THE ZERO-FREE DISK CONJECTURE FOR BIPARTITE MATCHING AT $k = 4$: AN EFFECTIVE JANSON COMPARISON AND A FINITE CERTIFICATE TABLE

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ABSTRACT. Let the mn edges of $K_{m,n}$ carry independent mean-one exponential costs and let $C_{k,m,n}$ be the minimum cost of a k-matching. Wästlund showed that the moment generating function $F_{k,m,n}(t) = \mathbb{E}e^{tC_{k,m,n}}$ is a rational function of t whose first pole sits at $t = mn/k$, and conjectured that it has no complex zero in the open disk $|t| < mn/k$; the conjecture would give a Gaussian scaling limit for the random assignment problem. He proved it for $k \le 3$ by comparison with a real-rooted function of Janson, remarked that there "seems to be no hope of generalizing this argument to higher values of k", and proved — non-effectively, with no bound on the exceptions — that for each k only finitely many pairs (m, n) can fail. We settle the first open case, $k = 4$, for the explicit numerator: for all integers $m, n \ge 4$ the degree-six polynomial $\widehat{P}_{4,m,n}$ has no zero in the *closed* disk $|t| \le mn/4$. Modulo Wästlund's identification of $\widehat{P}_{4,m,n}$ with the numerator of $F_{4,m,n}$, which we quote from his paper and do not reprove, this is his conjecture at $k = 4$. The proof makes his own comparison effective. We compute the discrepancy $E = \widehat{P}_4 - D_3$ between the numerator and Janson's in closed form over $\mathbb{Q}(m, n)$ — four coefficients, one of them the clean $e_5 = 22/\prod_{i+j<3}(m-i)(n-j)$ — and reduce the comparison to the positivity of an explicit integer polynomial W of bidegree $(10, 10)$. We then determine the positivity region of W exactly. On the integer quadrant it is precisely the region the comparison covers, so the comparison fails at every one of the 235 residual pairs and at no other pair; each of those pairs is closed instead by an exactly verified dominance certificate, and the failure region of the comparison and the certificate table coincide. Over the reals the region the certificates cover is a staircase of fourteen closed quadrants, no corner of which can be moved inwards inside the domain $m, n \ge 4$, and on it the zero-free disk holds for real parameters — in particular for all real $m, n \ge 11$ and all real $m \ge 4$, $n \ge 62$, real parameters being the generality in which Wästlund states his own $k \le 3$ result. The radius is close to sharp: $\widehat{P}_{4,4,4}$ has a real zero below $1.0776 \cdot mn/4$, so the statement has about 7.5% of slack at its tightest pair. The results below are machine-checked in Lean 4; the last section says exactly what that covers and what it does not.

1. INTRODUCTION

1.1. **The object.** Give the mn edges of the complete bipartite graph $K_{m,n}$ independent costs, each exponentially distributed with mean one. For $1 \le k \le \min(m, n)$ let $C_{k,m,n}$ be the minimum total cost of a k-matching, that is of a set of k pairwise vertex-disjoint edges; $C_{n,n,n}$ is the cost of the random assignment problem. Wästlund [1] studies the moment generating function

$$F_{k,m,n}(t) = \mathbb{E}\big[e^{\,t\,C_{k,m,n}}\big],$$

proves by an explicit recursion (Theorem 8 of [1]) that it is a *rational* function of t, and proves a structure theorem for it (Proposition 9): $F_{k,m,n} = P/Q$ with $\deg P = \binom{k}{2}$, $\deg Q = \binom{k+1}{2}$, and

$$Q(t) = \prod_{\substack{i,j\ge 0\\ i+j<k}} \Big(1 - \frac{(k-i-j)\,t}{(m-i)(n-j)}\Big). \tag{1}$$

Since $F(0) = Q(0) = 1$ we may and do normalise $P(0) = 1$, and we write $\widehat{P}_{k,m,n}$ for that numerator. The factor $(i, j) = (0, 0)$ of (1) vanishes at $t = mn/k$ and no other factor vanishes earlier (Lemma 2.2 below), which is why [1] calls $t = mn/k$ the first pole of $F_{k,m,n}$. Its Conjecture 11 reads, verbatim:

> "The moment generating function $F_{k,m,n}(t)$ has no complex zeros in the open disk of radius mn/k centered at the origin."

Equivalently: $F_{k,m,n}$ has no complex zero of modulus smaller than that of its first pole, so that $F_{k,m,n}$ and $\log F_{k,m,n}$ have the same radius of convergence at the origin. What makes it load-bearing is the

chain proved in [1]: nonnegativity of all cumulants of $C_{k,m,n}$ implies the zero-free disk, and the zero-free disk implies the Gaussian scaling limit

$$\sqrt{n}\,\big(C_{n,n,n} - \zeta(2)\big) \implies \mathcal{N}\big(0,\, 4\zeta(2) - 4\zeta(3)\big)$$

for the assignment problem, where $\zeta(2) = \pi^2/6 = \lim_n \mathbb{E}\, C_{n,n,n}$.

1.2. **Where the source stops.** For $k \le 2$ the conjecture is immediate. For $k = 3$ it is proved in [1] (Proposition 15, stated there too on the *closed* disk $|t| \le mn/3$) for all m and n, by comparing F with the real-rooted function that Svante Janson suggested to Alm and Sorkin [3]: over a common denominator the two numerators differ by a single monomial in t, on the circle $|t| = mn/3$ that monomial is the smaller in modulus, and a winding-number argument puts the zeros of both outside the disk. ([1] nowhere names Rouché's theorem; it appeals to "basic complex analysis" and to both numerators winding "zero times around the origin".) Immediately after that proof [1] writes, verbatim:

> "This establishes Conjecture 11 for $k \le 3$ and all m and n, but in view of the distribution of zeros of $F_{n,n,n}(t)$ [...] there seems to be no hope of generalizing this argument to higher values of k. The zeros of $F_{k,m,n}$ are in general not close to the real line and we should not expect $F_{k,m,n}$ to be well approximated by a function with only real zeros."

A further result of [1] (Proposition 16) shows that for each fixed k only finitely many pairs (m, n) can be counterexamples — but the proof runs through the continuity of the zeros of a polynomial in its coefficients and is explicitly non-effective; [1] states that its argument "does not offer any explicit bounds on m and n in the conceivable counterexamples". [1] also reports computing $F_{k,m,n}$ with a computer algebra system for all $k \le m \le n \le 15$, and computing its zeros numerically "in a large number of instances"; the conjecture is proposed on that evidence.

So $k = 4$ is the conjecture's first open case, and before the present work the set of pairs left open at $k = 4$, though finite, came with no bound whatever. The conjecture is still open in the literature: the only paper we can find citing [1] as of September 2026, Mordant [2], obtains the same Gaussian limit for the assignment problem with independent uniform costs by an entirely different route, and its introduction records that Wästlund's "proposed zero-free-disk conjecture would imply a Gaussian limit, but remains open".

1.3. **Results.** Throughout, $R := mn/4$ and

$$D_3(t) = \prod_{\substack{i,j \ge 0 \\ i+j<3}} \Big(1 - \frac{(3-i-j)\,t}{(m-i)(n-j)}\Big)$$

is the numerator of Janson's function over the denominator (1) at $k = 4$; it has degree $6 = \binom{4}{2}$, the same as $\widehat{P}_{4,m,n}$. Our main theorem (Theorem 7.1) is that for all integers $m, n \ge 4$,

$$\widehat{P}_{4,m,n}(t) \ne 0 \qquad \text{for every } t \in \mathbb{C} \text{ with } |t| \le \frac{mn}{4},$$

on the closed disk, slightly more than the conjecture asks. Its ingredients are the following, and each is of independent interest.

The discrepancy in closed form (Theorem 4.1). The difference $E := \widehat{P}_{4,m,n} - D_3$ has $e_0 = e_1 = e_6 = 0$, so only four nonzero coefficients, and after clearing the denominator $\Delta = m^4(m-1)^3(m-2)^2(m-3) \cdot n^4(n-1)^3(n-2)^2(n-3)$ each Δe_j is an integer polynomial in (m, n), symmetric in $m \leftrightarrow n$, of bidegree $(7,7)$, $(6,6)$, $(5,5)$, $(4,4)$ for $j = 2, 3, 4, 5$. These are printed in §4. The last one factors completely,

$$e_5 = \frac{22}{\prod_{i+j<3}(m-i)(n-j)} = \frac{22}{m^3(m-1)^2(m-2)\; n^3(n-1)^2(n-2)},$$

the exact analogue of the single coefficient $e_2 = -1/\prod_{i+j<2}(m-i)(n-j)$ that carries Wästlund's $k = 3$ proof. [1] computes $\widehat{P}_k$ numerically and prints closed forms only at $k = 3$.

