step inequality. In the W normalisation a single split is the inequality

$$(13) \quad W(\mathbf{d} \text{ with } d_i \text{ replaced by } k+1, d_i - k) > 3(2g - 2 + n) W(\mathbf{d}) = W(1, \mathbf{d}),$$

by the dilaton identity of Proposition 5.2(iii) — “splitting a part beats inserting a 1”. Both sides are values of the recursion (9) on tuples of the same genus and length $n+1$, so (13) is a statement about one step of the Dijkgraaf–Verlinde–Verlinde relation, and a proof of it for all $\mathbf{d}$ would prove Conjecture 1 of [1] with strictness by Corollary 4.3. We have no such proof. The source's own proof of the lower inequality proceeds by the same recursion with the quadratic term discarded and with slack; the upper inequality has no such slack to give away, since the differences $C(\mathbf{d}') - C(\mathbf{d})$ along a split are of relative size $O(1/g)$ at most, the total spread across a genus being $\sim 1/(4\pi g)$.

Conjecture. We state the refinement statement as a conjecture, as the form of (3) that the evidence supports.

Conjecture 6.1. *For every $g \geq 2$, every partition λ of $3g - 3$ and every single split $\lambda \rightarrow \mu$, $C(\mathbf{d}(\lambda)) < C(\mathbf{d}(\mu))$.*

By Corollary 4.3 and Lemma 3.1 it implies Conjecture 1 of [1] with strict inequalities off the extremes and unique extremisers. For the Brézin–Gross–Witten numbers of [2] the analogous statement is a consequence of their Conjecture 2, checked there to genus 40; whether the analogue of (13) is easier in that setting we have not examined.

Cost, and the next genera. The recursion memoises, cumulatively over genus 2 to g , about $1.7\text{--}1.9 \times 10^5$ multisets at $g = 13$, $0.9\text{--}1.1 \times 10^6$ at $g = 16$ and $2.7\text{--}2.9 \times 10^6$ at $g = 18$ (the lower figures for the compiled engine, 2685047 at $g = 18$; the higher for the recomputation of Remark 6.2, 2924991), and the wall time grows by a factor of about 2.1 per genus. The compiled evaluation behind the formal statements — the fifteen theorems together with the single evaluation they share — ran in one process in 364 seconds of wall time at a peak of 2.65 GB; that evaluation timed on its own through the shared library took 362 seconds and 1.34 GB. Both figures are the development record’s: the compiled check was not re-run for this paper, only its stored result read. The independent recomputation for this paper, in interpreted Python with integer arithmetic, is reported in Remark 6.2. Extrapolating, genus 19 and 20 cost roughly $2\times$ and $4.5\times$ genus 18 (about 4.4 and 7.3 million multisets, 2 and 3.5 GB); they are within reach and were not attempted. The eventual wall is memory, since the memo holds every multiset of every genus below the target.

Remark 6.2 (independent recomputation). For this paper the genus-2-to-18 sweep was re-run from a second, independently written implementation of (9) (Python, arbitrary-precision integers, the smallest entry as the distinguished index, a hash-map memo keyed on sorted tuples), together with a third, slower implementation using the largest entry as the distinguished index. The second implementation reproduced the counts $p(3g - 3)$, the split-step counts of Theorem 4.2, all extreme values, the one-point and dilaton identities and Zograf’s formula (11) as printed, at every genus $2 \leq g \leq 18$, and returned “no violation” for both inequalities of (3), the uniqueness of both extremisers, and refinement monotonicity at every genus, in 12 minutes 42 seconds of wall time with a peak resident set of 0.93 GB, memoising 2924991 multisets; the third agreed with the second on every primitive tuple of genus ≤ 7 , every two-point tuple of genus ≤ 9 , and the four tuples with zeros of Proposition 5.3.

