# THE THIN PROBLEM FOR UNIFORM WITNESSES: $W_{3,2}(8) = \binom{7}{3}$ AND $W_{5,3}(12) > \binom{11}{5}$

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Abstract. A family $\mathcal{F}$ of $(d+1)$-element subsets of $[n]$ is an *s-witness family* if every member $F$ carries a set $B_F \subseteq F$ with $|B_F| = s$ that no member of the family cuts out of $F$, i.e. $F \cap F' \neq B_F$ for all $F' \in \mathcal{F}$; $W_{d,s}(n)$ denotes the largest size of such a family. Chao, Xu, Yip and Zhang conjectured $W_{d,s}(n) \leq \binom{n-1}{d}$ for $n \geq 2(d+1)$ and $0 \leq s \leq d$, and in June 2026 Xu disproved it for $d \geq 4$ and $\lceil (d+2)/2 \rceil \leq s \leq d-1$, leaving open exactly one shape: whether $W_{2s-1,s}(n) > \binom{n-1}{2s-1}$ can hold for a fixed $s \geq 2$ and infinitely many $n$. We settle the first in-range cell of each of the first two branches of that question, and they disagree. At $s = 2$ the inequality fails at its first in-range $n$: $W_{3,2}(8) = 35 = \binom{7}{3}$, the star being optimal. At $s = 3$ it holds: $W_{5,3}(12) \geq 471 > 462 = \binom{11}{5}$, exhibited by a family with no point common to all its members. The latter is in particular a counterexample to the conjecture at a parameter set the published disproof does not reach. Around these we give the first table of $W_{d,s}(n)$ — all 48 cells with $n \leq 7$, all 27 with $n = 8$, and 36 of the 44 with $n = 9$ — prove that the whole space $\binom{[n]}{d+1}$ is an $s$-witness family exactly when $n + s < 2(d+1)$, close the two edges $s = 0$ and $s = d$ of the table by argument at every $n$, bound the column $s = d-1$ through the fibres of the trace map, and show that the colouring certificate an exact maximum-clique solver emits can never close the next open cell $W_{3,2}(9)$. Every statement below is machine-checked in Lean 4 except the exhaustive computations, which are named as such.

## 1. Introduction

Let $[n] = \{1, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the $k$-element subsets of $[n]$. We quote the following notion in the form printed by Chao, Xu and Zakharov [1]; it and the conjecture below are due to Chao, Xu, Yip and Zhang [3].

**Definition 1.1** ([1, Definition 1.1])**.** Let $n \geq d+1$ and $0 \leq s \leq d$. A family $\mathcal{F} \subseteq \binom{[n]}{d+1}$ is an *s-witness family* if for every $F \in \mathcal{F}$ there exists $B_F \subseteq F$ with $|B_F| = s$ such that $F \cap F' \neq B_F$ for every $F' \in \mathcal{F}$.

We call such a $B_F$ a *missing trace* of $F$, following [2], and write

$$W_{d,s}(n) \;=\; \max\bigl\{\, |\mathcal{F}| \;:\; \mathcal{F} \subseteq \tbinom{[n]}{d+1} \text{ is an } s\text{-witness family} \,\bigr\}.$$

The definition interpolates between two classical situations. At $s = 0$ the only candidate for $B_F$ is $\emptyset$, so the condition says that no two members are disjoint: an $s$-witness family is a uniform intersecting family and $W_{d,0}(n) = \binom{n-1}{d}$ for $n \geq 2(d+1)$ is the Erdős–Ko–Rado theorem [4]. If instead one asks only that *some* proper subset of each $F$ be missing, with no condition on its size, the resulting notion is "VC-dimension at most $d$", for which $\binom{n-1}{d}$ is false — Ahlswede and Khachatrian [5] constructed larger families. Fixing the size of the witness was proposed as the repair, and the natural guess is that the extremal example is the *star*

$$\mathcal{S}_a \;=\; \bigl\{ F \in \tbinom{[n]}{d+1} : a \in F \bigr\}, \qquad |\mathcal{S}_a| = \tbinom{n-1}{d},$$

which is an $s$-witness family for every $s \leq d$.

**Conjecture 1.2** (the uniform witness conjecture; [1, Conjecture 1.3], [2, Conjecture 1.1])**.** Let $n \geq 2(d+1)$ and $0 \leq s \leq d$. If $\mathcal{F} \subseteq \binom{[n]}{d+1}$ is an $s$-witness family then $|\mathcal{F}| \leq \binom{n-1}{d}$.

The conjecture was known for $s = 0$ (Erdős–Ko–Rado), for $s = d$ and for $s = 1$ with $n$ large [3], and for $1 \le s \le d/2$ with $n$ large [1, Theorem 1.4]. In June 2026 Xu [2] disproved it: for $d \ge 4$ and $\lceil (d+2)/2 \rceil \le s \le d-1$ he constructed $s$-witness families of size $\binom{n-1}{d} + \binom{n-2(d+1-s)-2}{2s-d-2}$, whose smallest instance is a family of 127 five-element subsets of $[10]$ against the conjectured $\binom{9}{4} = 126$. His construction misses one line of the parameter plane, because the excess $\binom{n-2(d+1-s)-2}{2s-d-2}$ has a negative lower index exactly when $d = 2s-1$, and this is what [2, Section 3] records as surviving:

> Together with the known results for $s \le \frac{d}{2}$ and for $s = d$, Theorem 1.2 leaves only a very thin problem whether $W_{2s-1,s}(n) > \binom{n-1}{2s-1}$ can hold for some fixed $s \ge 2$ and infinitely many $n$.

Neither [1] nor [2] prints a single numerical value of $W_{d,s}(n)$, and we found no matching sequence in the OEIS. This paper computes the function where it is reachable and answers the first in-range cell of each of the two smallest branches of the surviving question. The two answers are opposite.

**Results.** Write $d = 2s-1$, so that $2(d+1) = 4s$ is the first $n$ for which Conjecture 1.2 is even stated.

- **$s = 2$, $n = 8$: the inequality fails.** $W_{3,2}(8) = 35 = \binom{7}{3}$ (Theorem 5.2), so the star is optimal at the first in-range cell of the $s = 2$ branch. The lower bound is the star; the upper bound rests on exhaustive computation, carried out twice by unrelated algorithms.
- **$s = 3$, $n = 12$: the inequality holds.** $W_{5,3}(12) \ge 471 > 462 = \binom{11}{5}$ (Theorem 6.1), exhibited by an explicit family of 471 six-element subsets of $[12]$ with no common point. Since $(12, 5, 3)$ satisfies $n \ge 2(d+1)$ and $s \le d$, this is a second counterexample to Conjecture 1.2, at a parameter set outside the range of Xu's theorem. The existence claim needs no exhaustive search and is machine-checked.

So the answer to the surviving question depends on $s$, which its phrasing ("for some fixed $s \ge 2$") allows but on which there was no data; and $s = 2$, the instance one tries first, is the one where the inequality fails. Neither result answers the question as asked, which is about infinitely many $n$: at $s = 3$ we have one $n$, and the next cell $W_{5,3}(13)$ resisted the same methods (Section 6).

