VOUCHER COSTS OF PERMUTATIONS: THE SEVEN EXCEPTIONS ARE THE ONLY ONES

KOREA SUPERINTELLIGENCE LABS

Abstract. A buying order is a permutation $v = (v_1, \dots, v_n)$ of $\{1, \dots, n\}$, and its *voucher cost* is $C_V(v) = v_1 + v_1v_2 + v_2v_3 + \cdots + v_{n-1}v_n$. Call a positive integer c *achievable* if $c = C_V(v)$ for some buying order of some length. The authors of [1], who study this quantity through its extremes, observed that $2, 5, 6, 12, 13, 14, 22$ are not achievable and that a computer search found no further exceptions; the completeness of that list is left there as an observation supported by a counting heuristic, not as a theorem. We prove it: a positive integer is a voucher cost if and only if it is none of those seven. The proof splits at 215. Below that bound 207 explicit buying orders settle every value. Above it the argument is a *deficit calculus* anchored at the identity buying order: three exact reindexing identities move a permutation of $\{1, \dots, M\}$ to one of $\{1, \dots, M+2\}$ with a controlled change in cost, two tables of 266 inner orders at $M = 8, 9$ seed them, and the resulting cost bands overlap from $[215, 325]$ upward. All statements are machine-checked in Lean 4, and the completeness theorem uses no compiled evaluation: its finite content is 1346 explicit permutations verified by kernel reduction.

1. Introduction

The objects. A voucher with price tag N makes the next purchase cost N times its own price tag. If the tags $1, 2, \dots, n$ are bought in the order $v = (v_1, v_2, \dots, v_n)$, the first purchase costs v_1 and each later one costs $v_{i-1}v_i$, so the total is

$$C_V(v) \;=\; v_1 + v_1v_2 + v_2v_3 + \cdots + v_{n-1}v_n. \tag{1}$$

This is the *voucher cost* of the *buying order* v, in the sense of [1, Definition 1], where the quantity is drawn from a puzzle and then studied through its extremes: the minimum $m_V(n)$ and the maximum $M_V(n)$ of C_V over all $n!$ buying orders of $\{1, \dots, n\}$. Both are known in closed form,

$$m_V(n) = \frac{n^3 + 8n}{6} - \frac{15 - 3(-1)^n}{12}, \qquad M_V(n) = \frac{2n^3 + 3n^2 - 11n + 18}{6} \qquad (n \ge 2), \tag{2}$$

by [1, Example 15 and Corollary 14]; the values of m_V are the OEIS sequence [2], while M_V coincides with the maximal *loop* cost M_L, again by [1, Corollary 14], and its values are [3]. Extremal problems for sums of adjacent products of a permutation are classical — the maximum of the cyclic sum $x_1x_2 + \cdots + x_{n-1}x_n + x_nx_1$ is a Putnam problem [4], and the "smoothest and roughest permutations" are a Monthly problem [5] — but the quantity (1), with its unpaired leading term v_1, is the subject of [1].

The question. Beyond the extremes, [1, §6.1.1] asks which totals occur at all. Let

$$A \;=\; \{\, c \ge 1 \;:\; c = C_V(v) \text{ for some buying order } v \text{ of some length } n \ge 1 \,\}$$

be the set of *achievable* voucher costs, where n is not fixed: a cost is achievable if *some* number of vouchers realizes it. Computing the achievable set for each small n, the authors of [1] find seven positive integers that never occur and write that their

> "current list of non-achievable voucher costs is 2, 5, 6, 12, 13, 14, 22. … For a given n, the range of voucher costs is in the order of n^3. On the other hand, we have $n!$ permutations. It is not surprising that our program did not find any more non-achievable costs."

That is the whole of the evidence offered: a finite search and a count of permutations against a range. Write

$$\mathcal{G} = \{2,\ 5,\ 6,\ 12,\ 13,\ 14,\ 22\}.$$

The seven members of $\mathcal{G}$ are genuinely missing, and [1] establishes it: six of them fall in the gaps between $M_V(n)$ and $m_V(n+1)$ for $n \le 3$, and 22 is excluded there by appeal to that reference's classification of the orders of length 4, which lists their loop costs ([1, Example 3]). What is open is the converse.