7. VERIFICATION

Theorems 4.1 and 4.2 and Propositions 4.4, 4.5, 5.1, 5.2, 5.3, 5.4 and 5.5 rest on fifteen declarations checked by Lean 4 [18] (Lean 4.32.0; the surrounding repository builds against *Mathlib* [19], though neither module here uses it), whose statements are reproduced verbatim in Appendix A. The development is two modules. The first (398 lines, 33 definitions, no theorems) imports nothing and implements the recursion (9) on ascending lists of natural numbers with a hand-written hash table, the enumeration of partitions, the comparison integers (10), Zograf’s formula in integer form, and one function `genusReport` returning, for a genus g , the tuple $(g, p(3g - 3), \text{lower inequality}, \text{upper inequality}, \text{minimiser unique}, \text{maximiser unique}, \text{refinement monotonicity}, \text{controls})$; the constant `scanMain` is the list of these tuples for $g = 2, \dots, 18$. Because the module imports nothing it is compiled to a shared library, and the fifteen theorems of the second module (219 lines) are each proved by `native_decide`, i.e. by evaluating the compiled code and trusting the compiler; the report is evaluated once and read by all of them. Every proof therefore rests, besides `propext` and `Quot.sound`, on the native-evaluation axiom that Lean 4.32 attaches to each such theorem (printed as `<name>._native.native_decide.ax_1_1`); there is no `sorry`, no `Classical.choice`, and no axiom declared by hand, as confirmed by `#print axioms` on all fifteen declarations, read for this paper from the compiled modules. What the formal statements certify is the output of the recursion as implemented; that this output is the intersection number rests on the relation (5), on Lemma 2.1, and on the controls of Proposition 5.2. Outside the formal development, and labelled as such where they occur, are Lemma 2.1, Lemma 3.1, Corollary 4.3, the step counts in the proof of Theorem 4.2, the decimal expansions, and the recomputation of Remark 6.2. As recorded in the development record, the whole check ran in a single process in 364 seconds of wall time at a peak

of 2.65 GB. The development is currently in a private repository: repository reference to be added at submission.

APPENDIX A. THE FORMAL STATEMENTS

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). Lists are ascending; `Ceq d p q` decides $C(\mathbf{d}) = p/q$ and `Clf d e` decides $C(\mathbf{d}) < C(\mathbf{e})$ (false if the genera differ), both through the integer identities of Section 2; `report g` is the line of `scanMain` for genus `g`, and `upperAt`, `lowerAt`, `refineAt`, `ctrlAt` project it.

Definitions in the second module.

```
def report (g : Nat) : GRow := scanMain.getD (g - 2) (0, 0, false, false, false, false, false, false)

def upperAt (g : Nat) : Nat × Nat × Bool × Bool :=
  match report g with | (a, c, _, up, _, uh, _, _) => (a, c, up, uh)

def lowerAt (g : Nat) : Nat × Nat × Bool × Bool :=
  match report g with | (a, c, low, _, ul, _, _, _) => (a, c, low, ul)

def refineAt (g : Nat) : Nat × Nat × Bool :=
  match report g with | (a, c, _, _, _, sp, _) => (a, c, sp)

def ctrlAt (g : Nat) : Nat × Bool :=
  match report g with | (a, _, _, _, _, _, ct) => (a, ct)

and, from the engine module, def scanMain : List GRow := scanUpTo 18 with abbrev GRow := Nat ×
Nat × Bool × Bool × Bool × Bool × Bool × Bool, and

def Ceq (d : List Nat) (p q : Nat) : Bool :=
  let ds := sortAsc d
  let n := ds.length
  match genusOf (sumL ds) n with
  | none => false
  | some g => Wof ds * q == p * (6 ^ g * fct g * 3 ^ (2 * g - 2 + n) * fct (2 * g - 3 + n))

def Clf (d e : List Nat) : Bool :=
  let ds := sortAsc d
  let es := sortAsc e
  let nd := ds.length
  let ne := es.length
  match genusOf (sumL ds) nd, genusOf (sumL es) ne with
  | some g1, some g2 =>
    g1 == g2 &&
    (Wof ds * (3 ^ ne * fct (2 * g1 - 3 + ne)) < Wof es * (3 ^ nd * fct (2 * g1 - 3 + nd)))
  | _, _ => false
```

Theorem 4.1.

```
theorem witten_upper_bound_g14 : upperAt 14 = (14, 31185, true, true)
theorem witten_upper_bound_g15 : upperAt 15 = (15, 53174, true, true)
theorem witten_upper_bound_g16 : upperAt 16 = (16, 89134, true, true)
theorem witten_upper_bound_g17 : upperAt 17 = (17, 147273, true, true)
theorem witten_upper_bound_g18 : upperAt 18 = (18, 239943, true, true)
```