The supporting material is of three kinds. Section 3 determines the shape of the table: the star bound at every $(n, d, s)$, an equivalence saying that the whole of $\binom{[n]}{d+1}$ is an $s$-witness family precisely when $n + s < 2(d+1)$ (so that $W_{d,s}(n) = \binom{n}{d+1}$ there and $W_{d,s}(n) < \binom{n}{d+1}$ otherwise), and the two edges $s = 0$ and $s = d$, both equal to $\binom{n-1}{d}$, the second proved here with no hypothesis on $n$. The two edges do not squeeze the interior: $W_{d,s}(n)$ is not monotone in $s$. Section 5 gives the table itself and Theorem 5.2; Section 6 gives Theorem 6.1 and two further cells of the $s = 3$ branch. Section 7 collects the tools — a deletion recursion relating consecutive $n$, and its degeneracy exactly at $n = 2(d+1)$, which is a precise form of the word "thin"; a recasting of $W_{d,s}(n)$ as a maximum-clique number; a proof that the certificate an exact clique solver emits is too weak to close the next open cell; and a bound on the column $s = d-1$ obtained from the fibres of the map $F \mapsto B_F$.

**What rests on computation.** Every displayed statement is a theorem of Lean 4 (Section 9) except where the text says otherwise. Two kinds of exception occur, and they are different in nature. A *lower* bound is a certificate: one exhibits a family and its witnesses, and checking is quadratic in the size of the family, so every lower bound in this paper is machine-checked, however it was found. An *upper* bound at a cell where neither the star nor the whole space is optimal requires an exhaustive search, and beyond a measured wall those searches were run outside the proof assistant. Where that happens we say so, name the search, and give its size.

## 2. Preliminaries

Throughout, $\mathcal{F}, \mathcal{G}$ denote families of $(d+1)$-element subsets of $[n]$. Two immediate remarks about Definition 1.1.

**Lemma 2.1.** *In Definition 1.1 the quantifier "for every $F' \in \mathcal{F}$" may be taken over all of $\mathcal{F}$, the member $F$ itself included, without changing the notion: $F \cap F = F$ has $d+1$ elements while $|B_F| = s \le d$.*

**Lemma 2.2** (hereditariness)**.** *If $\mathcal{F}$ is an $s$-witness family and $\mathcal{G} \subseteq \mathcal{F}$, then $\mathcal{G}$ is an $s$-witness family.*

Lemma 2.2 is one line — shrinking $\mathcal{F}$ only weakens the requirement $F \cap F' \ne B_F$ — but it is what makes the whole subject computable: refuting one size refutes every larger size, and a partial family that already fails can never be repaired, so a depth-first search over subfamilies of a fixed size may be pruned at the first failure and remain complete.

Finally, $\mathcal{F} \subseteq \binom{[n]}{d+1}$ gives the trivial bound

$$W_{d,s}(n) \;\le\; \binom{n}{d+1}, \tag{1}$$

attained exactly in the range determined in Theorem 3.2 below.

## 3. The shape of the table

### 3.1. The star.

**Proposition 3.1.** *Let $n \ge 1$ and $0 \le s \le d$. For every $a \in [n]$ the star $\mathcal{S}_a$ is an $s$-witness family, and $|\mathcal{S}_a| = \binom{n-1}{d}$. Consequently $\binom{n-1}{d} \le W_{d,s}(n)$ for all such $n, d, s$.*

*Proof sketch.* This is [2, Section 1], in one sentence: for $F \in \mathcal{S}_a$ take any $s$-subset $B_F \subseteq F \setminus \{a\}$, which exists because $|F \setminus \{a\}| = d \ge s$. The centre $a$ lies in every member, hence in every intersection $F \cap F'$, and it does not lie in $B_F$; so $F \cap F' \ne B_F$. The map $F \mapsto F \setminus \{a\}$ is a bijection from $\mathcal{S}_a$ onto $\binom{[n]\setminus\{a\}}{d}$. No hypothesis on $n$ is used and no search is involved, so this supplies the lower half of every cell whose value is $\binom{n-1}{d}$, at every $n$. $\square$

### 3.2. The whole space.

**Theorem 3.2.** *Let $d+1 \le n$ and $0 \le s \le d$. The whole space $\binom{[n]}{d+1}$ is an $s$-witness family if and only if $n + s < 2(d+1)$. Consequently:*

(1) *if $n + s < 2(d+1)$ then $W_{d,s}(n) = \binom{n}{d+1}$;*
(2) *if $n + s \ge 2(d+1)$ then $W_{d,s}(n) < \binom{n}{d+1}$.*

*Part (1) holds under the weaker hypothesis $s \le d+1$.*

*Proof sketch.* Suppose $F, F' \in \binom{[n]}{d+1}$ and $F \cap F' = B$ with $|B| = s$. Then $B \subseteq F'$ and $F' \setminus B$, of size $d+1-s$, is disjoint from $F$, so it fits among the $n-(d+1)$ points outside $F$; hence $2(d+1) \le n+s$. Below that threshold no member cuts any $F$ in any $s$-set at all, so *every* $s$-subset of $F$ is a missing trace and the whole space works; with the trivial bound (1) this proves part (1). Conversely, if $2(d+1) - s \le n$ then for any $F$ and any $s$-subset $B \subseteq F$ there is room outside $F$ for a $(d+1-s)$-set $Y$, and $F' = B \cup Y$ is a legal member with $F \cap F' = B$, so no $B$ survives and the whole space fails. For (2): a family of size $\binom{n}{d+1}$ would have to *be* the whole space. Both directions are arguments, valid at every $(n, d, s)$ and involving no search. $\square$

Theorem 3.2(1) is an exact value at infinitely many cells. At $n = 9$ it settles 25 cells at once ($d = 4$, $s = 0$; $d = 5$, $s \le 2$; $d = 6$, $s \le 4$; $d = 7$, $s \le 6$; $d = 8$, $s \le 8$), and it settles a block of every later row; for instance

$$W_{6,3}(10) = 120, \qquad W_{6,2}(11) = 330, \qquad W_{7,3}(12) = 495.$$

The threshold is sharp on both sides: at $(9, 6, 4)$ one has $9 + 4 < 14$ and the whole space works, while at $(9, 6, 5)$ one has $9 + 5 = 14$ and it fails by exactly one.

### 3.3. **The two edges.**

**Theorem 3.3** (the edge $s = 0$)**.** *A family* $\mathcal{F} \subseteq \binom{[n]}{d+1}$ *is a* 0*-witness family if and only if it is intersecting, i.e.* $F \cap F' \neq \emptyset$ *for all* $F, F' \in \mathcal{F}$*. Consequently, for* $n \geq 2(d+1)$*,*

$$W_{d,0}(n) = \tbinom{n-1}{d}.$$

*Proof sketch.* The only set of size 0 is $\emptyset$, so the condition of Definition 1.1 at $s = 0$ reads $F \cap F' \neq \emptyset$ for all $F' \in \mathcal{F}$, in both directions. The upper bound is then the Erdős–Ko–Rado theorem [4] for $(d+1)$-uniform intersecting families, whose hypothesis $d+1 \leq n/2$ is exactly $n \geq 2(d+1)$; the lower bound is Proposition 3.1. □

**Theorem 3.4** (the diagonal $s = d$)**.** *For every* $d \geq 1$ *and every* $n \geq d + 1$*,*

$$W_{d,d}(n) = \tbinom{n-1}{d}.$$

That $W_{d,d}(n) \leq \binom{n-1}{d}$ for $n \geq 2(d + 1)$ is the case $s = d$ of Conjecture 1.2, which [2, Section 1] attributes to [3]; the point of Theorem 3.4 is that the proof below needs no hypothesis on $n$ whatsoever, so it also covers cells the conjecture does not state — $W_{6,6}(9) = 28$ with $9 < 2 \cdot 7$ is one.

