MULTIPLICABLE WALK POSETS: A TWO-FOLD AUTOMORPHISM CRITERION, THE DODECAHEDRON, AND THE TRIANGULAR GRAPH $T(6)$

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Abstract. The *walk poset* $P(G, v_0)$ of a finite graph G with base point v_0 has as elements the pairs (v, m) joined to v_0 by a walk of length m, ordered by the existence of a walk realising the rank difference. For vertex-transitive G it is a finite-type $\mathbb{N}$-graded upho poset, and Matsuoka used it to refute the conjecture of Fu, Peng and Zhang that every finitary upho poset is *multiplicable*: $P(\text{Petersen}, v_0)$ is not. He then asked for which finite vertex-transitive graphs $P(G, v_0)$ is multiplicable. Matsuoka's negative machinery rests on a spectral rigidity lemma, which forces every pair of bijections (φ, ψ) with $u \sim w \iff \varphi u \sim \psi w$ to be diagonal and therefore applies only to graphs whose spectrum contains no pair $\pm\lambda$. We remove that hypothesis. Our main theorem shows that a multiplication on $P(G, v_0)$ produces a *twisted regular representation*: a regular section of the group Γ of two-fold automorphisms of G whose partner family is conjugate back into it. Only symmetry of the adjacency relation, twin-freeness, and stabilisation of the rank sets are used—neither vertex-transitivity nor connectivity. Since a twisted regular representation is a finite object, non-multiplicability becomes decidable by a finite search whose completeness is proved rather than assumed. We apply this to the dodecahedron, where Γ has 240 elements against $|\operatorname{Aut}| = 120$: $P(\text{Dodecahedron}, v_0)$ is not multiplicable, and it is a cell that Matsuoka's spectral criterion provably cannot reach, the adjacency matrix being singular. On the positive side we show that $P(T(6), v_0)$ is multiplicable for the triangular graph $T(6) = L(K_6) = J(6, 2)$, which is neither a Cayley graph nor spectrally accessible, and we reprove Matsuoka's positive result for $L(\text{Petersen})$ from a single regular subgroup instead of a solver-found table. Fifteen of the sixteen graphs of a reference catalogue are thereby decided. Every statement is machine-checked in Lean 4.

1. Introduction

Upho posets and multiplicability. A poset P with minimum $\hat{0}$ is *upho* (upper homogeneous) when every principal order filter $U(a) = \{x : x \geq a\}$ is order-isomorphic to P itself. It is *finitary* if every element has finite height, $\mathbb{N}$-*graded* if it carries a rank function ρ with $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever $x \lessdot y$, and of *finite type* if each rank set P_n is finite.

Following Matsuoka [1], call P *multiplicable* if it carries a monoid structure $*$ that is left cancellative and invertible-free and whose left-divisibility order $a \leq b \iff \exists c,\ a * c = b$ is the given order. The same property carries three names in the literature: Fu, Peng and Zhang [2] call it *regular*, Hopkins and Lewis [4]—the sequel to [3]—call it *colorable*, and [1] coins *multiplicable*. The first two ask for a compatible colouring of the covering relations, which for a finitary upho poset is equivalent to multiplicability ([1, Proposition 2.7]). Fu, Peng and Zhang conjectured that every finitary upho poset is multiplicable ([2, Conjecture 3.12], stated there for colourings; the finite-type $\mathbb{N}$-graded case is [2, Conjecture 7.1]).

Walk posets. Let $G = (V, E)$ be a finite simple graph and $v_0 \in V$. The *walk poset* ([1, Definition 3.1]) is

$$P(G, v_0) \;=\; \{(v, m) \in V \times \mathbb{N} \;:\; \text{some walk of length } m \text{ runs from } v_0 \text{ to } v\},$$

ordered by $(v, m) \leq (w, m')$ if and only if $m \leq m'$ and some walk of length $m' - m$ runs from v to w. *Walks, not paths*: the rank sets are read off Boolean powers of the adjacency matrix. Covers are exactly the rank-one steps along edges, so a k-regular G gives every element exactly k covers; and for vertex-transitive G the poset is a finite-type $\mathbb{N}$-graded upho poset, independent of v_0 up to isomorphism ([1, Proposition 3.4, Remark 3.5]).

Matsuoka's counterexample to the Fu–Peng–Zhang conjecture is $P(\text{Petersen}, v_0)$ ([1, Theorem 3.7]), and his positive surprise is that the line graph $L(\text{Petersen})$—vertex-transitive and *not* a Cayley graph—has $P(L(\text{Petersen}), v_0)$ multiplicable ([1, Theorem 4.1]). He asks:

Question 1.1 ([1, Question 4.9])**.** For which finite vertex-transitive graphs G is the walk poset $P(G, v_0)$ multiplicable?

Two engines are available before the present work. On the positive side, if $G = \text{Cay}(H, S)$ then $(v, n) * (w, m) = (vw, n + m)$ works ([1, Example 3.6]), so Cayley graphs are settled. On the negative side, [1, Theorem 3.7 and Remark 3.10] give: $P(G, v_0)$ is not multiplicable whenever G is (1) connected vertex-transitive, (2) spectrally $\pm$-free, meaning that no real λ has both λ and $-\lambda$ as eigenvalues of the adjacency matrix—so in particular 0 is not an eigenvalue—and (3) not a Cayley graph.

Hypothesis (2) is what makes the negative engine narrow. Its essential use is a rigidity lemma ([1, Lemma 3.8], proved there for the Petersen graph out of its spectrum): if bijections $\varphi, \psi \colon V \to V$ satisfy $u \sim w \iff \varphi u \sim \psi w$ for all u, w, then $\varphi = \psi$. It is also what supplies twin-freeness and non-bipartiteness, which [1, Remark 3.10] records as consequences of (2) and which we shall instead assume outright. Such a pair (φ, ψ) is a *two-fold automorphism*, and a graph is called *unstable* when non-diagonal ones exist. Instability is common—the dodecahedron and $L(\text{Petersen})$ are both unstable—and wherever it occurs [1, Remark 3.10] is silent.