The obstruction is not merely that the search is finite. Consecutive ranges $[m_V(n), M_V(n)]$ overlap once $n \ge 5$ — [1, Proposition 30] shows that $M_V(n) > m_V(n+1)$ holds if and only if $n \ge 5$ — so no value above $m_V(5) = 26$ can be missed for want of range. But an overlap argument controls only the ends of the ranges. A value can fail to be achievable while sitting strictly *inside* a range, and 22 does exactly that: the $n = 4$ costs fill $[15, 25] \setminus \{22\}$. Internal holes are a real phenomenon in this problem, and the only evidence in [1] that they stop is the program's silence.

What is settled here.

Theorem 1.1 (Completeness)**.** *For every integer $c \ge 1$,*

$$c \in A \quad \Longleftrightarrow \quad c \notin \{2,\ 5,\ 6,\ 12,\ 13,\ 14,\ 22\}.$$

So the list of [1] is complete, and the achievable set is $\{1, 2, 3, \dots\} \setminus \mathcal{G}$ exactly. The forward implication is the part already known; we reprove it uniformly in n in Section 3. The content is the converse, and it splits at 215:

- for $c \le 214$, one explicit buying order per value (Section 4);
- for $c \ge 215$, a construction (Sections 5 and 6).

The construction is a *deficit calculus* anchored at the identity buying order. For a permutation s of $\{1, \dots, M\}$ we measure not the cost of s but its shortfall d below a fixed anchor $S(M) = \sum_{k \le M} k(k{+}1)$, and we show that two exact reindexings — append-two and complement — act on the pair (M, d) by $(M, d) \mapsto (M{+}2, d)$ and $(M, d) \mapsto (M{+}2, d{+}(M{+}2)^2{-}1)$, while a third, translate-and-frame, turns a shortfall at M into an achievable cost at $M{+}2$. Two seed tables at $M = 8$ and $M = 9$ then propagate to every $M \ge 8$, and the resulting cost bands $[\,S(M+1)+1-\Delta(M),\ S(M+1)-5\,]$ overlap consecutively from $[215, 325]$ on.

Anchoring at the identity is what makes the induction close. A more obvious route — grow a buying order one tag at a time and track the interval of costs reachable from each end-state — forces bookkeeping in which each end-state's interval is frozen once created, and the resulting growth recursion sits at a critical point where the margins neither grow nor decay. The deficit calculus removes the bookkeeping instead of solving it: all of its witnesses live in a single one-parameter family.

Every statement below is machine-checked in Lean 4 (Section 7), and Theorem 1.1 in particular uses no compiled evaluation: its finite content is $873 + 207 + 266 = 1346$ explicit permutations, each verified by kernel reduction.

2. Preliminaries

Throughout, a *buying order of length* n is a permutation $v = (v_1, \dots, v_n)$ of $\{1, \dots, n\}$, its voucher cost $C_V(v)$ is given by (1), and C_V of the empty order is 0. We write

$$P(v) = v_1 v_2 + v_2 v_3 + \cdots + v_{n-1} v_n, \qquad \text{so} \qquad C_V(v) = v_1 + P(v),$$

P being the *pairwise cost* of [1]; the same reference also studies the *loop cost* $C_L(v) = P(v) + v_n v_1$, which we need only when reproducing its printed data in Section 7. Achievability requires $n \ge 1$, so $0 \notin A$ and $1 \in A$ (buy the single voucher 1); this matches the convention of [1], whose exception list starts at 2.

The one inequality the whole paper runs on is elementary.

Lemma 2.1. *For every buying order v of length n,*

$$\frac{n(n+1)}{2} \le C_V(v).$$

Proof. Every tag is at least 1, so $v_i v_{i+1} \ge v_{i+1}$ for each i, and summing over $i = 1, \dots, n-1$ gives $P(v) \ge v_2 + \cdots + v_n$. Adding v_1 to both sides yields $C_V(v) \ge v_1 + \cdots + v_n = n(n+1)/2$. □

Lemma 2.1 is what makes the exception list a finite matter: a cost bounded above bounds the length of any order realizing it. It is not needed in [1], whose gap argument uses the exact minimum m_V of (2) instead; the weaker bound is the one that is uniform in n and decidable.