*Proof sketch.* Fix a $d$-witness family $\mathcal{F}$ and a missing trace $B_F \subsetneq F$ with $|B_F| = d$ for each member, so $F = B_F \cup \{y_F\}$ for a unique marked point $y_F$. Four steps. (i) $F \mapsto B_F$ is injective: if $B_F = B_{F'}$ with $F \neq F'$ then $y_F \notin F'$, so $F \cap F' = B_F$, which $B_F$ was chosen to avoid. (ii) For $z \in B_F$ the swapped set $\{y_F\} \cup (B_F \setminus \{z\})$ is not $B_{F'}$ for any $F' \in \mathcal{F}$: it would force either $y_F \in B_F$ or $F \cap F' = B_{F'}$. (iii) Fix a point $v$ and let $\mathcal{B} = \{B_F : F \in \mathcal{F}\}$. Send each $(d-1)$-set $A$ avoiding $v$ to $A \cup \{v\}$ when that lies outside $\mathcal{B}$, and otherwise — when $A \cup \{v\} = B_F$ — to $A \cup \{y_F\}$, which (ii) with $z = v$ puts outside $\mathcal{B}$. The two kinds are distinguished by containing $v$ or not, and (ii) makes each kind injective, so $\binom{n-1}{d-1} \leq \binom{n}{d} - |\mathcal{B}|$. (iv) $|\mathcal{F}| = |\mathcal{B}| \leq \binom{n}{d} - \binom{n-1}{d-1} = \binom{n-1}{d}$ by Pascal. The matching lower bound is Proposition 3.1. No search is used. □

Theorems 3.3 and 3.4 agree inside the conjecture's range, but they do not control what lies between them.

**Proposition 3.5.** $W_{d,s}(n)$ *is not monotone in* $s$*. Indeed* $W_{4,0}(10) = 126 < W_{4,3}(10)$*, both cells having* $n \geq 2(d + 1)$*; and the diagonal is a strict local minimum,* $W_{6,6}(9) = 28 < 33 = W_{6,5}(9)$*.*

*Proof sketch.* The first assertion is Theorem 3.3 against Xu's construction (Proposition 4.1) at the same $(n, d) = (10, 4)$; the second is Theorem 3.4 against Theorem 5.4. So no interpolation between the two proved edges can be expected. □

## 4. The disproof, re-derived, and the non-star extremal families

For completeness and as a check on the definitions we re-derive Xu's construction rather than quoting it. Given disjoint $C, U \subseteq [n]$ and a collection $\mathcal{P}$ of subsets of $C$, [2] sets $\mathcal{F}(\mathcal{P}, U) = \{P \cup A : P \in \mathcal{P},\ A \in \binom{U}{d+1-|P|}\}$ and gives, with $r = d + 1 - s$, disjoint $r$-sets $T, L \subseteq [n] \setminus \{1, 2\}$, $C = \{1, 2\} \cup T \cup L$, $U = [n] \setminus C$, and

$$\mathcal{P} = \Big(\big\{P \subseteq C : 1 \in P,\ |P| \leq d + 1\big\} \setminus \big\{\{1\} \cup T \cup Y : Y \subsetneq L\big\}\Big) \cup \big\{\{2\} \cup T \cup Y : Y \subseteq L\big\}.$$

The witness assignment is a five-case rule read off [2, Claim 2.3]; wherever the source says "choose" we take the numerically first choice.

**Proposition 4.1.** *At the five parameter sets* $(n, d, s) = (10, 4, 3)$*,* $(11, 4, 3)$*,* $(12, 5, 4)$*,* $(14, 6, 4)$*,* $(14, 6, 5)$ *the construction above yields* $s$*-witness families of sizes*

$$127, \quad 211, \quad 468, \quad 1717, \quad 1744,$$

*in agreement with* [2, Claim 2.2]*: the excesses over* $\binom{n-1}{d}$ *are* $1, 1, 6, 1, 28$*, equal to* $\binom{n-2(d+1-s)-2}{2s-d-2}$ *in each case.*

**Theorem 4.2.** *Conjecture 1.2 is false. At $n = 10$, $d = 4$, $s = 3$ — where $10 = 2(4+1)$ and $3 \le 4$ — there is a 3-witness family in $\binom{[10]}{5}$ of size $127 > 126 = \binom{9}{4}$.*

*Proof sketch.* The witnesses are carried with the family, so verification is $|\mathcal{F}|^2 = 16\,129$ intersection comparisons and no witness is ever searched for; a mistranscribed witness rule makes the check fail rather than make a false statement. An independent re-implementation, written from the source text and not from the formal development, agrees at eight parameter sets including $(16, 7, 5)$ and $(16, 7, 6)$. □

The extremal families at the boundary of Conjecture 1.2 are not stars, and [1] says so: for $d/2 < s \le d-1$ its Theorem 1.5 constructs $s$-witness families of size exactly $\binom{n-1}{d}$ that are not stars, "illustrating the barriers to adapting current methods as all the proofs of the known cases show that $\mathcal{F}$ must be a single star if equality is achieved". We reproduce that construction at the cell of Theorem 5.2.

**Proposition 4.3.** *At $(n, d, s) = (8, 3, 2)$ there is a 2-witness family of size $35 = \binom{7}{3}$ which is not a star: for every $a \in [8]$ some member misses $a$. Together with $\mathcal{S}_a$ this gives two structurally different extremal families at that cell, so $W_{3,2}(8) = 35$ cannot be proved by any argument that forces the extremal family to be a single star.*

*Remark* 4.4. The construction proving [1, Theorem 1.5] carries a parameter $m$, with printed range $m \in [s+2, n-1]$ in both v1 and v2. That range is wider than the construction allows. Its second block is $\mathcal{A}_2 = \left\{\{2\} \cup A : A \in \binom{[3,n]}{d},\ |A \cap [3,m]| < s\right\}$, which is empty unless $n - m \ge d - s + 1$; and when $\mathcal{A}_2$ is empty the family produced is $\{F : 1 \in F\}$, a star. At the top of the printed range this is what happens: for $m = n-1$ every $A$ has $|A \cap [3, n-1]| \ge d - 1 \ge s$, so $\mathcal{A}_2 = \emptyset$. Requiring $\mathcal{A}_2 \ne \emptyset$ leaves $m \in [s+2,\ n-d+s-1]$, which at $(8, 3, 2)$ is $\{4, 5, 6\}$. We take $m = 6$, the value singled out by v1's witness rule for $\mathcal{A}_2$ — there, "any subset of size $s$ containing $A \cap [m+1, n]$", which additionally needs $n - m \le s$, and both constraints hold together precisely because $s > d/2$. Version v2 replaces that rule by "any subset of size $s$ such that $|B_F \cap [m+1, n]| \ge d - s + 1$", which is well defined at every $m$ with $\mathcal{A}_2 \ne \emptyset$, so under v2 nothing singles out $m = 6$ at $(8, 3, 2)$. The family built with $m = 6$ is the one verified in Proposition 4.3.