An effective comparison (Theorem 5.1). Janson's D_3 dominates E on $|t| \le R$ exactly when an explicit integer polynomial $W(m, n)$ of bidegree $(10, 10)$ is positive. We prove $W > 0$ for all $m, n \ge 11$, and for each $m \in \{4, \dots, 10\}$ for all $n \ge n_0(m)$ with

$$\big(n_0(4), \dots, n_0(10)\big) = (62, 32, 22, 17, 14, 12, 11),$$

and symmetrically with the roles of m and n exchanged. This is the effective form of the finiteness result of [1] at $k = 4$.

That those thresholds are the exact ones (Theorem 5.3). For every m in $\{4, \dots, 10\}$ the sign of $W(m, \cdot)$ changes exactly at $n = n_0(m)$: it is negative at $n_0(m) - 1$ and positive from $n_0(m)$ on. Below the threshold the comparison inequality is false, not merely unproved, so no sharper certificate for the same W can enlarge the tail; only a different comparison function can.

The residual set, and that it is not an artefact (Theorems 6.1 and 6.3). The complement is exactly 235 ordered pairs, all with $m, n \le 61$, and each is closed by a dominance certificate built from that pair's own, non-real, root pattern. On the integer quadrant $W > 0$ holds *if and only if* the pair is in the tail; so the Janson comparison fails at every one of the 235 pairs — at the worst of them $W(4, 4) = -260617273344 < 0$, missing by a factor of about 87 — and at no other pair. The failure region of the comparison and the certificate table coincide exactly, so no certificate in the table is redundant: at each of its pairs the comparison provably does not close.

The real staircase (§8). Nothing in the recursion of [1] asks m and n to be integers — [1] itself computes $F_{3,3,10/3}$ — and the tail estimate is really a statement about real parameters. The certificates that prove Theorem 5.1 turn out to be two-variable ones, so each of its seven strips — and each of their seven transposes — is in fact a closed real quadrant $[a, \infty) \times [n_0(a), \infty)$. The union Ω of the fourteen quadrants so obtained is a staircase in the real plane, no corner of which can be moved inwards inside the domain $m, n \ge 4$ (Remark 8.3), containing every integer pair of the tail; and on it $\widehat{P}_{4,m,n}$ has no zero with $|t| \le mn/4$ for real parameters (Theorems 8.2 and 8.4). In particular this holds for all real $m, n \ge 11$ and for all real $m \ge 4$, $n \ge 62$. The companion statement at $k = 3$, Proposition 3.5, is then Wästlund's own Proposition 15 in the generality in which he states it.

Sharpness (Proposition 7.3). $\widehat{P}_{4,4,4}$ has a real zero strictly between $43/10$ and $431/100$, while $mn/4 = 4$ there. So the disk in the theorem cannot be enlarged by more than about 7.5% at $k = 4$, and the certificates are not vacuously satisfiable.

The one thing we do not prove is the identification of the explicit polynomial $\widehat{P}_{4,m,n}$ of Definition 2.3 with the numerator of $F_{4,m,n}$. That is Wästlund's own structure theorem, quoted here from [1]; granting it, Theorem 7.1 is his conjecture at $k = 4$ (Remark 7.2). Our definition of $\widehat{P}_4$ was checked against his recursion in exact rational arithmetic on every pair we use and on the whole box $[4, 25]^2$, but the identification itself is a statement about exponential random variables and is not part of the verified development.

1.4. **Method.** The engine is a dominance estimate on a closed disk (Lemma 3.1): if $P = A + B$ with $A(t) = p_0 \prod_i (1 - t/\rho_i)$ and all $|\rho_i| > R$, and if $\sum_j |b_j| R^j < |p_0| \prod_i (1 - R/|\rho_i|)$, then P has no zero with $|t| \le R$. It is two triangle inequalities, it is not Rouché, and it gives the closed disk directly. It is exactly Wästlund's $k = 3$ estimate with the comparison polynomial left free, and that freedom is the whole point: on the tail we take $A = D_3$ and recover an effective version of his argument, while on the 235 residual pairs, where his argument provably fails, we take for A a rational polynomial whose roots approximate the actual — non-real — roots of $\widehat{P}_4$ at that pair. Since the resulting inequality is checked exactly, an inaccurate approximation can only make a certificate fail; it can never make one wrongly succeed.

1.5. **What is not claimed.** Nothing here is proved for $k \ge 5$. Section 9 measures how the comparison degrades and prices the same scheme at $k = 5$ out of reach; that section is a report of computations, not a theorem. Nothing here bears on the cumulant-positivity conjecture of [1], which is strictly stronger than the zero-free disk and is an infinite family of inequalities with no finite core. And the theorem below is about the zeros of the numerator; the poles of $F_{4,m,n}$, all real and $\ge mn/4$ by (1), are not at issue. For *real* parameters we prove the zero-free disk only on the staircase Ω: the 235 residual pairs

are closed one at a time by certificates attached to individual pairs, and that argument says nothing at a nearby non-integer point. So the region between Ω and the quadrant $m, n \geq 4$ is settled at every integer point (Theorem 7.1) and open at the others.

1.6. **Changes from version 1.** This is a revised version. No statement of version 1 is withdrawn or corrected, and the numbering of version 1 is kept; what has changed is that three things which version 1 left outside the verified development — the first two of them labelled there as reported computations — are now theorems, and that the third turned out to be a strengthening rather than a restatement.

The seven thresholds. Version 1 proved only the positivity half of Theorem 5.1, and recorded the sharpness of its thresholds in Remark 5.2 as an exact computation performed outside the development. It is now Theorem 5.3, together with the other side of the threshold; Remark 5.2 is reprinted as it stood and its qualification is withdrawn in Remark 5.4.

The residual pairs. Version 1's Theorem 6.1 asserted the failure of the comparison only at $(4, 4)$, and Remark 6.2 reported the other 234 failures as a computation. Theorem 6.3 now proves the sharper statement that on the integer quadrant $W > 0$ holds exactly on the tail; Remark 6.2 likewise stands as printed, with Remark 6.4 recording what has been added.

Real parameters. Version 1 stated §2–§5 over $\mathbb{Q}$ and remarked that the same proofs work over $\mathbb{R}$. Section 8 is new and does more than carry them over: the tail certificates are two-variable, so the seven strips are seven closed real quadrants, and the region they cover is the staircase Ω of Definition 8.1. Proposition 3.5 and Theorems 8.2 and 8.4 are the real statements; Remark 8.6 identifies them with the rational ones at rational parameters. Lemma 3.4 in §3 is the real-coefficient form of the dominance estimate that they use.

Everything else is unchanged, including Theorem 7.1, the certificate table of §6, the $k \geq 5$ boundary of §9 and the errata of §10.

2. The numerator, Janson's comparison, and the discrepancy

We fix $k = 4$ except where stated. Until §8 we write m, n for *rational* parameters, restricting to integers only where the statement demands it; §8 carries the same estimates over $\mathbb{R}$, where they are what the coefficient certificates were proving all along, and Remark 8.6 identifies the two at rational parameters.

Definition 2.1. For $k \geq 2$ put $D_{k-1}(t) = \prod_{i+j<k-1}\bigl(1 - \frac{(k-1-i-j)t}{(m-i)(n-j)}\bigr)$, the product being over $i, j \geq 0$. At $k = 4$ this is the six-factor product

$$D_3(t) = \Bigl(1 - \tfrac{3t}{mn}\Bigr)\Bigl(1 - \tfrac{2t}{(m-1)n}\Bigr)\Bigl(1 - \tfrac{2t}{m(n-1)}\Bigr)\Bigl(1 - \tfrac{t}{(m-2)n}\Bigr)\Bigl(1 - \tfrac{t}{(m-1)(n-1)}\Bigr)\Bigl(1 - \tfrac{t}{m(n-2)}\Bigr).$$

It is the numerator, over the denominator (1), of the function of Janson that [1] compares F with.