What this paper settles. We replace the rigidity lemma by the two-fold automorphism group itself, and obtain a criterion with no spectral hypothesis at all.

- **The criterion** (Theorem 4.1). If $P(G, v_0)$ carries a multiplication then G admits a *twisted regular representation* based at v_0: a family $(h_v)_{v \in V}$ of two-fold automorphisms acting regularly on V, closed under composition, together with a single two-fold automorphism conjugating the partner family back into it. Only symmetry, twin-freeness and stabilisation of the rank sets are used. Contrapositively, the non-existence of a finite object refutes multiplicability.
- **The dodecahedron** (Theorem 6.4). $P(\text{Dodecahedron}, v_0)$ is not multiplicable. This is an instance of Question 1.1 that neither of the two engines above reaches: the fourth truncation of the poset is satisfiable, and the adjacency matrix of the dodecahedron is singular, so condition (2) of [1, Remark 3.10] fails. Here $|\Gamma| = 240$ against $|\operatorname{Aut}| = 120$.
- **New multiplicable instances** (Theorems 7.4 and 7.3). $P(T(6), v_0)$ is multiplicable for the triangular graph $T(6) = L(K_6) = J(6, 2)$, which is not a Cayley graph and whose spectrum $8, 2, -2$ contains a ± 2 pair, so it too lies outside [1, Remark 3.10]. The same construction reproves [1, Theorem 4.1] for $L(\text{Petersen})$ from one regular subgroup of the automorphism group of the bipartite double cover, with no search.
- **A decided catalogue** (Section 9). Fifteen of sixteen vertex-transitive graphs are settled; only the Desargues graph is left open by the two engines as they stand.

Everything below is machine-checked in Lean 4 against *Mathlib*; Section 10 says exactly which parts are kernel-checked and which delegate a finite verdict to compiled evaluation.

2. The walk poset and its multiplications

Throughout, G is a finite graph on the vertex set V, given by a symmetric Boolean adjacency relation, and $v_0 \in V$. We write $\rho(v, m) = m$ for the rank and $\text{vx}(v, m) = v$ for the vertex of a poset element, and $V_m \subseteq V$ for the set of vertices at rank m, that is, the vertices reachable from v_0 by a walk of length exactly m.

Definition 2.1. A *multiplication* on $P(G, v_0)$ is a monoid structure $(\,\cdot\,, 1)$ on the underlying set of $P(G, v_0)$ such that

(1) $x \cdot y = x \cdot z$ implies $y = z$ (left cancellativity), and
(2) $x \leq y$ if and only if $y = x \cdot z$ for some z (the order is left divisibility).

Definition 2.1 is [1, Definition 2.5] with the *invertible-free* clause deliberately omitted. Dropping a hypothesis makes the non-existence of a multiplication a *stronger* statement than non-multiplicability, so all negative results below are stated for the weaker structure; and by Lemma 2.2 the omitted clause is automatic, so the positive results lose nothing either.

Lemma 2.2 (Rank additivity)**.** *Let $\cdot$ be a multiplication on $P(G, v_0)$. Then $1 = (v_0, 0)$ and $\rho(x \cdot y) = \rho(x) + \rho(y)$ for all $x, y \in P(G, v_0)$.*

Proof sketch. The unit divides $\hat{0} = (v_0, 0)$, so it has rank 0 and equals $\hat{0}$. For additivity, nothing in Definition 2.1 mentions the rank, so it must be extracted from the order. Two facts about the walk poset suffice: a strict inequality strictly raises the rank, and every rank between two comparable elements is realised by an element between them (split the walk). Left multiplication L_x is an order embedding onto the up-set $[x, \infty)$, by the divisibility axiom and left cancellativity. A strong induction on $\rho(y)$ then bounds $\rho(x \cdot y)$ from below by a rank-$(\rho(y) - 1)$ element below y, and from above by a predecessor of $x \cdot y$ inside $[x, x \cdot y]$, which is $x \cdot w$ for some $w < y$. No chain or gradedness machinery is used. Finally $x \cdot y = 1$ forces $\rho(x) = \rho(y) = 0$, hence $x = y = 1$: a multiplication is automatically invertible-free. □

Remark 2.3. Matsuoka asks, of a poset already known to be upho, for a monoid structure that is left cancellative *and invertible-free* and whose left-divisibility relation is the given order ([1, Definition 2.5]). The last clause of Lemma 2.2 makes invertible-freeness automatic. Consequently, for vertex-transitive G—where $P(G, v_0)$ is upho by [1, Proposition 3.4], and where alone the word is defined—$P(G, v_0)$ is multiplicable in the sense of [1, Definition 2.5] exactly when it admits a multiplication in the sense of Definition 2.1. We use the two phrasings interchangeably from here on, with the direction that matters made explicit at each theorem.

3. Two-fold automorphisms and twisted regular representations

Definition 3.1. A *two-fold automorphism* of G is a pair (φ, ψ) of permutations of V with

$$u \sim w \iff \varphi(u) \sim \psi(w) \qquad \text{for all } u, w \in V.$$

These form a group $\mathrm{TF}(G)$ under componentwise composition, closed under the swap $(\varphi, \psi) \mapsto (\psi, \varphi)$; the diagonal pairs are exactly $\mathrm{Aut}(G)$. Equivalently $\mathrm{TF}(G)$ is the part-preserving subgroup of $\mathrm{Aut}(G \times K_2)$. The graph G is *twin-free* if distinct vertices have distinct neighbourhoods.

The name *TF-automorphism*, and the link with the classical theory of *stable* and *unstable* graphs, are due to Lauri, Mizzi and Scapellato [5], who trace the notion itself back to Zelinka's isotopies of digraphs. The identification of $\mathrm{TF}(G)$ with the part-preserving subgroup of $\mathrm{Aut}(G \times K_2)$ is [5, Lemma 3.1], and [5, Theorem 3.2] is exactly the statement that G is stable if and only if $\mathrm{TF}(G)$ is the diagonal.