For the constructions of Section 5 put

$$S(M) = \sum_{k=1}^{M} k(k+1) = \frac{M(M+1)(M+2)}{3}. \tag{3}$$

3. The seven exceptions

Theorem 3.1. *None of* $2, 5, 6, 12, 13, 14, 22$ *is an achievable voucher cost: for each such c and every $n \ge 1$, no buying order of* $\{1, \dots, n\}$ *costs c.*

Proof. Let $c \in \mathcal{G}$ and suppose $C_V(v) = c$ for a buying order v of length n. Since $c \le 22$, Lemma 2.1 gives $n(n+1) \le 2c \le 44$, hence $n \le 6$. It therefore suffices to check the $1! + 2! + \cdots + 6! = 873$ buying orders of length at most 6, and none of them has a cost in $\mathcal{G}$. This exhaustion is the only computation in the proof; it was carried out by kernel reduction inside the formal development, over the complete list of 873 permutations. □

This theorem is a reproduction, not a new result: [1] derives all seven rigorously. What the argument above adds is uniformity — a single finite search settles every n at once, because Lemma 2.1 caps the length — and a form suited to being combined with Section 6.

Remark 3.2. The neighbours of the hole are achievable: 21 and 23 occur at $n = 4$, and $26 = m_V(5)$, the first value above the last exception, occurs at $n = 5$. Also 22 is missing at $n = 4$ specifically, not merely unsearched there. These checks are recorded in the formal development as controls against an argument that overshoots or undershoots by one.

4. Witnesses in an initial range

The converse direction needs, for each achievable c, one buying order that realizes it. Below the point where the construction of Section 5 takes over, we simply exhibit them.

Theorem 4.1. *For every c with* $1 \le c \le 2000$*,*

$$c \in A \iff c \notin \mathcal{G}.$$

Proof. The implication $\Rightarrow$ is Theorem 3.1. For $\Leftarrow$ we use a table W indexed by $0 \le c \le 2000$ whose entry $W[c]$ is a list of positive integers, empty exactly at $c = 0$ and at the seven members of $\mathcal{G}$, and otherwise a buying order costing c. For each of the 1993 remaining indices one verifies that $W[c]$ is a permutation of $\{1, \dots, |W[c]|\}$, that $|W[c]| \ge 1$, and that $C_V(W[c]) = c$; each such verification produces $c \in A$ directly from the definition. The entries with $c \le 214$ are verified by kernel reduction, the sweep to 2000 by one compiled evaluation. □

The table was produced by a per-target hill climb, but nothing about the search enters the proof: each entry is re-verified from scratch, and no structural property of the chosen witnesses is used anywhere. Theorem 4.1 is again not a new mathematical statement — the program of [1] made the same observation — but it fixes an explicit bound, which that observation did not, and it supplies the finite input to Theorem 1.1. Only the initial segment matters for that purpose:

Corollary 4.2. *Every c with $1 \le c \le 214$ and $c \notin \mathcal{G}$ is achievable, by one of 207 explicit buying orders, all verified by kernel reduction.*

Remark 4.3. Outside the formal development, the achievable set was also computed exactly for each fixed $n \le 11$ by dynamic programming over (subset, last tag) with cost bitsets. The results: the sets for $n = 1, 2, 3$ are $\{1\}$, $[3, 4]$, $[7, 11]$; for $n = 4$ it is $[15, 25] \setminus \{22\}$; and for every n from 5 to 11 it is the *full* interval $[m_V(n), M_V(n)]$, of sizes $23, 40, 65, 97, 139, 190, 253$. So in this range all internal holes occur at $n = 4$, and the union of the missing values up to 487 is exactly $\mathcal{G}$. This computation is reported for orientation only; nothing below depends on it.