Lemma 2.2. *Let $k \geq 2$ and $m, n \geq k$. Every root of D_{k-1} is real, positive, and strictly larger than $R = mn/k$. Every root of the denominator (1) is real and at least mn/k, the factor $(i, j) = (0, 0)$ contributing mn/k itself.*

Proof. The roots of D_{k-1} are $t = (m-i)(n-j)/(k-1-i-j)$ with $i+j \leq k-2$, all positive. Writing $c = k-1-i-j \geq 1$ and using $(1 - i/m)(1 - j/n) \geq (1 - i/k)(1 - j/k)$ for $m, n \geq k$,

$$k(m-i)(n-j) - c\,mn = mn\Bigl[k\bigl(1 - \tfrac{i}{m}\bigr)\bigl(1 - \tfrac{j}{n}\bigr) - (k-1-i-j)\Bigr] \;\geq\; mn\Bigl[\tfrac{ij}{k} + 1\Bigr] > 0,$$

so $(m-i)(n-j)/c > mn/k$. The denominator (1) is the same product with $k-i-j$ in place of $k-1-i-j$ and $i+j < k$, and the same computation gives $k(m-i)(n-j) - (k-i-j)mn \geq mn\,\frac{ij}{k} \geq 0$, so each of its roots is at least mn/k, with equality for $(i, j) = (0, 0)$. □

In particular $\min_{|t|\leq R} |D_{k-1}(t)| = D_{k-1}(R) > 0$, and at $k = 4$ one computes from Definition 2.1 that

$$D_3\Bigl(\frac{mn}{4}\Bigr) = \frac{(3m-8)(3n-8)(3mn-4m-4n+4)}{1024\,(m-1)^2(n-1)^2} \;\xrightarrow[m,n\to\infty]{}\; \frac{27}{1024}. \tag{2}$$

So the entire question is the size of the discrepancy

$$E := \widehat{P}_{k,m,n} - D_{k-1}, \qquad E(t) = \sum_j e_j t^j.$$

At $k = 3$ that discrepancy is the single monomial $E = -t^2/\big(m^2n^2(m-1)(n-1)\big)$, which is what [1] exploits. At $k = 4$ it has four terms, and computing them is §4.

Definition 2.3. $\widehat{P}_{4,m,n} := D_3 + E$, where $E = e_2t^2 + e_3t^3 + e_4t^4 + e_5t^5$ with $e_j = g_j/\Delta$, the four integer polynomials $g_j = \Delta e_j$ being those printed in (3)–(6) below and

$$\Delta = \Delta(m,n) = m^4(m-1)^3(m-2)^2(m-3) \cdot n^4(n-1)^3(n-2)^2(n-3).$$

Everything below is a statement about this explicit polynomial family. Note that Δ vanishes on $m = 3$ and on $n = 3$: the coefficients e_j have poles exactly where a 4-matching ceases to exist, so the hypothesis $m, n \geq 4$ is not decoration.

3. Dominance on a closed disk

Lemma 3.1 (Dominance)**.** *Let $R \geq 0$, let $A(t) = p_0 \prod_{i=1}^{r}(1 - t/\rho_i)$ with $p_0 \neq 0$ and $|\rho_i| > R$ for every i, and let $B(t) = \sum_j b_j t^j$. If*

$$\sum_j |b_j|\, R^j \;<\; |p_0| \prod_{i=1}^{r} \Big(1 - \frac{R}{|\rho_i|}\Big),$$

then $(A + B)(t) \neq 0$ for every $t \in \mathbb{C}$ with $|t| \leq R$.

Proof. For $|t| \leq R$, the reverse triangle inequality $|1 - z| \geq 1 - |z|$ applied to each factor gives $|A(t)| \geq |p_0| \prod_i (1 - |t|/|\rho_i|) \geq |p_0| \prod_i (1 - R/|\rho_i|)$, and the triangle inequality gives $|B(t)| \leq \sum_j |b_j| R^j$. The hypothesis makes the first strictly larger than the second, so $A(t) + B(t) \neq 0$. □

The lemma is elementary and we claim no novelty for it; it is Wästlund's own $k \leq 3$ estimate with the comparison polynomial A left free. Two features matter for what follows. It concerns the *closed* disk, because it never passes to a boundary circle: this is why our theorem is slightly stronger than the conjecture. And it is not Rouché's theorem and uses no argument principle — where [1] closes its $k = 3$ proof by observing that P and P_J wind zero times around the origin, Lemma 3.1 concludes directly from $|A| > |B|$ — so it stays available when the comparison polynomial has nothing to do with the shape of $\widehat{P}_4$, in particular when its roots are not real.

For the residual pairs we need A to have rational coefficients while its roots come in non-real conjugate pairs, so we present A as a product of factors of two kinds: a *linear* factor $1 - ct$ with $c \in \mathbb{Q}$, contributing at least $1 - |c|R$ on $|t| \leq R$; and a *quadratic* factor $1 - bt + at^2$ with $a, b \in \mathbb{Q}$, contributing at least $(1 - uR)^2$ for any rational u with $a \leq u^2$ and $uR < 1$, *provided* $b^2 \leq 4a$.

Remark 3.2. The discriminant condition is load-bearing, not bookkeeping. Without $b^2 \leq 4a$ the two roots of $1 - bt + at^2$ are real with unequal moduli and the bound $(1 - uR)^2$ is false: for $b = 5/2$, $a = u = 1$ and $R = 9/10$ the quantity $(1 - uR)^2 = 1/100$ is positive, yet $1 - \frac{5}{2}t + t^2$ vanishes at $t = 1/2$, inside the disk. With $b^2 \leq 4a$ one has $1 - bt + at^2 = (1 - \gamma t)(1 - \bar{\gamma} t)$ with $\gamma = \big(b + i\sqrt{4a - b^2}\big)/2$, $|\gamma|^2 = a \leq u^2$, and the bound follows from Lemma 3.1 factor by factor.

As a first consequence, and as a check that the machinery reproduces its source, take $A = D_2$ at $k = 3$.

Proposition 3.3 (Wästlund)**.** *For all rational $m, n \geq 3$ and every $t \in \mathbb{C}$ with $|t| \leq mn/3$ one has $\widehat{P}_{3,m,n}(t) \neq 0$.*

Proof. Here $E = -t^2/\big(m^2n^2(m-1)(n-1)\big)$, so with $R = mn/3$ the left side of Lemma 3.1 is $R^2/\big(m^2n^2(m-1)(n-1)\big) = 1/\big(9(m-1)(n-1)\big)$, while the right side is $D_2(R) = (2m-3)(2n-3)/\big(27(m-1)(n-1)\big)$ by the computation of Lemma 2.2. The inequality is therefore equivalent to $(2m-3)(2n-3) > 3$, which holds for all $m, n \geq 3$. □

This is the result of [1] (its Proposition 15) and not a new one; we record it because it is the exact special case that our tail argument generalises, and because it shows what breaks at $k = 4$: the single monomial becomes four, and their contributions at $t = -R$ add rather than cancel. [1] states it for all m and n; the proof just given is written over $\mathbb{Q}$ because that is the coefficient ring in which the residual pairs of §6 have to be decided, and the next two statements remove that restriction.

Over $\mathbb{R}$ the comparison polynomials we need have no conjugate pairs — both D_2 and D_3 split into real linear factors by Lemma 2.2 — so the following version, with linear factors only, suffices; its proof is again the two triangle inequalities.

Lemma 3.4 (Dominance over $\mathbb{R}$)**.** *Let $R \geq 0$, let $c_1, \dots, c_r$ be real with $|c_i|R < 1$ for every i, and let $B(t) = \sum_j b_j t^j$ have real coefficients. If*

$$\sum_j |b_j|\, R^j \;<\; \prod_{i=1}^{r} \bigl(1 - |c_i|R\bigr),$$

then $\prod_{i=1}^{r}(1 - c_i t) + B(t) \neq 0$ for every $t \in \mathbb{C}$ with $|t| \leq R$.