Proposition 3.2. *Let G be twin-free. If (φ, ψ) and (φ, ψ') are two-fold automorphisms and φ is surjective, then $\psi = \psi'$. Dually, if (φ, ψ) and (φ', ψ) are two-fold automorphisms and ψ is surjective, then $\varphi = \varphi'$. Moreover every two-fold automorphism (φ, ψ) has φ an automorphism of the distance-2 graph G^2, in which distinct u, w are adjacent when they have a common neighbour.*

Proof sketch. For the first claim, φ surjective lets us write any z as $\varphi(t)$, and then $\mathrm{adj}(z, \psi w) = \mathrm{adj}(t, w) = \mathrm{adj}(z, \psi' w)$, so ψw and $\psi' w$ have the same neighbourhood; twin-freeness identifies them. The dual statement is the same argument after the swap, using symmetry of the adjacency relation. For the last claim, ψ carries $N(u) \cap N(w)$ bijectively onto $N(\varphi u) \cap N(\varphi w)$, so φ preserves the relation "distinct with a common neighbour". □

For twin-free G the second component of a two-fold automorphism is determined by the first, so $\mathrm{TF}(G)$ is carried by a subgroup $\Gamma = \Gamma(G) \leq \mathrm{Sym}(V)$ together with the induced involutive automorphism $s \colon \Gamma \to \Gamma$ sending φ to its partner; the swap-closure of $\mathrm{TF}(G)$ makes s an automorphism with $\mathrm{Fix}(s) = \mathrm{Aut}(G)$. Thus G is stable exactly when $\Gamma = \mathrm{Aut}(G)$.

Definition 3.3. A *twisted regular representation* of G based at v_0 consists of

- families $h, \bar{h}\colon V \to V^V$ with $(h_v, \bar{h}_v) \in \mathrm{TF}(G)$, both h_v and $\bar{h}_v$ injective, for every v;
- the *regularity* conditions $h_v(v_0) = v$ and $h_v \circ h_u = h_{h_v(u)}$ for all u, v;
- a two-fold automorphism $(x, y) \in \mathrm{TF}(G)$ with x, y injective, and a map $c\colon V \to V$, such that $x \circ \bar{h}_v = h_{c(v)} \circ x$ for every v.

The first two items say that $\{h_v\}$ is a regular subgroup of Γ, written one element per vertex; the third says that its image under s is conjugate back into it by a single element of Γ. The structure is not vacuous:

Proposition 3.4. *If* $\mathrm{Aut}(G)$ *contains a subgroup acting regularly on* V*—equivalently, by Sabidussi's characterisation* [6, §3.7]*, if* G *is a Cayley graph—then* G *admits a twisted regular representation based at any* v_0*, obtained by taking* $\bar{h} = h$*,* $x = y = \mathrm{id}$ *and* $c = \mathrm{id}$*. In particular the eight Cayley graphs of the catalogue of Section 9 all admit one.*

Conversely, a twisted regular representation is *strictly weaker* than a regular subgroup of $\mathrm{Aut}(G)$; see Proposition 9.3.

4. The criterion

Theorem 4.1. *Let* G *be a finite graph with symmetric adjacency relation, twin-free, and suppose there is an even* N_0 *such that every vertex of* G *is reachable from* v_0 *by a walk of every length* $m \geq N_0$*. If the walk poset* $P(G, v_0)$ *carries a multiplication, then* G *admits a twisted regular representation based at* v_0*. Equivalently: if* G *admits no twisted regular representation based at* v_0*, then* $P(G, v_0)$ *carries no multiplication, and in particular is not multiplicable.*

Note what is *not* assumed: neither vertex-transitivity nor connectivity, and no condition on the spectrum. The stabilisation hypothesis holds for every connected non-bipartite graph, and reduces to two finite checks:

Proposition 4.2. *Let* G *have symmetric adjacency and no isolated vertex. If every vertex is reachable from* v_0 *by a walk of length* N_0 *and by a walk of length* $N_0 + 1$*, then every vertex is reachable by a walk of every length* $m \geq N_0$*.*

Proof. Induct on $m - N_0$, appending a back-and-forth pair of steps at a vertex to pass from length m to length $m + 2$. □

Computationally the two hypotheses of Proposition 4.2 are evaluated level by level—the rank sets $V_0, V_1, \ldots$ are computed by repeated neighbourhood expansion—so the cost is quadratic in $|V|$ per level rather than exponential in N_0.

Proof sketch of Theorem 4.1. The argument is the proof of [1, Theorem 3.7] with the rigidity lemma [1, Lemma 3.8] replaced by $\mathrm{TF}(G)$. No search and no certificate enters; the whole proof is symbolic. Let $\cdot$ be a multiplication on $P(G, v_0)$ and write $E_m(v)$ for the element $(v, N_0 + m)$, which exists for every v by the stabilisation hypothesis.

Step 1. By Lemma 2.2 the ranks add, so for fixed $a \in P(G, v_0)$ left multiplication carries rank $N_0 + m$ to rank $\rho(a) + N_0 + m$. Read on vertices this is a map $\varphi_{a,m}\colon V \to V$, injective by left cancellativity, hence bijective.

Step 2. At rank difference one the order is exactly adjacency, and L_a is an order embedding onto an up-set in both directions; therefore $(\varphi_{a,m}, \varphi_{a,m+1}) \in \mathrm{TF}(G)$. *This is where* [1] *invokes rigidity to conclude* $\varphi_{a,m} = \varphi_{a,m+1}$*, and we do not.*

Step 3. By Proposition 3.2 the partner is unique, and $\mathrm{TF}(G)$ is swap-closed, so $\varphi_{a,m+2} = \varphi_{a,m}$: the sequence is 2-periodic, and the pair $(\varphi_{a,0}, \varphi_{a,1})$ carries all the information. Evenness of N_0 is what makes the rank shift by N_0 invisible.