5. The deficit calculus

Fix $M \ge 1$ and let $s = (s_1, \dots, s_M)$ be a permutation of $\{1, \dots, M\}$. Define its *block cost*

$$b(s) \;=\; s_1 s_2 + s_2 s_3 + \cdots + s_{M-1} s_M + (M+1) s_M, \tag{4}$$

that is, the pairwise cost of s together with one extra link joining its last entry to a phantom tag $M + 1$. The identity order $\mathrm{id}_M = (1, 2, \dots, M)$ has

$$b(\mathrm{id}_M) = \sum_{k=1}^{M-1} k(k+1) + (M+1)M = \sum_{k=1}^{M} k(k+1) = S(M),$$

so $S(M)$ of (3) is a natural anchor, and we measure every other order by how far it falls below it.

Definition 5.1. An integer $d \ge 0$ is a *deficit at M*, written $d \in \mathcal{D}(M)$, if some permutation s of $\{1, \dots, M\}$ satisfies $b(s) + d = S(M)$.

Thus $0 \in \mathcal{D}(M)$ for every $M \ge 1$, realized by id_M. The point of the definition is that deficits transform under two reindexings by an exact rule, and convert into voucher costs by a third.

Proposition 5.2 (Lift, nest, step)**.** *Let $M \ge 1$ and let $d \in \mathcal{D}(M)$, realized by s.*

- (L) (Lift.) *The list $v = (1,\ s_1{+}1,\ s_2{+}1,\ \dots,\ s_M{+}1,\ M{+}2)$ is a buying order of $\{1, \dots, M{+}2\}$ and*
$$C_V(v) + d \;=\; S(M+1) + 1.$$
- (N) (Nest.) $(s_1, \dots, s_M,\ M+1,\ M+2)$ *realizes d at $M + 2$; hence $\mathcal{D}(M) \subseteq \mathcal{D}(M+2)$.*
- (S) (Step.) $(M{+}2,\ M{+}2{-}s_1,\ \dots,\ M{+}2{-}s_M,\ 1)$ *realizes $d + (M+2)^2 - 1$ at $M + 2$; hence $d \in \mathcal{D}(M) \Rightarrow d + (M+2)^2 - 1 \in \mathcal{D}(M+2)$.*

Proof. All three are exact computations along s, using

$$\Sigma := \sum_{i=1}^{M} s_i = \frac{M(M+1)}{2} \qquad \text{and} \qquad S(m+1) = S(m) + (m+1)(m+2),$$

the latter from (3). Note also $S(M + 2) = S(M) + 2(M + 2)^2$.

(L) The entries of v are 1, the M values $s_i + 1 \in \{2, \dots, M + 1\}$, and $M + 2$, so v is a buying order of $\{1, \dots, M + 2\}$. Expanding $(s_i + 1)(s_{i+1} + 1) = s_i s_{i+1} + s_i + s_{i+1} + 1$ over $i = 1, \dots, M - 1$ gives

$$\sum_{i=1}^{M-1} (s_i + 1)(s_{i+1} + 1) = P(s) + 2\Sigma - s_1 - s_M + (M-1),$$

and therefore

$$C_V(v) = 1 + (s_1 + 1) + \bigl[P(s) + 2\Sigma - s_1 - s_M + M - 1\bigr] + (s_M + 1)(M+2) = b(s) + M(M+1) + 2M + 3,$$

where the terms $-s_M$ and $s_M(M + 2)$ have been recombined into the phantom link $(M + 1)s_M$ of $b(s)$. Substituting $b(s) = S(M) - d$ and $M(M + 1) + 2M + 3 = (M + 1)(M + 2) + 1$ turns this into $C_V(v) = S(M + 1) + 1 - d$.

(N) Appending $M+1, M+2$ adds the links $s_M(M+1)$ and $(M+1)(M+2)$ to the pairwise cost and replaces the phantom link $(M+1)s_M$ by $(M+3)(M+2)$, so the block cost increases by $(M+1)(M+2)+(M+3)(M+2) = 2(M+2)^2$. Since $S(M+2) = S(M)+2(M+2)^2$, the deficit is unchanged.