Proof. For $|t| \leq R$ the reverse triangle inequality gives $|1 - c_i t| \geq 1 - |c_i|R > 0$ factor by factor, so the product has modulus at least $\prod_i (1 - |c_i|R)$, while $|B(t)| \leq \sum_j |b_j| R^j$. □

Proposition 3.5 (Wästlund, real parameters)**.** *For all real $m, n \geq 3$ and every $t \in \mathbb{C}$ with $|t| \leq mn/3$ one has $\widehat{P}_{3,m,n}(t) \neq 0$.*

Proof. Word for word the proof of Proposition 3.3, with Lemma 3.4 in place of Lemma 3.1: the three reciprocal roots of D_2 are $2/mn$, $1/((m-1)n)$, $1/(m(n-1))$, whose per-factor bounds at $R = mn/3$ are $1/3$, $(2m-3)/(3(m-1))$, $(2n-3)/(3(n-1))$, all positive for $m, n \geq 3$ and with product $D_2(R)$; the discrepancy contributes $R^2/\bigl(m^2n^2(m-1)(n-1)\bigr) = 1/\bigl(9(m-1)(n-1)\bigr)$; and the inequality is again $(2m-3)(2n-3) > 3$, which holds for real $m, n \geq 3$ with room to spare, $9 > 3$ at $m = n = 3$. □

This is the statement of [1] in the generality in which [1] states it. Its Proposition 15 carries no hypothesis on m and n at all; its proof asks only that "in order for the optimization problem to make sense, both m and n have to be at least 3"; the sentence that closes it, quoted in §1, establishes the conjecture "for $k \leq 3$ and all m and n"; and [1] does evaluate the recursion at a non-integer parameter two pages earlier, remarking that "[i]n the recursion, m and n do not have to be integers" and printing $F_{3,3,10/3}$ in closed form. We record Proposition 3.5 for the same reason as Proposition 3.3: it is the control showing that the real-coefficient machinery reproduces the source's own result before we ask it for anything new.

Remark 3.6. The strict inequality $|c_i|R < 1$ in Lemma 3.4 cannot be relaxed to $|c_i|R \leq 1$. At $r = 1$, $c_1 = R = 1$ and $B = 0$ the hypothesis holds with equality, $0 = \prod_i (1 - |c_i|R)$, while $1 - t$ vanishes at $t = 1$, inside the closed disk.

4. The discrepancy at $k = 4$, in closed form

Theorem 4.1. *Let $E = \widehat{P}_{4,m,n} - D_3$. Then $e_0 = e_1 = e_6 = 0$, and for $j = 2, 3, 4, 5$ the products $g_j := \Delta\, e_j$ are integer polynomials in (m, n), symmetric under $m \leftrightarrow n$, of bidegrees $(7,7)$, $(6,6)$, $(5,5)$, $(4,4)$ and with 36, 36, 25, 16 terms respectively. Writing $g_j = \sum_{a,b} c^{(j)}_{a,b} m^a n^b$, the coefficient matrices are*

$$\bigl(c^{(2)}_{a,b}\bigr)_{2 \leq a,b \leq 7} = \begin{pmatrix} -288 & 1032 & -1500 & 1080 & -372 & 48 \\ 1032 & -3680 & 5306 & -3786 & 1294 & -166 \\ -1500 & 5306 & -7548 & 5303 & -1788 & 227 \\ 1080 & -3786 & 5303 & -3658 & 1213 & -152 \\ -372 & 1294 & -1788 & 1213 & -396 & 49 \\ 48 & -166 & 227 & -152 & 49 & -6 \end{pmatrix}, \tag{3}$$

$$\left(c^{(3)}_{a,b}\right)_{1\le a,b\le 6} = \begin{pmatrix} -288 & 672 & -120 & -600 & 408 & -72 \\ 672 & -884 & -2090 & 4130 & -2182 & 354 \\ -120 & -2090 & 7997 & -9407 & 4267 & -647 \\ -600 & 4130 & -9407 & 9092 & -3757 & 542 \\ 408 & -2182 & 4267 & -3757 & 1469 & -205 \\ -72 & 354 & -647 & 542 & -205 & 28 \end{pmatrix}, \tag{4}$$

$$\left(c^{(4)}_{a,b}\right)_{1\le a,b\le 5} = \begin{pmatrix} 288 & 480 & -1560 & 960 & -168 \\ 480 & -4312 & 6772 & -3512 & 572 \\ -1560 & 6772 & -8732 & 4172 & -652 \\ 960 & -3512 & 4172 & -1912 & 292 \\ -168 & 572 & -652 & 292 & -44 \end{pmatrix}, \tag{5}$$

all other coefficients vanishing, and

$$g_5 = 22\, m(m-1)(m-2)(m-3)\cdot n(n-1)(n-2)(n-3), \qquad \textit{i.e.} \qquad e_5 = \frac{22}{\prod_{i+j<3}(m-i)(n-j)}. \tag{6}$$

Moreover $e_2 < 0 < e_3$ *and* $e_4 < 0 < e_5$ *for all* $m, n \ge 4$.

Proof sketch. The four tables are *derived*, not fitted. The recursion of [1] for $F_{k,m,n}$ was run with m and n both symbolic, in exact arithmetic over $\mathbb{Q}(m,n)$, and $\widehat{P}_4 = F_{4,m,n}\cdot Q$ read off; subtracting D_3 gives E and hence the g_j after multiplication by Δ. The computation is short because every denominator the recursion can produce is an integer times a product of the eight linear forms $m-i$, $n-j$ with $0 \le i, j \le 3$, so $\mathbb{Q}(m,n)$ can be represented as a quotient of elements of $\mathbb{Z}[m,n]$ reduced by trial division by those eight forms and by integer content, with no multivariate gcd anywhere; likewise the rational functions of t are carried with their denominators factored, so no gcd in t arises either. The bidegree and term counts, the vanishing of e_0, e_1, e_6, and the factorisation (6) are then read off the result; (6) is a polynomial identity between the printed table and the stated product, and is verified as such.

The sign statement is proved by a shift: each of $-g_2$, g_3, $-g_4$, g_5 has all coefficients nonnegative after the substitution $(m,n) \mapsto (4+u, 4+v)$, with positive constant terms 3760128, 1792512, 271872, 12672 respectively, and a polynomial with nonnegative coefficients and positive constant term is positive on $u, v \ge 0$. These are finite exact checks on integer coefficient tables, not searches.

As an independent check, the polynomial of Definition 2.3 was evaluated against the recursion of [1] in exact rational arithmetic at every one of the 235 pairs of §6 and at every pair in $[4, 25]^2$; the two agree at every one of them. □

Remark 4.2. The vanishing of e_0 and e_1 says that $F_{4,m,n}$ and Janson's function agree in constant term and in mean — which is what makes the comparison natural in the first place — and $e_6 = 0$ says they share leading coefficients, so $\widehat{P}_4$ has degree exactly 6. The identity (6) is the exact analogue at $k = 4$ of the coefficient $e_2 = -1/\prod_{i+j<2}(m-i)(n-j)$ at $k = 3$; the other three coefficients are genuinely bivariate and do not factor.

5. The tail: where Janson's comparison suffices

Multiply the inequality of Lemma 3.1 for $A = D_3$, $B = E$, $R = mn/4$, namely $\sum_{j=2}^{5} |e_j| R^j < D_3(R)$, by $1024\,\Delta > 0$. Using (2) and the sign pattern of Theorem 4.1, it becomes $W(m,n) > 0$ for the integer polynomial

$$\begin{aligned} W := {} & m^4(m-1)(m-2)^2(m-3)\, n^4(n-1)(n-2)^2(n-3)\,(3m-8)(3n-8)(3mn-4m-4n+4) \\ & + 64\, g_2\,(mn)^2 - 16\, g_3\,(mn)^3 + 4\, g_4\,(mn)^4 - g_5\,(mn)^5, \end{aligned} \tag{7}$$

of bidegree $(10, 10)$.

Theorem 5.1. *Let* m, n *be rational. Then* $W(m,n) > 0$ *in each of the following cases, and consequently* $\widehat{P}_{4,m,n}(t) \ne 0$ *for every* $t \in \mathbb{C}$ *with* $|t| \le mn/4$*:*

(1) $m \geq 11$ *and* $n \geq 11$*;*
(2) $m \in \{4, \dots, 10\}$ *and* $n \geq n_0(m)$*, where* $\big(n_0(4), \dots, n_0(10)\big) = (62, 32, 22, 17, 14, 12, 11)$*;*
(3) $n \in \{4, \dots, 10\}$ *and* $m \geq n_0(n)$*.*

Proof sketch. That $W > 0$ is equivalent to the hypothesis of Lemma 3.1 for $A = D_3$, $B = E$ is the identity

$$\Big(D_3(R) - \sum_{j=2}^{5} |e_j| R^j\Big) \cdot 1024\,\Delta = W(m, n), \qquad R = \frac{mn}{4},$$

which is (2) together with $|e_2| = -e_2$, $|e_3| = e_3$, $|e_4| = -e_4$, $|e_5| = e_5$ from Theorem 4.1 and $\Delta > 0$ for $m, n \geq 4$. Lemma 3.1 applies because the six roots of D_3 exceed R by Lemma 2.2.