Step 4. A downward induction extends the description below rank N_0, where not every vertex carries a poset element. The step is that the neighbourhood of the vertex of $a \cdot z$ is exactly the $\varphi_{a,\rho(z)+1}$-image of the neighbourhood of the vertex of z: the inclusion $\supseteq$ transports upward comparabilities, and $\subseteq$ holds because every neighbour of $a \cdot z$ at the next rank lies above a and is therefore of the form $a \cdot z'$. Twin-freeness then identifies the two candidate vertices. At rank 0 this gives $\varphi_{a,0}(v_0) = \mathrm{vx}(a)$.

Step 5. Put $\psi_m(v) = \varphi_{E_m(v),0}$, a family $V \to \mathrm{Sym}(V)$. Associativity of the multiplication turns these into a graded system

$$\psi_{r+m}\big(\psi_r(v)(u)\big) \;=\; \psi_r(v) \circ \psi_m(u).$$

Step 6. The system determines ψ_{r+m} from ψ_r and ψ_m. There are finitely many families $V \to \mathrm{Sym}(V)$, so the even-indexed sequence is eventually periodic, and a pigeonhole gives an even J with $\psi_{N_0+2J} = \psi_J$. At that index the system reads $\psi_J(v)\circ\psi_J(u) = \psi_J(\psi_J(v)(u))$: the family $\{\psi_J(v)\}_{v\in V}$ is closed under composition, and $v \mapsto \psi_J(v)(v_0) = v$ makes it regular. This replaces the finite-semigroup and coset-counting steps of the informal argument; nothing here counts anything.

Step 7. Instantiating the graded system once at an even and once at an odd rank ($r = J, m = 1$ and $r = 1, m = J$) produces a single two-fold automorphism conjugating the partner family back into the group. That is the last item of Definition 3.3. □

Remark 4.3. Specialised to a stable graph, where $\Gamma = \mathrm{Aut}(G)$ and $s = \mathrm{id}$, a twisted regular representation is exactly a regular subgroup of $\mathrm{Aut}(G)$, and Theorem 4.1 degenerates to "$P(G, v_0)$ multiplicable $\Rightarrow G$ is a Cayley graph"—which is [1, Theorem 3.7 and Remark 3.10]. The content of the theorem is what it says at unstable graphs, where [1, Remark 3.10] has nothing to say at all.

5. Deciding the criterion: two complete searches

Theorem 4.1 reduces non-multiplicability to the emptiness of a finite set. We decide that emptiness by two nested depth-first searches, both with proved completeness: what has to be shown is not that the searches find the right answer, but that the prunes only ever discard candidates that could not have been solutions.

Proposition 5.1 (Prunes)**.** *Let (φ, ψ) be a two-fold automorphism of G with ψ bijective. Then*

1. *φ preserves the common-neighbour count: $|N(\varphi u) \cap N(\varphi w)| = |N(u) \cap N(w)|$ for all u, w, that is, φ preserves the entries of A^2;*
2. *φ carries neighbourhoods onto neighbourhoods, $\varphi(N(w)) = N(\psi w)$; in particular any partial image of a neighbourhood must still fit inside one.*

Proof. For (1), ψ maps $N(u) \cap N(w)$ bijectively onto $N(\varphi u) \cap N(\varphi w)$; for (2), unfold Definition 3.1 and use symmetry of the adjacency relation. □

Proposition 5.2 (Completeness)**.** *Fix an enumeration of V without repetitions. Let $\mathcal{G}$ be the output of the depth-first search that assigns images to vertices in that order, rejecting a partial assignment as soon as it violates (1) or (2) of Proposition 5.1. Then every first component of a two-fold automorphism of G lies in $\mathcal{G}$. Let $\mathcal{S}$ be the output of the second depth-first search, which builds a family $(h_v)_{v\in V}$ slot by slot, drawing the candidates for slot v from $\{g \in \mathcal{G} : g(v_0) = v\}$ and rejecting as soon as a composite $h_v \circ h_u$ fails to be the identity or fixed-point-free, or disagrees with an already-placed slot. If $\mathcal{S}$ is empty then G admits no twisted regular representation based at v_0.*

Both prunes are *necessary* conditions, which is why $\mathcal{G}$ may be a proper superset of Γ without harm, and why an empty $\mathcal{S}$ is conclusive. Restricting the candidates for slot v to $\{g : g(v_0) = v\}$ is legitimate because $h_v(v_0) = v$ is part of Definition 3.3; it was measured to be worth a factor 15 to 20 at no cost in the proof.

Remark 5.3 (What the searches cost). These are measurements, not theorems. With both prunes enabled the first search visits

$$2\,920 \text{ (Petersen)}, \quad 3\,940\ (T(5)), \quad 53\,235\ (KG(6,2)), \quad 76\,005\ (L(\text{Petersen})),$$

$$79\,120 \text{ (dodecahedron)}, \quad 847\,504 \text{ (Coxeter)}$$

nodes and returns candidate lists of sizes $120, 120, 720, 360, 240, 336$ respectively, which coincide with the separately measured $|\Gamma|$; against these, $|\operatorname{Aut}(G)|$ is $120, 120, 720, 120, 120, 336$, so the two unstable rows are L(Petersen) and the dodecahedron. The second search visits 1 356, 1 356, 189 840, 55 440, 28 884 and 4 092 nodes and returns $0, 0, 0, 6, 12, 0$ regular sections. Neither prune dominates the other: at $KG(6, 2)$ dropping the count prune raises the first search from 53 235 nodes to 2 634 195, while at L(Petersen) dropping it changes nothing and dropping the neighbourhood prune instead raises 76 005 to 5 993 325. A search over the pair (φ, ψ) simultaneously—trivially sound, needing no invariant—was measured and abandoned: it is 14 to 61 times worse on all six rows, reaching 51 416 372 nodes at the Coxeter graph. For the dodecahedron the formal development uses a specialised enumeration over $\operatorname{Aut}(G^2)$ instead, costing 98 560 nodes, and it agrees with the generic search on the section count to the node.