(S) Write $K = M+2$. The map $a \mapsto K-a$ carries $\{1,\dots,M\}$ onto $\{2,\dots,M+1\}$, so $w = (K, K-s_1, \dots, K-s_M, 1)$ is a buying order of $\{1,\dots,K\}$. Expanding $(K-s_i)(K-s_{i+1}) = K^2 - K(s_i+s_{i+1}) + s_i s_{i+1}$ over $i = 1,\dots,M-1$ and adding the two boundary links $K(K-s_1)$ and $(K-s_M)\cdot 1$ gives

$$P(w) = MK^2 - 2K\Sigma + Ks_M + P(s) + K - s_M,$$

and the phantom link of w contributes $(M+3)\cdot 1$, so

$$b(w) = P(s) + (K-1)s_M + MK^2 - 2K\Sigma + K + M + 3 = b(s) + MK^2 - KM(M+1) + 2M + 5.$$

Since $K - M - 1 = 1$ we have $MK^2 - KM(M+1) = KM(K-M-1) = M(M+2)$, whence $b(w) = b(s) + M^2 + 4M + 5$. Subtracting from $S(M+2) = S(M) + 2(M+2)^2$ leaves the deficit $d + M^2 + 4M + 3 = d + (M+2)^2 - 1$. □

Nest and step move M by two at a time and change d by 0 and by $(M+2)^2 - 1$ respectively, so define

$$\Delta(0) = \Delta(1) = 0, \qquad \Delta(M+2) = \Delta(M) + (M+2)^2 - 1. \tag{5}$$

Then $\Delta(8) = 116$ and $\Delta(9) = 160$. We do not claim $\Delta(M)$ is the largest deficit at M; only that deficits up to $\Delta(M)$ all occur, which is what the argument uses.

Proposition 5.3 (Seed tables). *Every d with $6 \le d \le 116$ lies in $\mathcal{D}(8)$, and every d with $6 \le d \le 160$ lies in $\mathcal{D}(9)$.*

Proof. Two explicit tables, of 111 and 155 permutations of $\{1,\dots,8\}$ and $\{1,\dots,9\}$: the i-th entry of the first has $b(s)+(6+i) = S(8) = 240$, the i-th entry of the second has $b(s)+(6+i) = S(9) = 330$. All 266 rows were verified by kernel reduction — permutation and cost identity for each row — which is the entire finite content of the tail construction. □

Remark 5.4. The left end 6 is not slack. Among the permutations of $\{1,\dots,5\}$ — checked exhaustively in the formal development — no order has deficit 1, 2, 4 or 5, while 0 and 3 do occur (realized by $(1,2,3,4,5)$ and $(2,1,3,4,5)$). Small deficits are genuinely sparse, and that is why the windows here start at 6 rather than at 0.

Proposition 5.5 (Covering). *For every $M \ge 8$ and every d with $6 \le d \le \Delta(M)$, we have $d \in \mathcal{D}(M)$.*

Proof. Induction on M in steps of two, so that the two parities are seeded separately by Proposition 5.3 at $M = 8$ and $M = 9$.

First a gluing inequality: for all $M \ge 8$,

$$\Delta(M) \;\ge\; M^2 + 4M + 8. \tag{6}$$

It holds at $M = 8$ ($116 \ge 104$) and at $M = 9$ ($160 \ge 125$), and if it holds at M then by (5),

$$\Delta(M+2) = \Delta(M) + (M+2)^2 - 1 \;\ge\; (M^2 + 4M + 7) + (M+2)^2,$$

which is at least $(M+2)^2 + 4(M+2) + 8 = (M+2)^2 + 4M + 16$ as soon as $M^2 \ge 9$; both parities are therefore covered for $M \ge 8$.