The positivity itself is proved by nonnegative-coefficient shifts, which is a finite exact computation and not a search. For (1): the shifted polynomial $W(11 + u, 11 + v)$ has all 121 of its coefficients ≥ 0 and constant term $W(11, 11) = 307327216678851283200 > 0$, so $W > 0$ on the quadrant $m, n \geq 11$. For (2): for each of the seven values $m \in \{4, \dots, 10\}$ the univariate polynomial $W(m, n_0(m) + v)$, of degree 10 in v, has all coefficients ≥ 0 and positive constant term. Case (3) is proved by the same seven checks performed in the other variable, so that no symmetry property of W is needed. Fifteen coefficient tables are inspected in total. □

Remark 5.2. The seven thresholds are best possible for this method: $W(m, n_0(m) - 1) < 0$ for each $m \in \{4, \dots, 10\}$, the values being about $-2.856 \cdot 10^{21}$, $-9.823 \cdot 10^{19}$, $-2.184 \cdot 10^{19}$, $-1.708 \cdot 10^{19}$, $-1.695 \cdot 10^{19}$, $-1.725 \cdot 10^{19}$, $-1.503 \cdot 10^{19}$. These seven evaluations are exact rational computations *reported* in this remark, not part of the machine-checked development of §11: what is checked there is the positivity half of Theorem 5.1, in the form stated, together with the single negative value $W(4, 4) < 0$ of Theorem 6.1. The theorem is a theorem; the sharpness of its thresholds is a labelled computation.

That is no longer where the matter rests.

Theorem 5.3 (Sharpness of the thresholds)**.** *For every* $m \in \{4, \dots, 10\}$*,*

$$W\big(m,\, n_0(m) - 1\big) < 0 < W\big(m,\, n_0(m)\big),$$

so the sign of $W(m, \cdot)$ *changes exactly at* $n = n_0(m)$*: each of the seven thresholds of Theorem 5.1 is the least integer from which its strip can start. The seven negative values are*

m	$n_0(m) - 1$	$W\big(m,\, n_0(m) - 1\big)$
4	61	-2855815750231550853120
5	31	-98233633771313400000
6	21	-21837476603522764800
7	16	-17076160393799270400
8	13	-16952052402251366400
9	11	-17245491208654705920
10	10	-150255544320000000000

Proof sketch. Fourteen exact evaluations of the integer polynomial (7), performed on its coefficient table by the same decision procedure on integer lists that proves Theorem 5.1. No search and no approximation is involved: each value is a single integer, computed once. □

What this says about the method, and not merely about the table, is that below the threshold the dominance inequality $\sum_j |e_j| R^j < D_3(R)$ is *false*, not merely unproved. A finer positivity certificate for the same W therefore cannot enlarge the tail by a single pair; only a different comparison polynomial can. This is the quantitative form of Wästlund's remark that his argument does not generalise, read at the level of the individual strip rather than at the level of k.

Remark 5.4. Remark 5.2 is reprinted above as version 1 of this paper published it; the qualification it makes — that the seven evaluations are reported rather than verified — is withdrawn by Theorem 5.3. Two further finite checks say that the thresholds are a boundary of the mathematics and not of the search: for each $m \in \{4, \dots, 10\}$ the strip certificate itself fails, coefficientwise, when its threshold is

lowered by one, and the pair $(4, 61)$ is outside the tail region T while $(4, 62)$ is inside it. The quadrant corner of Theorem 5.1(1) is sharp in the same sense: the two-variable certificate at $(10, 10)$ fails, and indeed $W(10, 10) = -15025554432000000000 < 0$, which is the $m = 10$ entry of the table above since $n_0(10) - 1 = 10$.

6. The residual pairs

Write T for the set of integer pairs covered by Theorem 5.1 and $\mathcal{C} = \{(m, n) : m, n \geq 4\} \setminus T$.

Theorem 6.1. *$\mathcal{C}$ consists of exactly 235 ordered pairs, all with $4 \leq m, n \leq 61$, namely the 121 pairs with $m \leq n$ listed by*

m	4	5	6	7	8	9	10
n	$4 \dots 61$	$5 \dots 31$	$6 \dots 21$	$7 \dots 16$	$8 \dots 13$	$9 \dots 11$	10

together with their transposes. For every $(m, n) \in \mathcal{C}$ there is an explicit comparison polynomial $A_{m,n}$ with rational coefficients, a product of linear and conjugate-pair quadratic factors as in §3, satisfying the hypothesis of Lemma 3.1 for $P = \widehat{P}_{4,m,n}$ at $R = mn/4$; consequently $\widehat{P}_{4,m,n}(t) \neq 0$ for $|t| \leq mn/4$. Moreover

$$W(4, 4) = -260617273344 < 0,$$

so the Janson comparison of §5 does not reach $(4, 4)$ and the certificates are doing real work.

Proof sketch. That $\mathcal{C}$ is as described, and in particular finite and inside $[4, 61]^2$, is immediate from the definition of T: outside the box the first clause of Theorem 5.1 applies, and inside it the seven strip thresholds cut the seven rows and the seven columns. The count $235 = 2 \cdot 121 - 7$ follows from the table, the seven being the pairs on the diagonal.

The certificates were *constructed* numerically and *verified* exactly. For each of the 235 pairs the six roots of the degree-six polynomial $\widehat{P}_{4,m,n}$ were computed in floating point, rounded to rationals with denominator 10^6, and assembled into $A_{m,n}$ as linear factors for the real roots and quadratic factors $1 - bt + at^2$ for the conjugate pairs, together with a rational $u \geq \sqrt{a}$; the largest rational entry appearing is of size about 10^{12}, and the 235 certificates use 1136 factors in total. The hypothesis of Lemma 3.1 — the side conditions of §3 together with $\sum_j |b_j| R^j < |p_0| \prod_i (1 - R/|\rho_i|)$ for $B = \widehat{P}_{4,m,n} - A_{m,n}$ — is then a single decidable inequality between explicit rationals for each pair, and it is that inequality, not the numerics, that is checked. A poor root approximation therefore makes a pair fail; it cannot make one wrongly succeed. The verification is exhaustive over the box: all 3844 pairs of $[0, 61]^2$ are enumerated, and for each the check is performed exactly when the pair lies in $\mathcal{C}$. All 235 succeed at denominator 10^6. The value $W(4, 4)$ is a single exact evaluation of (7). □

Remark 6.2. The comparison fails on the whole of $\mathcal{C}$, not only at $(4, 4)$: exact evaluation of (7) gives $W < 0$ at every one of the 235 pairs, the ratio $\rho := \sum_j |e_j| R^j / D_3(R)$ running from $1.0166 \dots$ at $(4, 61)$ up to 87.3 at $(4, 4)$. Those 235 evaluations are reported here, outside the machine-checked development; the one verified in it is $W(4, 4) < 0$. At $(4, 4)$ the failure is not marginal, so no sharpening of the estimates inside Lemma 3.1 could rescue the comparison there. This is the quantitative content of Wästlund's remark that his argument does not generalise — and also the reason it is only an obstruction at $k = 4$ rather than a defeat: see §9.

The whole of that remark, apart from the ratios ρ, is now a theorem, and in a sharper form: the region where the comparison works and the region where W is positive are the same region.

Theorem 6.3 (The comparison's exact range)**.** *Let $m, n \geq 4$ be integers. Then*

$$W(m, n) > 0 \iff (m, n) \in T,$$

and $W(m, n) < 0$ for every $(m, n) \in \mathcal{C}$; in particular W has no zero on the integer quadrant $m, n \geq 4$. Moreover $\mathcal{C}$ has exactly 235 elements, and for $m = 4, \dots, 10$ the number of n with $(m, n) \in \mathcal{C}$ is

$$58,\ 28,\ 18,\ 13,\ 10,\ 8,\ 7$$

respectively — these count every such n, not only those with $n \geq m$.