6. The dodecahedron

Proposition 6.1. *For the dodecahedron,* Γ *has exactly* 240 *elements, and every one of them is the first component of a two-fold automorphism with the partner forced by Proposition 3.2. Since* $|\operatorname{Aut}| = 120$*, the dodecahedron is unstable, and* [1, Lemma 3.8] *fails for it.*

Proof sketch. By Proposition 3.2 every two-fold automorphism has its first component in $\operatorname{Aut}(G^2)$, where G^2 is the distance-2 graph—for the dodecahedron a 6-regular graph on 20 vertices. A depth-first enumeration of $\operatorname{Aut}(G^2)$, whose completeness is proved in the sense of Proposition 5.2, produces a list of candidates in 98 560 nodes; the two facts that the list has 240 entries and that each is genuinely a two-fold automorphism are finite verdicts about that list, and are the only parts of this proposition delegated to exhaustive computation. As a cross-check the value 240 was obtained in three independent ways: as the part-preserving half of $\operatorname{Aut}(\text{Dodecahedron} \times K_2)$, which has order 480; by a complete search over $\operatorname{Sym}(V)$ pruned only by Proposition 5.1(2), in 6 593 680 nodes; and as $\operatorname{Aut}(G^2)$. □

Theorem 6.2. *The dodecahedron admits no twisted regular representation based at any vertex.*

Proof sketch. By Proposition 6.1 the group Γ is the explicit 240-element list. The section search of Proposition 5.2 enumerates the regular sections of Γ in 28 884 candidate trials and finds exactly 12; a further exhaustive check over the 240×12 pairs finds that for none of them is the partner family conjugate back into the section by an element of Γ. These three finite verdicts—the size of Γ, the two-fold property of its members, and the emptiness of the twisted-stable sections—are the whole computational content; the reduction of Definition 3.3 to them is symbolic. □

Remark 6.3 (The structure behind the verdict). The following description was computed outside the formal development and is recorded for intuition only. The group Γ(Dodecahedron) of order 240 has element orders $1^1 2^{51} 3^{20} 4^{60} 5^{24} 6^{60} 10^{24}$—no element of order 20—and exactly 18 subgroups of order 20, in two classes: six dihedral ones, which are s-stable but have fixed points and so are not regular, and twelve Frobenius groups $F_{20} = Z_5 \rtimes Z_4$, which are regular but whose s-images lie in the other Γ-conjugacy class, the twelve splitting into two classes of six interchanged by s. Regularity and s-stability up to conjugacy are therefore incompatible here.

Theorem 6.4. $P(\textit{Dodecahedron}, v_0)$ *carries no multiplication; in particular it is not multiplicable. Moreover* 0 *is an eigenvalue of the adjacency matrix of the dodecahedron, so condition (2) of* [1, Remark 3.10] *fails and that criterion cannot decide this cell.*

Proof. The dodecahedron is symmetric, twin-free, and every vertex is reachable from v_0 by a walk of every length ≥ 6, by Proposition 4.2 and two evaluations of the rank sets: $V_6 = V_7 = V$. Theorem 4.1 with $N_0 = 6$ and Theorem 6.2 give the first claim. For the second, an explicit integer vector in the kernel of the adjacency matrix is exhibited and checked entrywise; it is nonzero, so the matrix is singular. □

Remark 6.5. Both of the source's engines are silent at the dodecahedron, and both facts are machine-checked. The truncation engine is silent because Φ_4 is satisfiable there (Proposition 8.6), and the spectral criterion is silent because of the kernel vector. Computed exactly over $\mathbb{Q}$ outside the formal development: the nullity of the adjacency matrix is 4, and $\deg\gcd\big(p(x), p(-x)\big) = 10$ for the characteristic polynomial p, so condition (2) fails twice over; among the other catalogue graphs decided negatively, all have $\gcd(p(x), p(-x)) = 1$ and $p(0) \neq 0$, so [1, Remark 3.10] does cover them. Note that failing condition (2) carries no sign either way—L(Petersen) also fails it and *is* multiplicable.

Corollary 6.6. *The dodecahedron is not a Cayley graph.*

Proof. Immediate from Theorem 6.2 and Proposition 3.4. □

This recovers, by a route that does not consult the classification, the exceptional case $GP(10, 2)$ of [8]: a generalized Petersen graph $GP(n, k)$ is a Cayley graph exactly when $k^2 \equiv 1 \pmod{n}$, and $2^2 = 4 \not\equiv 1 \pmod{10}$.

7. The positive side

Theorem 7.1. *If* $\mathrm{Aut}(G)$ *contains a subgroup acting regularly on* V*—if* G *is a Cayley graph—then* $P(G, v_0)$ *carries a multiplication, namely* $(v, n)\cdot(w, m) = (v\cdot w,\ n+m)$. *In particular the eight Cayley graphs* C_5*,* K_4*,* Q_3*,* $K_{3,3}$*, Heawood, Möbius–Kantor,* Q_4 *and the Pappus graph all have multiplicable walk posets.*

This is [1, Example 3.6], and it is the non-vacuity control for Definition 2.1: the structure whose emptiness Theorems 6.4 and 9.2 assert is inhabited ten times over, by these eight together with Theorem 7.3. The verification that a product is again in the poset is the statement that walks transport along an automorphism in both directions. No stabilisation hypothesis is needed, so bipartite graphs are covered.

The interesting positive instances are not Cayley graphs. Write $\hat{G} = G \times K_2$ for the bipartite double cover and $\hat{v_0} = (v_0, 0)$. A regular subgroup of the part-preserving automorphisms of $\hat{G}$ can be recorded on V alone, as a pair of families $\alpha, \beta\colon V \times (\mathbb{Z}/2\mathbb{Z}) \to V^V$ with

$$h_{(v,i)}(u, 0) = (\alpha_{v,i}(u),\ i), \qquad h_{(v,i)}(u, 1) = (\beta_{v,i}(u),\ 1 + i),$$

subject to $\alpha_{v,i}(v_0) = v$, to $(\alpha_{v,i}, \beta_{v,i}) \in \mathrm{TF}(G)$, and to the two composition laws of the group that act on part 1, namely $\beta_{v,i} \circ \beta_{u,0} = \beta_{\alpha_{v,i}(u),\, i}$ and $\alpha_{v,i} \circ \beta_{u,1} = \beta_{\beta_{v,i}(u),\, i+1}$. Call such data a *cover representation* of G.