## 5. The table, and the thin problem at $s = 2$

### 5.1. Exact values.

**Theorem 5.1.** *All 48 values $W_{d,s}(n)$ with $n \le 7$ are as in Table 1, forty-six of them proved outright and two with an exhaustively computed upper half. Exactly five of them are attained neither by a star nor by the whole space:*

$$W_{2,1}(5) = 7, \quad W_{3,2}(6) = 12, \quad W_{3,1}(7) = 28, \quad W_{3,2}(7) = 22, \quad W_{4,3}(7) = 18,$$

*and every one of these five has $n < 2(d+1)$, so none of them contradicts Conjecture 1.2.*

*Proof sketch.* Each cell is a matched pair. The lower half exhibits a family with its witnesses: the star in most cells, the whole space where $n+s < 2(d+1)$, and, in the five cells listed, an explicit family given by a list of sets. The upper half is either the trivial bound (1) together with Theorem 3.2, when the whole space is optimal, or else the pruned depth-first exhaustion over subfamilies of size $W+1$ licensed by Lemma 2.2. Forty-six of the forty-eight cells are proved outright. Two upper halves are exhaustive computations run outside the proof assistant, because their node counts exceed what its evaluator reaches at the relevant per-node cost: $W_{3,1}(7) = 28$ (refutation at 29, $2\,396\,571$ nodes) and $W_{3,2}(7) = 22$ (refutation at 23, $40\,876\,011$ nodes). Their lower halves are proved. □

The two patterns visible in Table 1 are now theorems: the ■ cells are exactly those with $n+s < 2(d+1)$ (Theorem 3.2) and the last entry of each row, $W_{d,d}(n) = \binom{n-1}{d}$, is Theorem 3.4. At $n = 8$ all 27 cells are known; twenty-one of them are proved by Theorems 3.2, 3.3 and 3.4, and the remaining

| $n$ | $d$ | $s=0$ | $s=1$ | $s=2$ | $s=3$ | $s=4$ | $s=5$ | $s=6$ |
|---|---|---|---|---|---|---|---|---|
| 4 | 1 | $3^\star$ | $3^\star$ | | | | | |
| 4 | 2 | $4^\blacksquare$ | $4^\blacksquare$ | $3^\star$ | | | | |
| 5 | 1 | $4^\star$ | $4^\star$ | | | | | |
| 5 | 2 | $10^\blacksquare$ | **7** | $6^\star$ | | | | |
| 5 | 3 | $5^\blacksquare$ | $5^\blacksquare$ | $5^\blacksquare$ | $4^\star$ | | | |
| 6 | 1 | $5^\star$ | $5^\star$ | | | | | |
| 6 | 2 | $10^\star$ | $10^\star$ | $10^\star$ | | | | |
| 6 | 3 | $15^\blacksquare$ | $15^\blacksquare$ | **12** | $10^\star$ | | | |
| 6 | 4 | $6^\blacksquare$ | $6^\blacksquare$ | $6^\blacksquare$ | $6^\blacksquare$ | $5^\star$ | | |
| 7 | 1 | $6^\star$ | $6^\star$ | | | | | |
| 7 | 2 | $15^\star$ | $15^\star$ | $15^\star$ | | | | |
| 7 | 3 | $35^\blacksquare$ | **28** | **22** | $20^\star$ | | | |
| 7 | 4 | $21^\blacksquare$ | $21^\blacksquare$ | $21^\blacksquare$ | **18** | $15^\star$ | | |
| 7 | 5 | $7^\blacksquare$ | $7^\blacksquare$ | $7^\blacksquare$ | $7^\blacksquare$ | $7^\blacksquare$ | $6^\star$ | |
| 8 | 1 | $7^\star$ | $7^\star$ | | | | | |
| 8 | 2 | $21^\star$ | $21^\star$ | $21^\star$ | | | | |
| 8 | 3 | $35^\star$ | $35^\star$ | $35^\star$ | $35^\star$ | | | |
| 8 | 4 | $56^\blacksquare$ | $56^\blacksquare$ | **49** | **40** | $35^\star$ | | |
| 8 | 5 | $28^\blacksquare$ | $28^\blacksquare$ | $28^\blacksquare$ | $28^\blacksquare$ | **25** | $21^\star$ | |
| 8 | 6 | $8^\blacksquare$ | $8^\blacksquare$ | $8^\blacksquare$ | $8^\blacksquare$ | $8^\blacksquare$ | $8^\blacksquare$ | $7^\star$ |

TABLE 1. $W_{d,s}(n)$ for $n \le 8$. A star $\star$ marks $W = \binom{n-1}{d}$ (the star is optimal), $\blacksquare$ marks $W = \binom{n}{d+1}$ (the whole space is an $s$-witness family), and boldface marks a cell where neither is optimal. Every boldface cell has $n < 2(d+1)$. The entry $W_{3,2}(8) = 35$ is Theorem 5.2.

six — $W_{2,1}(8) = 21$, $W_{3,1}(8) = 35$, $W_{3,2}(8) = 35$, $W_{4,2}(8) = 49$, $W_{4,3}(8) = 40$, $W_{5,4}(8) = 25$ — have exhaustively computed upper halves. Inside the conjecture's range every computed value at $n \le 8$ equals $\binom{n-1}{d}$.

5.2. $W_{3,2}(8)$**.** The cell $(8, 3, 2)$ is the first instance of the surviving question: $d = 2s-1$ with $s = 2$, and $n = 8 = 2(d+1)$ is the first $n$ for which Conjecture 1.2 is stated.

**Theorem 5.2.** $W_{3,2}(8) = 35 = \binom{7}{3}$*. In particular the inequality* $W_{2s-1,s}(n) > \binom{n-1}{2s-1}$ *of* [2, Section 3] fails *at* $s = 2$ *and* $n = 8$*: the star is optimal there.*

*Proof sketch.* The lower bound $35 \le W_{3,2}(8)$ is Proposition 3.1 and is proved; a second, non-isomorphic family of the same size is Proposition 4.3. The upper bound $W_{3,2}(8) \le 35$ rests on exhaustive computation and is not machine-checked. It was obtained twice, by unrelated methods.

*(a) A satisfiability refutation.* The formula has a variable $x_F$ for each of the 70 four-element subsets of [8] and a variable $w_{F,B}$ for each of the six pairs $B \subseteq F$, with clauses $x_F \to \bigvee_B w_{F,B}$, $w_{F,B} \to x_F$, and $w_{F,B} \to \neg x_{B \cup Y}$ for each of the six sets $Y$ of size $d+1-s = 2$ outside $F$. The last group replaces the quantifier over the whole family by an explicit list of blockers, and Proposition 7.2 below proves that replacement is an equivalence. A sequential counter [9] adds $\sum_F x_F \ge 36$. Two symmetry-breaking devices are used, both sound: one unit clause fixing a chosen 4-set to be in the family (legitimate because a non-empty family may be relabelled and the symmetric group is transitive on 4-sets), and lexicographic-leader clauses over the 575 non-identity elements of its stabiliser $S_4 \times S_4$. The plain formula (2 870 variables, 7 805 clauses) gave no verdict in 1 303 s; the symmetry-broken one (40 885 variables, 120 701 clauses) was refuted in 266 s at 1 012 531 conflicts [8]. Before the verdict was used, the encoding was validated on all 27 cells with $n \le 7$ and a non-trivial value: asked for $W$ and for $W+1$ it answered satisfiable and unsatisfiable respectively in 27 of 27, agreeing with an independent branch-and-bound implementation at every cell. A later encoding,

which also fixes the witness of the chosen set (sound by Proposition 7.3) and uses a totalizer counter, reproduced the same refutation in 40 s on a formula 5.3 times smaller.