Now let $M \ge 10$, assume the claim at $M-2$, and take $6 \le d \le \Delta(M)$. If $d \le \Delta(M-2)$ then $d \in \mathcal{D}(M-2) \subseteq \mathcal{D}(M)$ by Proposition 5.2(N). Otherwise $d > \Delta(M-2)$, and (6) at $M-2$ gives

$$d \;\ge\; \Delta(M-2) + 1 \;\ge\; (M-2)^2 + 4(M-2) + 9 \;=\; M^2 + 5,$$

so $d' = d - (M^2 - 1) \ge 6$; also $d' \le \Delta(M) - (M^2 - 1) = \Delta(M-2)$ by (5). Hence $d' \in \mathcal{D}(M-2)$ by the inductive hypothesis, and $d = d' + (M^2 - 1) \in \mathcal{D}(M)$ by Proposition 5.2(S) applied at $M-2$. □

6. Completeness

Proposition 6.1 (Bands). *Let $M \geq 8$. Every integer c with*

$$S(M+1)+1-\Delta(M) \;\leq\; c \;\leq\; S(M+1)-5$$

is achievable, by a buying order of length $M+2$.

Proof. Put $d = S(M+1)+1-c$. The two inequalities say exactly $6 \leq d \leq \Delta(M)$, so $d \in \mathcal{D}(M)$ by Proposition 5.5, and Proposition 5.2(L) produces a buying order v of $\{1,\dots,M+2\}$ with $C_V(v) = S(M+1)+1-d = c$. □

Write $B_M = [\,S(M+1)+1-\Delta(M),\; S(M+1)-5\,]$ for the band at M. Since $S(9) = 330$ and $\Delta(8) = 116$, the first band is $B_8 = [215, 325]$.

Theorem 6.2 (The tail). *Every integer $c \geq 215$ is an achievable voucher cost.*

Proof. It suffices that the bands $B_8, B_9, B_{10}, \dots$ cover $[215, \infty)$, since their left endpoint at $M = 8$ is 215 and $S(M+1) \to \infty$. Consecutive bands meet iff the left endpoint of B_{M+1} is at most one more than the right endpoint of B_M, that is, iff

$$S(M+2)+1-\Delta(M+1) \;\leq\; S(M+1)-4,$$

and since $S(M+2) - S(M+1) = (M+2)(M+3)$ by (3), this is the second gluing inequality

$$\Delta(M+1) \;\geq\; (M+2)^2 + (M+2) + 5 \qquad (M \geq 8). \tag{7}$$

As with (6), (7) follows from (5) by induction in steps of two. It holds at $M = 8$, where it reads $\Delta(9) = 160 \geq 115$, and at $M = 9$, where it reads $\Delta(10) = \Delta(8)+99 = 215 \geq 137$. Passing from M to $M+2$ increases the left side by $(M+3)^2-1$ and the right side by $(M+4)^2-(M+2)^2+2 = 4M+14$, and $(M+3)^2 - 1 \geq 4M+14$ for all $M \geq 2$.

Given $c \geq 215$, choose the least $M \geq 8$ with $c+5 \leq S(M+1)$; such an M exists because S is unbounded. If $M = 8$ then $c \leq S(9)-5 = 325$, and $c \geq 215$ places c in B_8. If $M > 8$ then minimality gives $S(M) \leq c+4$, while (7) at $M-1$ reads $\Delta(M) \geq (M+1)^2+(M+1)+5 = \big(S(M+1)-S(M)\big)+5$; combining,

$$S(M+1)+1-\Delta(M) \;\leq\; S(M)-4 \;\leq\; c,$$

so again $c \in B_M$. Either way Proposition 6.1 applies. □

Proof of Theorem 1.1. ($\Rightarrow$) If $c \in \mathcal{G}$ then $c \notin A$ by Theorem 3.1.
($\Leftarrow$) Let $c \geq 1$ with $c \notin \mathcal{G}$. If $c \geq 215$, apply Theorem 6.2. If $c \leq 214$, apply Corollary 4.2. □

Three finite computations enter Theorem 1.1, and no others: the 873 buying orders of length at most 6 behind Theorem 3.1; the 207 tabulated witnesses for $1 \leq c \leq 214$ behind Corollary 4.2; and the 266 tabulated inner orders at $M = 8, 9$ behind Proposition 5.3. Everything else — the three identities of Proposition 5.2, the covering induction, and the band chaining — is uniform in M. The larger sweep of Theorem 4.1, which is the only place a compiled evaluation is used, is not needed for Theorem 1.1.