Theorem 7.2. *Let* G *be symmetric and twin-free and let* (α, β) *be a cover representation of* G *at* v_0*. Then* $P(G, v_0)$ *carries a multiplication, given by*

$$(v, m)\cdot(w, m') \;=\; \big(\alpha_{v,\, m \bmod 2}(w),\ m+m'\big) \text{ if } m' \text{ is even}, \qquad \big(\beta_{v,\, m \bmod 2}(w),\ m+m'\big) \text{ if } m' \text{ is odd}.$$

Proof sketch. The motivating picture—which the statement of the theorem does not need—is that for connected non-bipartite G the walk poset of $\hat{G}$ at $\hat{v_0}$ *is* $P(G, v_0)$: rank m forces the part m mod 2, so the part coordinate is redundant. The formal proof never writes the double cover down; it uses instead the fact that walks transport along a two-fold automorphism *alternating* between its two halves—a walk $u = x_0 \sim \cdots \sim x_d = w$ maps to $\varphi x_0 \sim \psi x_1 \sim \varphi x_2 \sim \cdots$, so the far end lands under φ or ψ according to the parity of d. This supplies both the well-definedness of the product and the divisibility axiom. Associativity needs all four composition laws of the double cover's group, of which a cover representation records only the two acting on part 1; the other two are *derived*, not assumed, from the uniqueness of the first component of a two-fold automorphism (Proposition 3.2), which is where twin-freeness enters. Consequently a witness recorded for the weaker data already determines the whole monoid, and no new certificate is required. □

Theorem 7.3. $P(T(6), v_0)$ *and* $P(L(\mathit{Petersen}), v_0)$ *both carry a multiplication; both are multiplicable.*

Proof. Explicit cover representations are exhibited for both graphs and every defining identity is checked exhaustively over the finite index sets; twin-freeness and symmetry are likewise finite checks. Theorem 7.2 then applies. For $L(\text{Petersen})$ the underlying regular subgroup has order 30 inside $\mathrm{Aut}(L(\text{Petersen}) \times K_2)$, which has order 720; for $T(6)$ it has order 30 inside $\mathrm{Aut}(T(6) \times K_2)$, which has order 40 320. □

For $L(\text{Petersen})$ this is [1, Theorem 4.1], whose published proof is a certificate: an 8×15 table of permutation pairs found by an SMT solver, lifted to automorphisms of the double cover, together with four finite verifications. Here it is one regular subgroup fed to a general theorem, with no search.

The case of $T(6) = L(K_6) = J(6,2)$ is new. It is out of reach of both engines of [1]: it is not a Cayley graph, by [7, Corollary 4.6]—$T(n)$ is a Cayley graph exactly when $n \in \{2,3,4\}$ or $n \equiv 3 \pmod 4$ with n a prime power—and it is $\mathrm{SRG}(15,8,4,4)$ with spectrum $8, 2, -2$, so it has a ± 2 eigenvalue pair and condition (2) of [1, Remark 3.10] fails. It is also out of reach of any search for a labelling of the truncated poset, the branching factor being $8! = 40\,320$. A second, self-contained reason for non-Cayleyness: $\mathrm{Aut}(T(6))$ has order 720, a regular subgroup would have order 15, every group of order 15 is cyclic, and S_6 has no element of order 15. A third: $T(6)$ is the complement of $KG(6,2)$, hence has the same automorphism group, and $KG(6,2)$ admits no twisted regular representation (Theorem 9.2), which by Proposition 3.4 rules out a regular subgroup of its automorphism group.

The same witness settles the truncated question, which is the form in which the problem is posed in [1, Appendix A.1]:

Theorem 7.4. *Every finite truncation of $P(T(6), v_0)$ is satisfiable: for every height H there is a labelling of the elements of rank $< H$ by bijections onto the corresponding neighbourhoods that satisfies the transport constraints Φ_H of Section 8.*

Proof. The cover representation of Theorem 7.3 is in particular a rank-periodic transport model of period two, and such a model satisfies Φ_H at every H (Proposition 8.4). □

Remark 7.5. The regular subgroup used for $T(6)$ was verified three independent ways before it was formalised: by seven structural checks (its size, that all entries are permutations, that it contains the identity, that all are graph automorphisms of the double cover, closure under composition, fixed-point-freeness of every non-identity element, and bijectivity of $h \mapsto h(\hat{v_0})$); by rebuilding the Cayley graph from it and comparing elementwise with $T(6) \times K_2$; and by pushing the induced labelling onto $P(T(6), v_0)$ and checking the word form of Φ_H directly at $H = 2, \ldots, 6$, over 69 variables and the 84 elements of rank at most 6.

8. The truncations Φ_H

The engine of [1, Appendix A.1] works with a finite truncation. Fix H and a k-regular G. A *labelling* of height H assigns to every element $x = (v, m)$ with $m < H$ a bijection $c_x \colon \{1, \ldots, k\} \to N(v)$. Following a word w of labels from an element gives an element $x \cdot w$ of rank $\rho(x) + |w|$, and the labelling is *valid* when

$$\hat{0} \cdot w = \hat{0} \cdot w' \iff x \cdot w = x \cdot w' \qquad \text{for all } x \text{ and all label words } w, w' \text{ with } |w| = |w'|.$$

Write Φ_H for this constraint system and say that Φ_H is satisfiable when a valid labelling of height H exists. Restricting a multiplication on $P(G, v_0)$ to its products by atoms gives a valid labelling at every H—this is the argument [1, Appendix A.4] runs for the encoding used there—so unsatisfiability at one H refutes multiplicability. *That restriction argument is not part of the formal development;* every statement in this section is about Φ_H alone.