Proof sketch. The implication $\Leftarrow$ is Theorem 5.1. For the converse, a pair of $\mathcal{C}$ lies in $[4,61]^2$ by Theorem 6.1, so it suffices to check the box: all 3844 pairs of $[0,61]^2$ are enumerated, membership in $\mathcal{C}$ is decided at each, and at each of the 235 members the integer $W(m,n)$ is evaluated exactly and found negative. The count 235 and the seven row lengths are read off the same enumeration. This is an exhaustive finite computation on integer coefficient tables, not a search: every pair of the box is visited, and no pair is visited twice. The seven row lengths account for 142 of the 235 pairs; the other 93 have $m \geq 11$, and then $n \leq 10$, since $m, n \geq 11$ is the first clause of the definition of T. □

Remark 6.4. Remark 6.2 is reprinted above as version 1 published it. What Theorem 6.3 adds is the exactness: the failure region of the Janson comparison at $k = 4$ and the certificate table of Theorem 6.1 coincide cell for cell. So the 235 certificates are not padding around a bad corner — each of them closes a pair at which the comparison provably does not close, and dropping any one of them leaves a genuine gap. The ratios ρ of Remark 6.2, which measure *by how much* the comparison fails, remain computations reported outside the verified development; what is verified is the sign. Both halves of the equivalence are inhabited: $(4,4) \in \mathcal{C}$ and $(11,11) \notin \mathcal{C}$.

7. The theorem at $k = 4$, and its sharpness

Theorem 7.1. *Let $m, n \geq 4$ be integers. Then*

$$\widehat{P}_{4,m,n}(t) \neq 0 \qquad \text{for every } t \in \mathbb{C} \text{ with } |t| \leq \frac{mn}{4}.$$

Proof. Let (m,n) be an integer pair with $m, n \geq 4$. If it lies in T, Theorem 5.1 applies. Otherwise it lies in $\mathcal{C}$, hence in $[4,61]^2$ by Theorem 6.1, and that theorem applies. The two cases exhaust the quadrant. □

Remark 7.2 (Consequence for the conjecture). By the structure theorem of [1] — his recursion (Theorem 8) together with Proposition 9, that $F_{4,m,n} = P/Q$ with Q as in (1) and P of degree $\binom{4}{2} = 6$ normalised by $P(0) = 1$ — the polynomial of Definition 2.3 is that numerator P, and the zeros of $F_{4,m,n}$ are among its zeros. Granting that identification, which we quote from [1] and do not reprove here, Theorem 7.1 says that $F_{4,m,n}$ has no complex zero with $|t| \leq mn/4$ for any integers $m, n \geq 4$: the conjecture of [1] at $k = 4$, its first open case, and on the closed disk rather than the open one. This step is a statement about exponentially distributed edge costs; it is the one link in the chain that is not part of the verified development of §11, and the reader who prefers to keep it separate may read Theorem 7.1 as a theorem about the explicit polynomial family of Definition 2.3 alone. As evidence that Definition 2.3 is the right polynomial, it agrees with the recursion of [1] in exact rational arithmetic at every pair of $\mathcal{C}$ and at every pair of $[4,25]^2$.

Proposition 7.3 (Sharpness)**.** *$\widehat{P}_{4,4,4}$ has a real zero t_0 with $43/10 < t_0 < 431/100$. Since $mn/4 = 4$ at $(4,4)$, the closed disk of radius $c \cdot mn/4$ contains a zero of $\widehat{P}_{4,4,4}$ for every $c \geq 1.0776$; so the radius in Theorem 7.1 cannot be enlarged by more than about 7.5%, and $(4,4)$ is where the statement is tightest.*

Proof. The two exact rational evaluations $\widehat{P}_{4,4,4}(43/10) > 0$ and $\widehat{P}_{4,4,4}(431/100) < 0$ together with the intermediate value theorem give t_0. Then $t_0 < 431/100 < 1.0776 \cdot 4$. □

Remark 7.4. The certificates of Theorem 6.1 are correspondingly not vacuous: widening the radius at $(4,4)$ by 10%, from $mn/4$ to $\frac{11}{10} \cdot mn/4$, makes that pair's certificate fail the check. A formal control also records the consequence of Proposition 7.3 in the concrete form "$\widehat{P}_{4,4,4}$ has a zero in the closed disk of radius $1.08 \cdot mn/4$".

Remark 7.5. Numerically, the ratio $\min\{|t| : \widehat{P}_{4,m,n}(t) = 0\}/(mn/4)$ appears to increase in both m and n at $k = 4$, from $1.0753\ldots$ at $(4,4)$ towards $4/3$; that monotonicity is a computation and is not claimed here as a theorem. The limit is the value the source's own asymptotics predict: in the rescaled variable $s = t/(mn/k)$ the proof of Proposition 16 of [1] concludes that for fixed k and m, n both large the zeros of $F_{k,m,n}$ cluster around the points $s = k/(i' - j')$, "with one zero close to $k/(k-1)$, two zeros close to $k/(k-2)$" and so on — and $k/(k-1) = 4/3$ at $k = 4$. [1] gives no zero locations at $k = 4$ itself; its two pictures of zeros are of $F_{20,20,20}$ and $F_{5,12,20}$.

8. Real parameters: the staircase

Nothing in Definition 2.3 requires m and n to be integers. The four coefficients $e_j = g_j/\Delta$ are rational functions of (m, n) with poles only on the eight lines $m = i$, $n = j$, $0 \le i, j \le 3$, so $\widehat{P}_{4,m,n}$ is a well-defined polynomial in t at every real (m, n) with $m, n \ge 4$. Neither does [1] require it, and it says so: asking whether factors of Q can cancel against factors of P, it observes that "[i]n the recursion, m and n do not have to be integers", takes $k = m = 3$ and $n = 10/3$, and exhibits a cancellation there. Two questions therefore separate, and only the first of them was answered in §5: at which real (m, n) does the Janson comparison close, and does the closed-disk conclusion follow there.

The certificates of Theorem 5.1 answer more than they were asked. Each of the seven strips was proved by fixing $m \in \{4, \dots, 10\}$, specialising W in its first variable, and shifting only the second; but the *two-variable* shift at the same corner already has nonnegative coefficients. A strip is therefore a quadrant.

Definition 8.1. The *staircase* is the closed subset of $\mathbb{R}^2$

$$\Omega := \bigcup_{a=4}^{10} \Big([a, \infty) \times [n_0(a), \infty) \;\cup\; [n_0(a), \infty) \times [a, \infty) \Big),$$

the union of the fourteen closed quadrants with corners $(4, 62)$, $(5, 32)$, $(6, 22)$, $(7, 17)$, $(8, 14)$, $(9, 12)$, $(10, 11)$ and their transposes. Since $[10, \infty) \times [11, \infty) \subseteq \Omega$, the quadrant $m, n \ge 11$ lies in Ω.

Theorem 8.2. *$W(m, n) > 0$ at every real point of Ω. Every integer pair of T lies in Ω, and $(4, 4) \notin \Omega$.*

Proof sketch. For each of the fourteen corners (a, b) the shifted polynomial $W(a+u, b+v)$ has all 121 of its coefficients ≥ 0 and constant term $W(a, b) > 0$; such a polynomial is positive at every real $u, v \ge 0$, which is the quadrant with corner (a, b). Fourteen finite exact inspections of integer coefficient tables, one per corner, and no search. For the containment: the first clause of T lies in the quadrant with corner $(10, 11)$, and the two strip clauses are the quadrants with corners $(a, n_0(a))$ and $(n_0(a), a)$. Finally no corner is $\le (4, 4)$ in both coordinates, so $(4, 4) \notin \Omega$ — as it must not be, $W(4, 4)$ being negative. □

The containment is an equality on the integers: an integer point of Ω with $m, n \ge 4$ has $W > 0$ by Theorem 8.2 and therefore lies in T by Theorem 6.3. So Ω is a region of the real plane whose integer points are exactly the tail.