*(b) An exact maximum-clique search.* Under the recasting of Theorem 7.6, $W_{3,2}(8)$ is the independence number of a graph on $\binom{8}{4} \cdot \binom{4}{2} = 420$ vertices. A bitset branch-and-bound with a greedy colouring bound in the style of [10], using the symmetry of [8] at the top levels and the deletion bound of Theorem 7.4 at every node, returned 35 after 58 108 595 nodes in 57 s, with no solver and no certificate file. The same program reproduces all 48 values of Table 1 with $n \leq 7$ — the 45 that were theorems here when it was run, together with the three then recorded as computations — and 26 of the 27 cells of the row $n = 8$; the twenty-seventh, $W_{4,2}(8) = 49$, it did not finish, reaching an incumbent of 48 in 1 376 190 464 nodes. ☐

### 5.3. **The next cell of the $s = 2$ branch.**

**Proposition 5.3.** *Granting $W_{3,2}(8) \leq 35$, one has $56 \leq W_{3,2}(9) \leq 63$, the lower bound being the star and the upper bound the deletion recursion of Theorem 7.4; and, granting further $W_{3,2}(9) \leq 56$, one has $84 \leq W_{3,2}(10) \leq 93$. Moreover, at $n = 9$ the following four cells strictly beat their stars,*

$$W_{4,1}(9) \geq 97, \quad W_{4,2}(9) \geq 81, \quad W_{5,4}(9) \geq 61, \quad W_{6,5}(9) \geq 33,$$

*against the star values $70, 70, 56, 28$; by Theorem 3.2 each is also strictly below its whole space $126, 126, 84, 36$. All four have $n < 2(d + 1)$.*

The exact value of $W_{3,2}(9)$ is open. It is bracketed in $[56, 63]$, so seven values are at issue. Both available routes have been measured past their frontier on it: a quiet hour of satisfiability solving on the best available 203 403-clause formula returned no verdict at 12 309 635 conflicts, three times the depth of any previous run, and the exact clique search of Theorem 5.2(b) burned 1 729 495 040 nodes in fifty minutes without clearing the first of eighteen symmetry classes, where the corresponding instance at $(8, 3, 2)$ — same code, same bound, also eighteen classes — finishes in under a minute. Six local searches totalling 23.2 million steps never left the star's 56. Theorem 7.7 below explains why one natural certificate for the upper half cannot exist.

**Theorem 5.4.** $W_{6,5}(9) = 33$ *and* $W_{7,7}(9) = 8$.

*Proof sketch.* For $(9, 6, 5)$ the lower bound is the explicit family of 33 seven-element subsets of $[9]$ of Proposition 5.3, found by local search on the 756-vertex flag graph; that 33 is optimal was first shown by the exact clique search of Theorem 5.2(b) on that same graph (1 981 222 nodes, 1.25 s); the upper bound proved here is instead the pruned exhaustion of Lemma 2.2, which is cheap because only $\binom{9}{7} = 36$ candidate sets exist, so a family of 34 discards just two of them. The value sits strictly between the star $\binom{8}{6} = 28$ and the whole space 36, and $9 < 2(6 + 1)$, so this cell too lies outside Conjecture 1.2. For $(9, 7, 7)$ the lower bound is the star and the upper bound is Theorem 3.2, since $9 + 7 = 2 \cdot 8$; the value $8 = \binom{8}{7}$ agrees with Theorem 3.4. Both cells are proved. ☐

Together with Theorems 3.2, 3.3 and 3.4, this makes 36 of the 44 cells of the row $n = 9$ known. (The row is all $(d, s)$ with $1 \leq d \leq 8$ and $0 \leq s \leq d$. Its nine cells with $d = 8$ are trivial, $\binom{[9]}{9}$ having a single member; Table 1 suppresses the corresponding trivial cells $d = n - 1$, which is why its rows $n \leq 8$ are counted without them.) The eight that remain are $(9, 2, 1)$, $(9, 3, 1)$, $(9, 3, 2)$, $(9, 4, 1)$, $(9, 4, 2)$, $(9, 4, 3)$, $(9, 5, 3)$, $(9, 5, 4)$, all with $1 \leq s \leq d - 1$.

## 6. The thin problem at $s = 3$

At $s = 3$ the surviving question concerns $d = 2s - 1 = 5$, and the first $n$ inside the conjecture's range is $n = 2(d + 1) = 12$. Here the inequality holds.

**Theorem 6.1.** $W_{5,3}(12) \geq 471 > 462 = \binom{11}{5}$. *In particular $W_{2s-1,s}(n) > \binom{n-1}{2s-1}$ holds at $s = 3$, $d = 5$, $n = 12$, and Conjecture 1.2 — with $n \geq 2(d + 1)$ and $s \leq d$ as printed — fails at $(12, 5, 3)$.*

*Proof sketch.* We exhibit an explicit family of 471 six-element subsets of [12]. It is presented as a bare list of sets: the witnesses are not supplied but searched for and then checked against Definition 1.1, so a wrong witness assignment makes the check fail rather than make a false statement. The verification is a certificate check, not a search, and it is machine-checked; nothing in this theorem rests on exhaustive computation.

How the family was found, which the statement does not depend on: a satisfiability model at target $463 = \binom{11}{5} + 1$ on a 731 210-clause formula, obtained in 390 s of solver time, extended by greedy addition to 471, a fixed point of twelve randomised passes. The formula's lexicographic-leader clauses used generators only, since the residual group has order $3! \cdot 3! \cdot 6! = 25\,920$; that is a weaker symmetry break but a sound one, and in any case symmetry breaking can only turn a satisfiable formula into an unsatisfiable one, never the reverse, so a positive answer is not at risk from it. The family was independently re-verified three ways: by decoding the model and re-reading Definition 1.1; inside the proof assistant, as above; and by parsing the list of sets back out of the compiled formal source, so that the object verified is the object proved about. ☐

*Remark* 6.2. $(12, 5, 3)$ lies outside the range $d \geq 4$, $\lceil (d + 2)/2\rceil \leq s \leq d - 1$ of Xu's theorem, which at $d = 5$ forces $s = 4$; and his size formula does not apply there, its lower index $2s - d - 2$ being $-1$ when $d = 2s - 1$. So the family of Theorem 6.1 is not an instance of that construction, and the theorem is a counterexample to Conjecture 1.2 at a new parameter set. The excess is 9, against the excess 6 of Xu's own family at the neighbouring cell $(12, 5, 4)$. The family is not a star and not a star with a few sets added: no point lies in all 471 members, and its degree sequence, the number of members through each of the twelve points, is

$$202,\ 202,\ 203,\ 345,\ 210,\ 314,\ 203,\ 204,\ 201,\ 202,\ 204,\ 336,$$

against a star's $462, 0, \ldots, 0$. That it is not a star with additions is forced: for a star centred at $p$ and any added $G \not\ni p$, every 3-subset $B \subseteq G$ is cut out of $G$ by the star member $\{p\} \cup B \cup \{\text{two more points}\}$, so $G$ has no witness at all.