Corollary 6.3. *The achievable set A has complement $\mathcal{G}$ in $\{1,2,3,\dots\}$; in particular $|A^c| = 7$, every $c \geq 23$ is achievable, and the largest non-achievable voucher cost is 22.*

Remark 6.4. In a search run on 2026-08-28, neither the sequence $2, 5, 6, 12, 13, 14, 22$ nor its complement in the positive integers returned a match in the OEIS, in either comma- or space-separated form; a control query in the same format did return matches, so the queries were well formed.

7. Verification

Every statement in Sections 2–6 is formalized and machine-checked in Lean 4 against Mathlib, with the definitions transcribed directly from [1]: eleven modules covering the cost functions, the bound of Lemma 2.1, the exception theorem, the witness tables, the deficit calculus and its seed tables, and the completeness assembly. Theorem 1.1 (`achievable_iff`) depends only on `propext`, `Classical.choice` and `Quot.sound`: every finite check underneath it — the 873 short orders, the 207 witnesses for $c \leq 214$, the 266 seed rows — is performed by kernel reduction, with no appeal to compiled evaluation. Compiled evaluation (`native_decide`) enters the development in exactly two places: the extension of the witness sweep to 2000 (Theorem 4.1), and one validation sweep over the 5040 and 40320 orders of length 7 and 8. Neither lies under Theorem 1.1. Two further groups of checks guard the formalization itself rather than proving anything: a validation group that reproduces the printed values of [1] — the introductory example $C_V(2,3,4) = 20$, the $n = 3$ cost table, the loop costs $\{21, 24, 25\}$ at $n = 4$, the closed forms (2) as exact extrema for $2 \leq n \leq 8$, and the extremal orders at $n = 8$ — so that a misreading of the definitions would fail there rather than propagate; and a control group (Remarks 3.2 and 5.4) that refutes the natural too-strong and too-weak variants of each claim and confirms that the permutation constraint and the quantifier over n are both load-bearing. The development compiles in roughly 96 seconds. Repository reference to be added at submission.

References

[1] Chris Chen, Vivian Chen, Ray Cui, Ermin Dong, Alexander Radul, Lev Radul, Jack Shan, Arthur Shu, Kenneth Sun, Kenneth Wood, William Zelevinsky, Brian Zhao (PRIMES STEP), and Tanya Khovanova (MIT), *From a Voucher Puzzle to Extremal Sums of Adjacent Products*, arXiv:2606.16775v1 [math.CO], 15 June 2026. All numbered references above are to v1, which as of 2026-08-28 is the only version listed.

[2] The On-Line Encyclopedia of Integer Sequences, sequence A393145, "Minimal voucher cost for n distinct positive integer vouchers `v_1, v_2, ..., v_n`, where the cost is defined as `v_1 + v_1*v_2 + v_2*v_3 + ... + v_(n-1)*v_n`." Terms $1, 3, 7, 15, 26, 43, 65, 95, 132, 179, 235, \ldots$; contributed by Tanya Khovanova and the PRIMES STEP junior group, 2 February 2026. Retrieved 2026-08-28.

[3] The On-Line Encyclopedia of Integer Sequences, sequence A110610, "Maximal value of `sum(p(i)p(i+1),i=1..n)`, where `p(n+1)=p(1)`, as p ranges over all permutations of $\{1, 2, \ldots, n\}$." Terms $1, 4, 11, 25, 48, 82, 129, 191, \ldots$, with $a(n) = (2n^3 + 3n^2 - 11n + 18)/6$ for $n \geq 2$; contributed by Emeric Deutsch, 30 July 2005. Retrieved 2026-08-28.

[4] The Fifty-Seventh William Lowell Putnam Mathematical Competition (7 December 1996), Problem B-3. Competition report by L. F. Klosinski, G. L. Alexanderson and L. C. Larson, *Amer. Math. Monthly* **104** (1997), no. 8, 744–754.

[5] V. Mihai and M. Woltermann, *The Smoothest and Roughest Permutations: 10725*, *Amer. Math. Monthly* **108** (2001), no. 3, 272–273.