Proposition 8.1. *The word form of Φ_H above is equivalent to the transport form: for every element x of rank at most H and every $\ell \leq H - \rho(x)$, the relation $\{(\hat{0} \cdot w,\ x \cdot w) : |w| = \ell\}$ is a well-defined injective partial function. Both forms are decidable, and permuting the label alphabet carries valid labellings to valid labellings.*

The transport form is what a search computes incrementally, and the alphabet symmetry is the free factor $k!$ that lets the row at the base point be pinned.

Proposition 8.2 (Search soundness)**.** *Let a partial labelling be extended depth-first, rejecting as soon as one of the transport relations computed from the assigned rows fails to be a well-defined injective partial function. If the search exhausts its space then Φ_H is unsatisfiable. The soundness does not require the list of variables or the list of elements to be complete: a short list weakens the pruning but never produces a wrong refutation.*

Proof sketch. The transport relation computed from a partial assignment is a sub-relation of the one computed from any total labelling extending it; a sub-relation of a well-defined injective partial function is one, so the check can never reject an assignment extending a valid labelling. Everything else is bookkeeping over the fold. Together with the alphabet symmetry of Proposition 8.1 this also justifies pinning the row at the base point, which divides the search by exactly $k!$. □

Proposition 8.3. *For the Petersen graph, Φ_4 is unsatisfiable, hence so is Φ_H for every $H \geq 4$; and Φ_2 and Φ_3 are satisfiable. So $H = 4$ is the exact threshold.*

Proof sketch. Satisfiability at $H = 2, 3$ is witnessed by explicit labellings, checked by kernel evaluation. Unsatisfiability at $H = 4$ is Proposition 8.2 applied to a complete pruned search, which visits 242 179 nodes and about 1.69×10^7 elementary transport steps with the base row pinned—exactly $1\,453\,074 = 6 \times 242\,179$ without pinning, the free relabelling factor $k! = 6$ being visible in the measurement. Two independently written solvers reproduce both counts to the node. This reproduces the single sentence of [1, Appendix A.4], which records one SMT run with the same verdict. □

A whole-space decision procedure is not available here: the constraints on ranks 0 to 2 are exactly the satisfiable Φ_3, and the shortest unsatisfiable prefix in rank order still involves 15 of the 20 variables, i.e. $6^{14} \approx 7.8 \times 10^{10}$ assignments. Hence the pruning, and hence the need for Proposition 8.2.

Proposition 8.4. *Suppose the labelling and a family of transport permutations depend on the rank only modulo $p+1$, each transport is injective, the transport starts at the right place, and the one-step recurrence $T^x_{\ell+1}(c_{(u,\ell)}(a)) = c_{(T^x_\ell(u),\, \ell+\rho(x))}(a)$ holds. Then Φ_H is satisfiable for every H. A regular subgroup of $\mathrm{Aut}(G)$ is the case $p = 0$, and a cover representation is the case $p = 1$.*

Proposition 8.5. *Every finite truncation of $P(L(Petersen), v_0)$ is satisfiable. Moreover the Desargues graph is the bipartite double cover of the Petersen graph, by an explicit relabelling.*

Proposition 8.6 (Calibration)**.** *For the Coxeter graph, Φ_3 and Φ_4 are both satisfiable, found in* 1 105 *and* 1 161 *search nodes; for the dodecahedron likewise, in* 1 393 *and* 1 619 *nodes.*

This is the calibration that governs every positive statement in this paper. The Coxeter graph is cubic, connected, vertex-transitive, spectrally $\pm$-free and not a Cayley graph, so [1, Remark 3.10] already declares its walk poset non-multiplicable—and Theorem 9.2 now proves it. *A satisfiable truncation is therefore no evidence whatsoever of multiplicability*, and no positive result above is obtained from a search: each comes from a general theorem valid at every H, or, better, from Theorem 7.2, which produces the infinite object itself. The same search machinery supplies the remaining controls: corrupting one row of the L(Petersen) labelling by transposing two entries leaves it a labelling but destroys validity; a constant row and an injective row landing outside the neighbourhood are both rejected; the claim "every non-Cayley vertex-transitive graph has an unsatisfiable truncation" is refuted by $T(6)$ and L(Petersen), and the claim "every vertex-transitive graph has all truncations satisfiable" is refuted by the Petersen graph.

9. The catalogue

Proposition 9.1. *The sixteen graphs C_5, K_4, Q_3, $K_{3,3}$, Petersen, $T(5)$, Heawood, $L(Petersen)$, $KG(6,2)$, $T(6)$, Möbius–Kantor, Q_4, Pappus, Desargues, dodecahedron and Coxeter are regular of the stated degrees and vertex-transitive, with an explicit automorphism carrying the base point to each* *vertex; hence each walk poset is independent of the base point and is a finite-type* $\mathbb{N}$*-graded upho poset (*[1, Proposition 3.4, Remark 3.5]*). Their rank sequences* $|V_0|, \dots, |V_6|$ *are as computed; for instance* $1, 3, 7, 9, 10, 10, 10$ *for the Petersen graph,* $1, 4, 13, 15, 15, 15, 15$ *for* L*(Petersen),* $1, 8, 15, 15, 15, 15, 15$ *for* $T(6)$*,* $1, 3, 7, 15, 19, 20, 20$ *for the dodecahedron and* $1, 3, 7, 15, 25, 27, 28$ *for the Coxeter graph. Moreover Desargues* = *Petersen* $\times K_2$*,* $T(5)$ *is the complement of the Petersen graph, and* $T(6)$ *is the complement of* $KG(6, 2)$*.*

The Petersen and L(Petersen) rank sequences reproduce the values recorded in [1, §4.2 and Lemma 4.2].