Remark 8.3 (the corners are sharp). No corner of Ω can be moved down or to the left inside the domain $m, n \ge 4$ on which $\widehat{P}_4$ is defined. In the second coordinate this is Theorem 5.3: $W(a, n_0(a) - 1) < 0$, so $[a, \infty) \times [n_0(a) - 1, \infty)$ is not a region of positivity, for every $a \in \{4, \dots, 10\}$. In the first coordinate, and for $a \in \{5, \dots, 10\}$, it is Theorem 6.3: n_0 is strictly decreasing on $\{4, \dots, 10\}$, so $n_0(a) < n_0(a-1)$ and the pair $(a - 1, n_0(a))$ lies in $\mathcal{C}$, whence $W(a - 1, n_0(a)) < 0$. At $a = 4$ the first coordinate cannot be lowered at all inside the domain: $m \ge 4$ is where $\widehat{P}_4$ is defined, Δ vanishing identically on $m = 3$. The transposed corners are the same statements with the variables exchanged. What is sharp is this family of quadrants, and not the positivity region of W over $\mathbb{R}$: nothing proved here says that $W \le 0$ at a non-integer point outside Ω, and Ω is where our certificates apply, not where W happens to be positive.

Theorem 8.4. *Let $m, n \ge 4$ be real with $(m, n) \in \Omega$. Then*

$$\widehat{P}_{4,m,n}(t) \ne 0 \qquad \text{for every } t \in \mathbb{C} \text{ with } |t| \le \frac{mn}{4}.$$

Proof sketch. Lemma 3.4 applied to the six linear factors of D_3, whose reciprocal roots are $3/mn$, $2/((m-1)n)$, $2/(m(n-1))$, $1/((m-2)n)$, $1/((m-1)(n-1))$, $1/(m(n-2))$. At $R = mn/4$ the six per-factor bounds $1 - |c_i|R$ are

$$\tfrac{1}{4}, \quad \tfrac{m-2}{2(m-1)}, \quad \tfrac{n-2}{2(n-1)}, \quad \tfrac{3m-8}{4(m-2)}, \quad \tfrac{3mn-4m-4n+4}{4(m-1)(n-1)}, \quad \tfrac{3n-8}{4(n-2)},$$

each positive for real $m, n \ge 4$, and their product is $D_3(mn/4)$ as computed in (2). The four coefficients of E keep the signs $e_2 < 0 < e_3$, $e_4 < 0 < e_5$ over $\mathbb{R}$, by the same nonnegative-coefficient shifts as in

Theorem 4.1 — a shift table is a statement about coefficients, so it certifies positivity over $\mathbb{R}$ exactly as it does over $\mathbb{Q}$. So $|e_2| = -e_2$, $|e_3| = e_3$, $|e_4| = -e_4$, $|e_5| = e_5$, the identity $\big(D_3(R) - \sum_j |e_j| R^j\big) \cdot 1024\,\Delta = W(m,n)$ of §5 holds verbatim over $\mathbb{R}$, and $\Delta > 0$ for $m, n \geq 4$. Theorem 8.2 then supplies the hypothesis of Lemma 3.4. □

Corollary 8.5. *$\widehat{P}_{4,m,n}$ has no zero with $|t| \leq mn/4$ in each of the following cases: for all real $m, n \geq 11$; and, for each $a \in \{4, \ldots, 10\}$, for all real $m \geq a$ and $n \geq n_0(a)$, as well as for all real $m \geq n_0(a)$ and $n \geq a$. In particular it holds for all real $m \geq 4$, $n \geq 62$.*

Remark 8.6. These statements are not obtained from their rational counterparts by a limit, and could not be: a dense set of *strict* inequalities yields only a non-strict one in the limit, and non-strict is exactly what the dominance estimate cannot use. They are re-proved, over $\mathbb{R}$, from Lemma 3.4; the certificates themselves are integer coefficient tables and are not recomputed. At rational parameters the real polynomials are the rational ones: the coefficients of $\widehat{P}_{3,m,n}$ and of $\widehat{P}_{4,m,n}$ are given by the same formulas in m and n over either field, so at rational m, n the two agree coefficient by coefficient, and Proposition 3.5 restricts to Proposition 3.3, and Theorem 8.4 restricts to the tail half of Theorem 7.1. The hypothesis $m, n \geq 4$ is not decoration over $\mathbb{R}$ either: $\Delta(3, n) = 0$ for every n. Two limits of scope should be stated plainly. First, the residual pairs are closed one at a time, by certificates built from an individual pair's own root pattern, so they contribute nothing at real parameters; over $\mathbb{R}$ we prove the zero-free disk on Ω and nowhere else, even though every integer point outside Ω with $m, n \geq 4$ is settled by Theorem 7.1. Second, at non-integer m, n the identification of $\widehat{P}_4$ with the numerator of a moment generating function is even more clearly a statement about the analytic continuation of the recursion of [1] than it is at integer parameters, the source's own $F_{3,3,10/3}$ being such an evaluation; everything proved in this section is a statement about the explicit polynomial family of Definition 2.3.

9. The boundary of the method: $k \geq 5$

This section reports computations; it contains no theorem, and none of it is part of the verified development.

The quantity that decides whether the Janson comparison closes a pair at level k is the ratio $\rho(k, m, n) := \sum_j |e_j| R^j / D_{k-1}(R)$ with $R = mn/k$ and $E = \widehat{P}_k - D_{k-1}$; Lemma 3.1 applies exactly when $\rho < 1$. Computed exactly from the recursion of [1] — in rational arithmetic throughout, floating-point root finding being unreliable on this family above degree about 15 — it behaves as follows.

k	$\rho(k,k,k)$	$\rho(k,k,8k)$	$\rho(k,k,1024k)$
4	87.3	—	—
5	$1.013 \cdot 10^5$	$5.015 \cdot 10^2$	3.01
6	$7.143 \cdot 10^8$	$3.937 \cdot 10^5$	$2.018 \cdot 10^3$
7	$3.531 \cdot 10^{13}$	$1.098 \cdot 10^9$	$4.603 \cdot 10^6$

Along a strip $m = k$ the ratio decays like $1/n$, so each level has a finite residual set; at $k = 4$ the strip threshold read off this way is $n = 62$, in agreement with Theorem 5.1 ($\rho(4,4,62) = 0.99899 < 1$ and $\rho(4,4,61) > 1$), and $\rho(4,11,11) = 0.804$. At $k = 5$ the thresholds $n^*(m)$ in the strips are about $1.54 \cdot 10^4$ at $m = 5$ and about $5.75 \cdot 10^3/m$ for $m \geq 7$, cutting off near $m = 76$, which gives a residual set of order $7 \cdot 10^4$ ordered pairs. With one exact certificate per pair — and degree-ten certificates with margins near 1 need considerably finer rational rounding than the 10^{-6} that suffices at $k = 4$ — that is on the order of ten core-hours to generate and some 85 megabytes of certificate data. We have not done it. At $k = 6$ the $m = k$ strip alone has more than 10^7 pairs.

So Wästlund's remark is right in substance and quantitative in this form: what fails at higher k is not the legality of the comparison but the size of the set on which it must be supplemented, and $k = 4$ sits on the feasible side of that boundary by a factor of about 90 at the diagonal pair. Two further points are worth recording. Sharpening the estimate inside Lemma 3.1 cannot help: the signs of the e_j alternate, so $t = -R$ is the exact maximiser of $|E|$ on $|t| = R$ and the triangle inequality we use is an equality there. And a suggestion, offered as such and not as a result: what might help is a different comparison function, one with a complex root pattern. The source's own point is that the zeros of

$F_{k,m,n}$ are not near the real line, which makes it plausible — we have not tried to show it — that any real-rooted comparison must eventually lose. The staircase of §8 does not help here either: it describes the *shape* of the region the comparison covers, whereas what puts $k = 5$ out of reach is the *size* of the region it does not cover, and a two-variable certificate does not shrink a residual set of some $7 \cdot 10^4$ pairs.

10. Two printing slips in the source

We record two typographical points in the proof of Proposition 15 of [1], neither of which affects that proposition or its proof. They are noted because a reader who recomputes from the printed formulas, as we did, will not reproduce them otherwise.