*Remark* 6.3. Theorem 6.1 settles that the inequality *can* hold, at one $n$. It does not answer the question of [2, Section 3], which asks for infinitely many $n$. The next cell did not fall: at $n = 13$ the target is $793 = \binom{12}{5} + 1$ among $\binom{13}{6} = 1716$ candidate sets, a 2 440 925-clause formula returned no verdict in 2 400 s of wall time at 1 055 616 conflicts, and a greedy pass seeded with the family of Theorem 6.1 reached only 725, below the star. Imposing a cyclic symmetry, which is sound for a lower bound and collapses the search to one variable per orbit, did not make the problem cheap either: at $n = 12$ with the group of order 12 and target 468, a 717 131-clause formula gave no verdict in 1 800 s. What the 471-set family lacks is structure: a list of sets proves an inequality at one $n$ and explains nothing.

Two further cells of the same branch, both below the conjecture's range, are worth recording because they show the branch is not degenerate.

**Proposition 6.4.** *$W_{5,3}(9) \geq 77$ and $W_{5,3}(10) \geq 155$, against the star values $\binom{8}{5} = 56$ and $\binom{9}{5} = 126$. By Theorem 3.2, $W_{5,3}(9) < \binom{9}{6} = 84$, so $(9, 5, 3)$ is a cell where neither the star nor the whole space is optimal. Granting $W_{5,3}(9) \leq 77$, the deletion recursion gives $W_{5,3}(10) \leq 192$.*

The matching upper halves are exhaustive computations and are not proved: $W_{5,3}(9) = 77$ (the target 78 was refuted in 0.5 s) and, at $(10, 5, 3)$, no verdict at target 156 in 150 s. Both cells have $n < 2(d+1) = 12$, so like the boldface cells of Table 1 they bear on the hypothesis of Conjecture 1.2 rather than on its conclusion.

## 7. Methods, and two obstructions

### 7.1. **Two decision procedures.**

**Proposition 7.1.** *Let $n, d, s$ be given.*

(1) *(Lower halves.) Given a list of pairs* $(F, B_F)$ *of subsets of* $[n]$*, the checks* $|F| = d+1$, $B_F \subseteq F$, $|B_F| = s$ *and* $F \cap F' \neq B_F$ *for all listed* $F'$*, if all passed, prove that the listed sets form an* $s$*-witness family, of size the number of distinct listed sets. The cost is* $|\mathcal{F}|^2$ *comparisons and no witness is searched for.*

(2) *(Upper halves.) If the pruned depth-first enumeration of* $(k+1)$*-element subfamilies of* $\binom{[n]}{d+1}$*, pruned as soon as the chosen part already fails, reports failure, then* $W_{d,s}(n) \leq k$.

*Both statements are proved. Completeness of the prune in (2) is exactly Lemma 2.2.*

All computations here run on integers whose binary digits encode subsets of $[n]$, and are transported to sets by the map sending an integer to the positions of its set bits. Three facts carry the transport — bitwise conjunction is intersection; the map is injective below $2^n$; and every subset is the image of some integer — and they, too, are proved, so that the arithmetic never has to be trusted. If a witness is wrong, or the transport is misused, the check returns failure; it cannot return success for a false statement.

### 7.2. **The blocker form and the symmetry breaks.**

**Proposition 7.2.** *Let* $\mathcal{F} \subseteq \binom{[n]}{d+1}$*, let* $F \in \binom{[n]}{d+1}$ *and let* $B \subseteq F$ *with* $|B| = s$*. Then* $B$ *is a missing trace of* $F$ *for* $\mathcal{F}$ *if and only if* $B \cup Y \notin \mathcal{F}$ *for every* $(d+1-s)$*-element* $Y \subseteq [n] \setminus F$*.*

*Proof sketch.* One direction is immediate, since such a $B \cup Y$ cuts $F$ in $B$. For the other, if $F' \in \mathcal{F}$ has $F \cap F' = B$ then $F' = B \cup (F' \setminus F)$, and $|F' \setminus F| = d+1-s$ is forced by $|F'| = d+1$: this is where uniformity is used. The proposition is what makes the finite clause list of the encodings in Theorem 5.2 equivalent to Definition 1.1; it is the only step of those encodings that is not a transcription, and it is the step at which a wrong encoding would give a wrong answer. □

**Proposition 7.3.** $s$*-witness families are preserved by relabelling of* $[n]$*. Moreover, if* $B \subseteq F$ *and* $B' \subseteq F'$ *with* $|B| = |B'|$ *and* $|F| = |F'|$*, some permutation of* $[n]$ *carries* $(B, F)$ *to* $(B', F')$*. Consequently, if some* $s$*-witness family has size* $k > 0$ *then some* $s$*-witness family of size* $k$ *contains a prescribed* $(d+1)$*-set* $S$ and *has a prescribed* $s$*-subset* $B \subseteq S$ *as a missing trace of* $S$*.*

This is the joint soundness of the two unit clauses used by the encodings; the second of them shrinks the residual symmetry group by a factor $\binom{d+1}{s}$, from 576 to 96 at $(8,3,2)$ and from 2880 to 480 at $(9,3,2)$, and it is worth about a factor of five in solving time. Since an unsound narrowing looks exactly like a good optimisation, it is proved rather than argued.

### 7.3. **The deletion recursion, and why the problem is thin.**

**Theorem 7.4.** *For all* $n, d, s$*,*

$$\big(n+1-(d+1)\big)\, W_{d,s}(n+1) \;\leq\; (n+1)\, W_{d,s}(n).$$

*Proof sketch.* Delete a point $v$: by Lemma 2.2 the members avoiding $v$ form an $s$-witness family, and after deleting $v$ from the ground set it lives on $n$ points, so it has at most $W_{d,s}(n)$ members. Each of the $n+1$ points gives one such family, and each member of a family on $n+1$ points avoids exactly $n+1-(d+1)$ of them. Summing gives the claim. This is the only statement here relating two different values of $n$. □

**Proposition 7.5.** *For every* $d$*,* $2\binom{2d+1}{d+1} = \binom{2d+2}{d+1}$*. Hence at* $n = 2(d+1)$ *the recursion of Theorem 7.4 returns exactly the trivial bound* (1) *and says nothing.*

Proposition 7.5 is a precise form of the word "thin" in [2, Section 3]: at the only $n$ that the surviving question is about — the first one inside the conjecture's range — induction on $n$ gives nothing, and a search is forced. This is why both Theorem 5.2 and Theorem 6.1 are computations rather than inductions.

7.4. **A maximum-clique recasting.** Call a pair $(F, B)$ with $B \subseteq F$, $|B| = s$, $|F| = d + 1$ a *flag*, and call a set $\Phi$ of flags a *flag clique* when distinct flags of $\Phi$ have distinct first coordinates and $F \cap F' \neq B$ for every ordered pair $(F, B), (F', B') \in \Phi$.