Theorem 9.2. *None of the four graphs Petersen,* $T(5)$*,* $KG(6, 2)$*, Coxeter admits a twisted regular representation, and none of their walk posets carries a multiplication; so none of the four is multiplicable.*

Proof. Proposition 5.2 for the first claim, the section search returning nothing in each case (Remark 5.3); then Theorem 4.1 with $N_0 = 4, 2, 2, 6$ respectively, the hypotheses being finite checks discharged through Proposition 4.2. □

All four cells are already covered by [1]: the Petersen graph is [1, Theorem 3.7], and $T(5)$, $KG(6, 2)$ and the Coxeter graph satisfy conditions (1)–(3) of [1, Remark 3.10]—each is connected vertex-transitive, spectrally ±-free with 0 not an eigenvalue, as verified here by an exact computation over $\mathbb{Q}$, and non-Cayley, which the empty section search itself establishes through Proposition 3.4. What is new about them is only that the implication is now checked end to end from the definition of the poset, with no informal step.

Proposition 9.3 (The criterion is not non-Cayleyness)**.** L*(Petersen) admits a twisted regular representation, exhibited explicitly, in which ten of the fifteen maps* h_v *are* not *graph automorphisms; the conjugator may be taken to be the identity.*

L(Petersen) is not a Cayley graph ([7, Proposition 5.1]) and its walk poset *is* multiplicable ([1, Theorem 4.1], and Theorem 7.3 above). So the criterion of Theorem 4.1 is strictly weaker than Sabidussi's regular subgroup of $\mathrm{Aut}(G)$, and an empty section search is not an artefact of non-Cayleyness. This is the non-vacuity control for Proposition 5.2: the section search is not empty by construction. Numerically, $\Gamma(L(\text{Petersen}))$ has 360 elements against $|\mathrm{Aut}| = 120$, and exactly 6 regular sections.

Collecting the results, fifteen of the sixteen graphs of Proposition 9.1 are decided:

graph	$P(G, v_0)$	route
C_5, K_4, Q_3, $K_{3,3}$, Heawood, Möbius–Kantor, Q_4, Pappus	multiplicable	Theorem 7.1 ([1, Ex. 3.6])
L(Petersen)	multiplicable	Theorem 7.3 ([1, Thm 4.1])
$T(6) = L(K_6) = J(6, 2)$	multiplicable	Theorem 7.3
Petersen	not multiplicable	Theorem 9.2 ([1, Thm 3.7])
$T(5)$, $KG(6, 2)$, Coxeter	not multiplicable	Theorem 9.2 ([1, Rem. 3.10])
dodecahedron	not multiplicable	Theorem 6.4
Desargues	open	—

Remark 9.4 (The remaining cell). The Desargues graph is bipartite, so the stabilisation hypothesis of Theorem 4.1 fails by construction; and it is not a Cayley graph ($GP(10, 3)$, $3^2 = 9 \not\equiv 1 \pmod{10}$, [8]), so Theorem 7.1 does not apply either. It is nonetheless not a hard cell: by Proposition 9.1 it is the bipartite double cover of the Petersen graph, and for connected non-bipartite G the walk poset of $G \times K_2$ at $(v_0, 0)$ is $P(G, v_0)$, because rank m forces the part m mod 2. Transporting Theorem 9.2 along that isomorphism would settle it. We record this as an observation: the transport of Definition 2.1 along a rank-preserving order isomorphism is not part of the machine-checked development, and we make no formal claim about the Desargues graph.

Remark 9.5 (The positive engine at the dodecahedron). The dodecahedron admits no regular subgroup of $\mathrm{Aut}(G)$ (Corollary 6.6), so the period-one case of Proposition 8.4 is unavailable there; and

Theorem 6.2 shows that no cover representation exists either. Since a rank-periodic transport model of period P exists precisely when the layered digraph $G \otimes \vec{C}_P$ is a Cayley digraph, which for odd P needs G itself to be a Cayley graph and for even P needs a regular subgroup of Γ whose s-image is Γ-conjugate to it, no such model exists at *any* period. (The case $P = 2$ is the statement that Dodecahedron $\times K_2$ is not a Cayley graph, which is also the smallest example of [9].) The equivalence just quoted is an argument on paper; what is machine-checked is the non-existence of the finite objects.

10. Verification

Every statement above has been formally verified in Lean 4 against *Mathlib*; the development is free of `sorry` and free of user-declared axioms. The distinction that matters is between statements proved by the Lean kernel and statements that delegate a single finite verdict to compiled evaluation via `native_decide`. *Kernel-checked throughout*, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: Lemma 2.2; Theorem 4.1 and Proposition 4.2 in their entirety—the criterion adds no computational axiom to anything it is composed with; Propositions 3.2, 5.1, 5.2, 8.1 and 8.2, so that the *completeness* of both searches and the *soundness* of the truncation search are symbolic facts, not measurements; Theorem 7.1 and all eight of its instances; Theorem 7.2 and Theorem 7.3, end to end, including the entrywise checks of the two cover representations; Proposition 9.3; and Proposition 8.4 with its instances. *Delegated to compiled evaluation* is exactly the finite verdict of each search: three Booleans for the dodecahedron—the size of the two-fold list, that each of its entries is a two-fold automorphism, and that no regular section is twisted-stable—which are the only computational axioms behind Theorem 6.2 and hence Theorem 6.4; and exactly one Boolean per graph, the emptiness of the section search, behind Theorem 9.2. Proposition 8.3 carries two such Booleans (the truncation search itself and the completeness of its candidate table), while the satisfiability halves of Propositions 8.3 and 8.6 are kernel evaluations of explicit witnesses. Proposition 9.1 is kernel-evaluated for fifteen of the sixteen graphs, the two on twenty vertices included; only the three statements about the Coxeter graph, the largest at twenty-eight vertices—its 3-regularity, its rank sequence and its vertex-transitivity witness—are delegated to compiled evaluation. Every numerical value quoted in Remarks 5.3 and 6.3 and in Propositions 8.3 and 8.6 is a measurement, not a theorem; each was produced by an independently written implementation before the formal proof was attempted, and the truncation node counts agree to the node across three separate solvers. The spectral computations of Remark 6.5 were carried out exactly over $\mathbb{Q}$ by the Faddeev–LeVerrier algorithm followed by a gcd of $p(x)$ and $p(-x)$; only the kernel vector of Theorem 6.4 is formalised.

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