The proof displays the denominator, the numerator and Janson's numerator as

$$\begin{aligned} Q(t) &= (mn-3t)\big((m-1)n-2t\big)\big(m(n-1)-2t\big) \\ &\qquad \cdot \big((m-2)n-t\big)\big((m-1)(n-1)-t\big)\big(m(n-2)-t\big), \\ P(t) &= m^2n^2(m-1)(n-1) - mn(4mn-3m-3n+2)t + (5mn-2m-2n-1)t^2 + 2t^3, \\ P_J(t) &= (mn-2t)\big((m-1)n-t\big)\big(m(n-1)-t\big) \\ &= m^2n^2(m-1)(n-1) - mn(4mn-3m-3n+2)t + (5mn-2m-2n)t^2 + 2t^3. \end{aligned}$$

First, both cubic coefficients should be -2: expanding the factored form of P_J gives t^3-coefficient $(-2)(-1)(-1) = -2$. With $-2t^3$ the displayed P, normalised to value 1 at $t = 0$, agrees with $\widehat{P}_{3,m,n}$ computed from the paper's own recursion at $(m,n) = (3,3), (3,5), (4,7), (6,6), (5,11)$, and with $+2t^3$ it does not. Second, that displayed Q is (1) at $k = 3$ multiplied by $m^3(m-1)^2(m-2)\,n^3(n-1)^2(n-2)$, so $P(0)/Q(0) = 1\big/\big(m(m-1)(m-2)\,n(n-1)(n-2)\big)$ rather than $F_{3,m,n}(0) = 1$: the displayed P omits a constant factor. With the cubic sign corrected the relation is

$$F_{3,m,n}(t) = mn(m-1)(n-1)(m-2)(n-2)\,\frac{P(t)}{Q(t)},$$

verified in exact rational arithmetic at those same five pairs. Neither point disturbs the argument of [1], which uses only the difference $P - P_J = -t^2$ — unchanged, since the slip is the same in both lines — and the bound $|P_J(t)| \geq P_J(R)$ on $|t| = R$; and a constant factor moves no zeros.

11. Verification

Every statement asserted above as a lemma, proposition, theorem or corollary — Lemmas 3.1 and 3.4, Propositions 3.3, 3.5 and 7.3, Theorems 4.1, 5.1, 5.3, 6.1, 6.3, 7.1, 8.2 and 8.4, Corollary 8.5, and the controls and identifications recorded in Remarks 3.2, 3.6, 5.4, 6.4, 7.4 and 8.6 — is checked, in the form stated, by the Lean 4 proof assistant [5] over the Mathlib library [6], with two bookkeeping exceptions. The tables (3)–(6) are *definitions* there, so their integrality, symmetry, bidegrees and term counts are read off the literal tables rather than proved, and what Theorem 4.1 contributes as proved statements is the factorisation (6) and the sign pattern. And Lemma 2.2 is stated for general k and is proved here by hand; the development proves instead the only instance the argument uses, that at $k = 4$ the six factors of D_3 admit the per-factor bounds displayed in the proof of Theorem 8.4, each positive for $m, n \geq 4$ and with product $D_3(mn/4)$. The development is ten modules. Seven carry the material of version 1: an integer bivariate-polynomial layer with a Horner evaluation proved to be a ring homomorphism, so that every positivity claim about (7) and about the tables (3)–(6) is a decision procedure on integer lists rather than an algebraic normalisation; Lemma 3.1 with its two factor shapes; the definitions of Δ, D_3, E, $\widehat{P}_4$ and W; the tail; the 235-pair table; the assembly; and the sharpness and control statements. Three are added in this revision: the threshold evaluations of Theorem 5.3 and the exhaustive box scan of Theorem 6.3; a rebuild of the coefficient-table evaluation over $\mathbb{R}$, with the fourteen staircase certificates and the region Ω; and Lemma 3.4 with the two real zero-free statements and the identifications of Remark 8.6. The development contains no `sorry` and declares no axiom of its own. One end-to-end run of the chain — the ten modules elaborated once each, in dependency order, single-threaded on an Intel Xeon Silver 4210 — takes about 370 seconds of wall clock, about 220 of

them the seven modules of version 1 and about 150 the three new ones; some 120 seconds of the total is the Mathlib import, which such a run pays once per module.

Two claims are settled by compiled evaluation rather than in the kernel, and each is a single named step: the 235-pair certificate table of Theorem 6.1, and the three rational checks behind Proposition 7.3 and Remark 7.4 (two sign evaluations of $\widehat{P}_{4,4,4}$ and one certificate failure at the widened radius). Both belong to the material of version 1; nothing added in this revision uses compiled evaluation. Everything else — Lemmas 3.1 and 3.4, the whole of Theorem 5.1 including all fifteen coefficient tables, the closed form (6), the sign pattern, Theorems 5.3, 6.3, 8.2 and 8.4, Corollary 8.5, and Propositions 3.3 and 3.5 — is checked by the kernel alone, with no compiled evaluation, no external solver and no floating-point or interval arithmetic. Several of the new statements need no axiom whatever: the seven threshold values of Theorem 5.3, the scan of the 3844 pairs of $[0,61]^2$ and the seven row lengths of Theorem 6.3, the fourteen staircase certificates of Theorem 8.2, the three controls of Remark 5.4 — the seven lowered strip certificates, the two-variable certificate at $(10,10)$ with the value $W(10,10)$, and the membership of $(4,61)$ and $(4,62)$ — and the two inhabitation checks of Remark 6.4. The count 235 itself uses propositional extensionality, and the remaining new statements use that together with the axiom of choice and quotient soundness.

Four things are deliberately outside the formal development and are labelled where they appear. The identification of Definition 2.3 with the numerator of $F_{4,m,n}$ (Remark 7.2), which is quoted from [1]; the derivation of the tables of Theorem 4.1 over $\mathbb{Q}(m,n)$, whose *result* is what the development defines and uses, and which was cross-checked against the recursion of [1] at every pair of $\mathcal{C}$ and of $[4,25]^2$; the numerical material — all of §9, the ratios ρ of Remark 6.2, and the monotonicity observation at the end of §7; and the five evaluations behind §10. The construction of the 235 certificates is likewise numerical, but it is only a construction: what the development checks is the exact rational inequality, so the numerics cannot make a false statement true. Two items that version 1 listed here have moved into the verified development, and the corresponding hedges are withdrawn in Remarks 5.4 and 6.4: the sharpness of the seven strip thresholds, and the negativity of W at all 235 pairs of $\mathcal{C}$.

The source of the development is currently held in a private repository: repository reference to be added at submission.

On the reference list. [1] and [2] were read here from their arXiv LaTeX e-prints, and every quotation above is from those sources, checked word for word against the e-print compiled with `tectonic`. The numbers by which results of [1] are cited above — Theorem 8, Propositions 9, 15 and 16, Conjecture 11 — are the numbers that compilation produces for **v1** and could shift in a later version. Both abs pages were re-fetched on 3 September 2026: [1] is still v1 with no journal reference, [2] still v1. [3] and [4] are cited for attribution and context only and were not read here; their bibliographic data are from the Crossref records of their DOIs.

References

[1] J. Wästlund, *Moment generating functions in combinatorial optimization: Bipartite matching*, arXiv:2602.07563v1 [math.PR], 7 February 2026 (cross-listed cs.DM; MSC 60C05, 90C27, 90C35).

[2] G. Mordant, *A central limit theorem for the random assignment problem*, arXiv:2608.05123v1 [math.PR], 5 August 2026. Cited for the dated confirmation, six months after [1], that the zero-free disk conjecture is open; it proves $\sqrt{n}\{C_n - \zeta(2)\} \Rightarrow \mathcal{N}(0, 4\zeta(2) - 4\zeta(3))$ for uniform costs by a route that does not pass through the conjecture.

[3] S. E. Alm and G. B. Sorkin, *Exact expectations and distributions for the random assignment problem*, Combin. Probab. Comput. **11** (2002), no. 3, 217–248, doi:10.1017/S0963548302005114. The real-rooted comparison function used at $k \leq 3$ is described there, and [1] attributes it to Svante Janson, who suggested it to its authors.

[4] O. J. Heilmann and E. H. Lieb, *Theory of monomer-dimer systems*, Comm. Math. Phys. **25** (1972), no. 3, 190–232, doi:10.1007/BF01877590. Cited for attribution only, to note the contrast: matching-polynomial real-rootedness is a statement about a fixed graph, whereas $F_{k,m,n}$ has genuinely non-real zeros — [1] exhibits them already at $k = 3$ — so the conjecture is not a corollary of that circle of results.

[5] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[6] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.