**Theorem 7.6.** *For all $n, d, s, k$: some $s$-witness family has at least $k$ members if and only if some flag clique has at least $k$ flags. Equivalently, $W_{d,s}(n)$ is the largest size of a flag clique. No hypothesis on $s$ is needed; a non-empty flag clique forces $s \leq d$ by its own diagonal instance.*

*Proof sketch.* In one direction forget the witnesses; in the other choose one missing trace per member. Both are one-line constructions, and the diagonal instance $F' = F$ is free by Lemma 2.1. This is not a new bound but a change of search space: it turns the problem into a maximum independent set problem on a graph with $\binom{n}{d+1}\binom{d+1}{s}$ vertices — 756 at $(9, 3, 2)$, 420 at $(8, 3, 2)$ — which is what the searches of Theorem 5.2(b) and Theorem 5.4 operate on, and what the local search producing the families of Proposition 5.3 operates on. The distinctness requirement is load-bearing: two flags on the same set with different witnesses satisfy the other conditions but present one member, not two. □

7.5. **The colouring certificate cannot close $(9, 3, 2)$.** An exact clique solver that terminates emits, for its upper bound, a colouring: a partition of the vertices into classes each of which is a clique of the complementary graph. In the language of flags, call a set $\Psi$ of flags a *block* if any two distinct members of it conflict, i.e. share their first coordinate or cut it in one of the two witnesses, and call a set $C$ of blocks a *cover* if every flag lies in one. Such an object is finite, checkable by evaluation, and would be the natural way to transport an upper bound into a proof assistant. It cannot work at the cell that matters.

**Theorem 7.7.** (1) *(Soundness.) If $C$ is a cover then $W_{d,s}(n) \leq |C|$.*

(2) *(Blocks are small.) Let $m$ bound the number of pairwise disjoint $(d + 1 - s)$-sets that fit outside a $(d + 1)$-set, i.e. $n - (d + 1) < (m + 1)(d + 1 - s)$. Then every block has at most $\binom{d+1}{s} + m + 1$ flags.*

(3) *(Hence the certificate is too weak.) Every cover at $(9, 3, 2)$ has at least 84 blocks, and every cover at $(8, 3, 2)$ has at least 47 blocks.*

*Proof sketch.* (1) A flag clique meets each block in at most one flag, since each of the three ways two flags conflict is forbidden inside a flag clique; so $|\Phi| \leq \sum_{\Psi \in C} |\Phi \cap \Psi| \leq |C|$, and Theorem 7.6 converts this to a bound on $W_{d,s}(n)$. (2) Fix $(F_0, B_0) \in \Psi$. If some first coordinate repeats in $\Psi$, then every other flag $(F, B)$ has $F_0 \cap F = B$, and the tails $F \setminus F_0$ are pairwise disjoint $(d+1-s)$-sets outside $F_0$, so at most $m$ of them occur, while the repeated fibre contributes at most $\binom{d+1}{s}$. If no first coordinate repeats, at most one flag can have $F_0 \cap F = B_0$ with $B \neq B_0$, and the same disjointness applies to the rest. Either way $|\Psi| \leq \binom{d+1}{s} + m + 1$. (3) At $(9, 3, 2)$ the bound reads $\binom{4}{2} + 2 + 1 = 9$ against $\binom{9}{4}\binom{4}{2} = 756$ flags, so at least $\lceil 756/9 \rceil = 84$ blocks are needed; at $(8, 3, 2)$ it reads 9 against 420, so at least 47. □

The consequence is that no cover can certify $W_{3,2}(9) \leq 63$, let alone the value 56 at issue — and the bound 63 is one the deletion recursion already yields from $W_{3,2}(8) \leq 35$. The same count at the cell where the answer is not in doubt gives at least 47 blocks against the true value 35, so it is the shape of the certificate that fails, not the cell. The obstruction was measured as well as proved: the true maximum block size, computed exactly, is $7, 7, 8, 8$ at $n = 6, 7, 8, 9$ with $d = 3$, $s = 2$, so the proved bound is off by exactly one — as it is at all seven cells measured, which suggests $\binom{d+1}{s} + m$ is the sharp form — and the measured obstruction is stronger, at least 95 blocks at $(9, 3, 2)$. Since every block holds at most 8 of the 756 flags, even a fractional cover has weight at least $756/8 = 94.5$, so weighting does not help. A greedy cover built in practice uses 119 blocks at $(9, 3, 2)$ and 68 at $(8, 3, 2)$.

The certificate is not useless in general: at $(9, 4, 1)$, $(9, 4, 2)$, $(9, 5, 3)$ and $(9, 5, 4)$ the same count still leaves room to beat the trivial bound (1). What kills it at $(9, 3, 2)$ is that $d = 2s - 1$ with $s = 2$

makes $\binom{d+1}{s} = 6$ small while $\binom{n}{d+1} = 126$ is large — so the surviving question of [2] is precisely the shape this certificate is worst at. The definition is not vacuous: the cover by the fibres of $(F, B) \mapsto F$ has exactly $\binom{n}{d+1}$ blocks and recovers the trivial bound (1) through part (1).

7.6. **The column** $s = d - 1$**.** The proof of Theorem 3.4 began with the injectivity of $F \mapsto B_F$, which holds because $|B_F| = |F| - 1$. One column to the left the map has fibres, and they can be described exactly.

**Theorem 7.8.** *Let $\mathcal{F} \subseteq \binom{[n]}{d+1}$ be a $(d-1)$-witness family with chosen missing traces $B_F$, and let $C$ be a $(d-1)$-set. Then $\{F \in \mathcal{F} : B_F = C\}$ has at most $\max(n-d, 3)$ members, and the value $n - d$ is attained — in the star centred at $a$, every $(d-1)$-set $C$ avoiding $a$ is a missing trace of all $n - d$ members containing $C \cup \{a\}$. Consequently, at every $n$ and every $d \geq 1$,*

$$W_{d,d-1}(n) \;\leq\; \max(n-d, 3) \cdot \binom{n}{d-1},$$

*and in particular $W_{2,1}(n) \leq (n-2)n$ for $n \geq 5$, which gives $W_{2,1}(9) \leq 63$ and hence the bracket $28 \leq W_{2,1}(9) \leq 63$.*

*Proof sketch.* Write $P_F = F \setminus B_F$, a 2-set. If $B_F = B_{F'} = C$ then $F \cap F' = C \cup (P_F \cap P_{F'})$, so the defining condition $F \cap F' \neq C$ says exactly that $P_F$ and $P_{F'}$ meet. The fibre over $C$, transported by $F \mapsto F \setminus C$, is therefore an intersecting family of 2-sets on the $n-d+1$ points outside $C$, and such a family is a star or a triangle: if no point is common, take $P_0 = \{a, b\}$; a member missing $a$ is $\{b, c\}$, a member missing $b$ is then $\{a, c\}$, and any further member meeting all three is inside $\{a, b, c\}$. So a fibre has at most $\max(n - d, 3)$ members; grouping $\mathcal{F}$ by its trace and counting the possible traces gives the displayed bound, with no search and no hypothesis. □

The bound of Theorem 7.8 beats the trivial bound (1) exactly for $n \geq d^2 + 2d$, with a tie at $n = d^2 + 2d - 1$, so it is useful only in the column $d = 2$; there it gives the first bound below $\binom{n}{3}$ at four cells, of which $(9, 2, 1)$ is one of the eight cells of the row $n = 9$ still open. It does not, however, extend to the column what Theorem 3.4 gives to the diagonal, and this is not an accident of the method.