Theorem 4.2.

```
theorem witten_refinement_le13 :
  (List.range' 2 12).map refineAt =
    [(2, 3, true), (3, 11, true), (4, 30, true), (5, 77, true), (6, 176, true),
     (7, 385, true), (8, 792, true), (9, 1575, true), (10, 3010, true), (11, 5604, true),
     (12, 10143, true), (13, 17977, true)]

theorem witten_refinement_g14_g18 :
```

```
(List.range' 14 5).map refineAt =
  [(14, 31185, true), (15, 53174, true), (16, 89134, true), (17, 147273, true),
   (18, 239943, true)]
```

Propositions 4.4 and 4.5.

```
theorem witten_upper_bound_le13 :
  (List.range' 2 12).map upperAt =
    [(2, 3, true, true), (3, 11, true, true), (4, 30, true, true), (5, 77, true, true),
     (6, 176, true, true), (7, 385, true, true), (8, 792, true, true), (9, 1575, true, true),
     (10, 3010, true, true), (11, 5604, true, true), (12, 10143, true, true),
     (13, 17977, true, true)]
```

```
theorem witten_lower_bound :
  (List.range' 2 17).map lowerAt =
    [(2, 3, true, true), (3, 11, true, true), (4, 30, true, true), (5, 77, true, true),
     (6, 176, true, true), (7, 385, true, true), (8, 792, true, true), (9, 1575, true, true),
     (10, 3010, true, true), (11, 5604, true, true), (12, 10143, true, true),
     (13, 17977, true, true), (14, 31185, true, true), (15, 53174, true, true),
     (16, 89134, true, true), (17, 147273, true, true), (18, 239943, true, true)]
```

Propositions 5.1 and 5.2.

```
theorem witten_table_one :
  [Ceq [4] 35 144, Ceq [2,3] 1015 3888, Ceq [2,2,2] 175 648,
   Ceq [7] 25025 93312, Ceq [2,6] 77077 279936, Ceq [3,5] 38731 139968,
   Ceq [4,4] 4249 15552, Ceq [2,2,5] 6545 23328, Ceq [2,3,4] 39235 139968,
   Ceq [3,3,3] 714175 2519424, Ceq [2,2,2,4] 6625 23328, Ceq [2,2,3,3] 179375 629856,
   Ceq [2,2,2,2,3] 120625 419904, Ceq [2,2,2,2,2,2] 546875 1889568,
   Ceq [1,4] 35 144, Ceq [1,1,2,2,2] 175 648] = List.replicate 16 true
```

```
theorem witten_source_remarks :
  [Ceq [0,0,0,0,0,10] 1616615 6718464, Ceq [0,0,6] 5005 15552,
   Ceq [2,2,2,2,2,8] 727759375 2448880128, Ceq [3,3,3,3,5] 419588015525 1410554953728,
   Clt [0,0,0,0,0,10] [4], Clt [2,2,2] [0,0,6],
   Clt [2,2,2,2,2,8] [3,3,3,3,5], Clt [4,4] [3,5]] = List.replicate 8 true
```

```
theorem witten_controls :
  (List.range' 2 17).map ctrlAt =
    [(2, true), (3, true), (4, true), (5, true), (6, true), (7, true), (8, true), (9, true),
     (10, true), (11, true), (12, true), (13, true), (14, true), (15, true), (16, true),
     (17, true), (18, true)]
```

The controls component of a report line is the conjunction of the one-point identity $w1 == df0dd$ ($3 * g - 1$), the two-point identity $(w * g : \text{Nat}) == \text{zograpS } g \text{ d1}$ for $d1$ in $\text{List.range } (3 * g)$, the dilaton identity $w1d == 3 * (2 * g - 2 + d.\text{length}) * wd$ at every enumerated d , and $\text{enumOK } g$.

Propositions 5.3, 5.4 and 5.5.

```
theorem witten_zeros_break_both_bounds :
  [Clt [0,0,0,0,0,10] [4], Clt [2,2,2] [0,0,6],
   Clt (List.replicate 6 0 ++ [46]) [40],
   Clt (List.replicate 39 2) [0,0,42]] = List.replicate 4 true
```

```
theorem witten_bounds_are_attained :
  [Clt [7] [2,6], Clt [2,2,2,2,3] [2,2,2,2,2]] = List.replicate 2 true
```

```
theorem witten_dominance_fails_both_ways :
  [Clt [4,4] [3,5], Clt [7] [2,2,2,2,2]] = List.replicate 2 true
```

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