**Proposition 7.9.** (1) *There is no bound $W_{d,d-1}(n) \leq \binom{n-1}{d}$, even inside the conjecture's range: it fails at $(10, 4, 3)$, which is an $s = d - 1$ cell with $n = 2(d + 1)$, by Theorem 4.2.*

(2) *The route through fibres cannot reach $\binom{n-1}{d}$ in any case: the product of the two tight factors, the fibre bound $n - d$ and the count $\binom{n-1}{d-1}$ of traces available to a star member, satisfies the identity $(n-d)\binom{n-1}{d-1} = d\binom{n-1}{d}$ — exactly $d$ times the conjectured answer, because every member of a star has exactly $d$ valid traces of size $d - 1$ and so is counted $d$ times.*

(3) *By Proposition 4.3 the extremal family at $(8, 3, 2)$ is not unique and is not a star, so no star-rigidity argument can settle that cell either.*

*The measured slack of the bound of Theorem 7.8 at the three cells of interest is large:* 84 *against* 22 *at* $(7, 3, 2)$*,* 140 *against* 35 *at* $(8, 3, 2)$*,* 216 *against* $[56, 63]$ *at* $(9, 3, 2)$ *— worse than the trivial bound* (1) *at all three.*

## 8. Controls

Formal statements can be true and vacuous, and searches can be complete and about the wrong object. The following are recorded to exclude the corresponding failures.

**Proposition 8.1.** (1) *(The size $s$ is load-bearing.) Xu's family at $(10, 4, 3)$ is a 3-witness family and is an $s$-witness family for no $s \in \{0, 1, 2, 4\}$; and it is not $(d + 1)$-uniform for $d = 5$. A definition reading "a witness of size at most $s$" or "at least $s$" would not reproduce these.*

(2) *(The hypothesis $n \geq 2(d + 1)$ is not decoration.) Conjecture 1.2 without it is false already at $(7, 3, 2)$, where $W_{3,2}(7) = 22 > 20 = \binom{6}{3}$ — a failure independent of Xu's construction, which needs $d \geq 4$. Likewise Theorem 3.3 fails without it: at $(9, 4, 0)$ the answer is the whole space, 126, not $\binom{8}{4} = 70$.*

(3) *(The trivial bound is loose.)* $126 < W_{4,3}(10) \leq 252 = \binom{10}{5}$.
(4) *(Both procedures are non-vacuous.) The exhaustion of Proposition 7.1(2) reports success at the exact value in four cells where it reports failure one higher; and the check of Proposition 7.1(1) rejects Xu's family when each witness is replaced by the numerically first s-subset of its set, rejects the whole of* $\binom{[6]}{3}$ *as a* 1*-witness family, and rejects the* 471*-set family of Theorem 6.1 and the* 77*-set family of Proposition 6.4 under the same substitution.*
(5) *(Bracketing.) At the cells carrying an exact value, both the claim one larger and the claim one smaller are refuted. Where only one half is a theorem, only the corresponding side is claimed: at* $(12, 5, 3)$*, for instance, the assertion* $W_{5,3}(12) \leq 470$ *is refuted while no upper bound at all is asserted.*

## 9. Verification

All statements above are formalised in Lean 4 [6] against Mathlib [7], and the development is free of `sorry`. The formal statements are the ones displayed here: Definition 1.1 is [1, Definition 1.1] with no change but notation, Conjecture 1.2 is stated with its hypotheses as printed and refuted in that form (Theorems 4.2 and 6.1), and $W_{d,s}(n)$ is defined as the supremum of the attained sizes rather than through any surrogate. The exceptions to full proof are named explicitly and are of one kind only, namely upper bounds obtained by exhaustive search outside the proof assistant: the upper half of Theorem 5.2, the upper halves of $W_{3,1}(7) = 28$ and $W_{3,2}(7) = 22$ in Theorem 5.1, the six exhaustively computed cells of the row $n = 8$, and the upper halves quoted after Proposition 6.4. Lower bounds are certificates and are all machine-checked, including the 471-set family of Theorem 6.1, for which the proof assistant searches for the witnesses itself rather than trusting the ones a solver produced. Statements whose proof invokes compiled evaluation of a decision procedure — the explicit families, and the two small exhaustions behind Theorem 5.4 — depend on the corresponding evaluation principle in addition to the ambient axioms; all general theorems (Propositions 3.1, 7.2, 7.3, Theorems 3.2, 3.3, 3.4, 7.4, 7.6, 7.7, 7.8) use none of it. The repository is private; repository reference to be added at submission.

**Scope of the literature search.** Both new results are claimed new against the search recorded here, whose corpus is a local full-text mirror of arXiv: 880 750 papers in 365 shards, identifiers running through `2608.26093`. The months `2607` and `2608` were added to the mirror and swept on 28 August 2026. Fixed-string sweeps for *uniform witness*, *s-witness*, *witness family*, *missing trace*, *uniform set systems*, $W_{d,s}$, $W_{2s-1,s}$ and the two arXiv identifiers return: exactly two papers treating $s$-witness families, namely [1] and [2]; exactly one occurrence of $W_{2s-1,s}$, in the closing remark of [2] quoted in the introduction; no paper giving a value, a bound or a construction for $W_{d,s}(n)$ at any $(n, d, s)$; and no forward citation of either source inside the mirror other than [2] citing [1]. On 28 August 2026 Semantic Scholar likewise listed [2] as the only citation of [1]. The same pass found no OEIS sequence for $W_{d,s}(n)$.

The gaps are worth weighing. Journal-only versions were not searched, and neither were later versions of papers already in the mirror — the mirror's copy of [1] is v1, and v2 differs in the respects noted in Remark 4.4. Google Scholar was not available, so forward citations were checked only in the mirror and on Semantic Scholar. Nothing off arXiv and nothing in another language was searched, nor was any treatment of the same question under other terminology that cites none of the papers above. One near-collision was checked and excluded: the conjecture is stated in arXiv:2501.13850, the preprint of [3], and not only in the journal version, but that preprint never uses the word "witness", so a keyword sweep on "witness" alone does not reach it.

## References

[1] Ting-Wei Chao, Zixuan Xu and Dmitrii Zakharov, *Uniform set systems with uniform witnesses*, arXiv:2602.17459. Version v2 (23 March 2026) is the one used here, and the numbering cited above — Definition 1.1, Conjecture 1.3, Theorem 1.4 ($s \leq d/2$), Theorem 1.5 (the non-star construction) — is v2's; v1 (19 February 2026) carries the same

numbering but stops at Theorem 1.5, v2 adding a Theorem 1.6 on non-star families of arbitrary piercing number. The second author is not the author of [2]: this paper thanks Zixiang Xu in its acknowledgements.
[2] Zixiang Xu, *A disproof of the uniform witness conjecture*, arXiv:2606.24776 (v1, 23 June 2026).
[3] Ting-Wei Chao, Zixiang Xu, Chi Hoi Yip and Shengtong Zhang, *Uniform set systems with small VC-dimension*, Int. Math. Res. Not. IMRN **2025** (2025), no. 17, Paper No. rnaf269, 20 pp.; arXiv:2501.13850. The second author is the author of [2], so the conjecture is disproved by one of the four who made it.
[4] P. Erdős, Chao Ko and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320.
[5] R. Ahlswede and L. H. Khachatrian, *Counterexample to the Frankl–Pach conjecture for uniform, dense families*, Combinatorica **17** (1997), no. 2, 299–301.
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