FIVE VOTERS INDUCE THE ORDER-21 SUBTOURNAMENTS OF THE PALEY TOURNAMENT ON 23 VERTICES

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ABSTRACT. A tournament is *m-inducible* if it is the majority tournament of m linear orders of its vertices, and the *margin* of an arc in such a profile is the number of voters ranking it forward minus the number ranking it backward. Chindelevitch and Harutyunyan [1] exhibited Paley(43) — the first explicit tournament of small size that is not 5-inducible — conjectured that the classes $\mathcal{I}_{m,t}$ of tournaments inducible with all margins at most t form a strictly increasing hierarchy, and left the 5-inducibility of Paley(23) open, reporting that they could not decide even its order-21 subtournament. We settle that subtournament: it *is* 5-inducible, and by a profile with no unanimous arc, so it lies in $\mathcal{I}_{5,3}$; the same holds for all 253 two-vertex deletions of Paley(23) simultaneously, obtained by transporting one witness along the affine automorphisms. We also record explicit five-voter certificates for the seven order-20 subtournaments and for Paley(11), prove McG(Paley(11)) = 5 with a certified lower bound, and land the hierarchy's base level $\mathcal{I}_{3,1} \subsetneq \mathcal{I}_{3,3}$ on a named 9-vertex tournament. Two negative measurements delimit the next step: the extension route suggested in [1] is exhaustively rigid — no order-22 extension of the order-21 profile exists, and none of 37 nearby order-21 profiles extends either — and a complete census shows every tournament on at most nine vertices lies in $\mathcal{I}_{5,1}$, so a witness for the conjecture at $m = 5$ needs at least ten vertices. All theorems are machine-checked in Lean 4.

1. INTRODUCTION

Let T be a tournament on a finite vertex set V. A *voter* is a linear order of V; a *profile* is a finite list of voters; and the profile *induces* T when, for every ordered pair $u \to v$ of T, a strict majority of the voters rank u above v. The tournament T is *m-inducible* if some profile of m voters induces it, and its *McGarvey number* McG(T) is the least such m. McGarvey [6] proved that McG(T) is finite for every tournament; Bachmeier et al. [2] call it the majority dimension. Following [1] we write $N(m)$ for the least n such that some tournament on n vertices is not m-inducible. Every m-inducible tournament is $(m+2)$-inducible, and McG(T) is always odd [1, §1, §7].

The quantitative refinement that organises this paper is the *margin*. In a profile of m voters the margin of an arc is the number of voters ranking it forward minus the number ranking it backward; for odd m it is an odd number in $\{1, 3, \ldots, m\}$, and an arc of margin m is *unanimous*. For odd m and odd $t \leq m$, [1, §1] defines

$$\mathcal{I}_{m,t} \;=\; \{\, T : T \text{ has an inducing profile of } m \text{ voters, every arc of margin} \leq t \,\},$$

so that $\mathcal{I}_{m,1} \subseteq \mathcal{I}_{m,3} \subseteq \cdots \subseteq \mathcal{I}_{m,m} = \{m\text{-inducible}\}$. At $m = 3$ the only margins are 1 and 3, so a profile is margin-1 exactly when no arc is unanimous; from $m = 5$ on the two notions separate, an arc split 4:1 having margin 3.

Chindelevitch and Harutyunyan [1] study $m = 3$ and $m = 5$ in detail. Their headline is that the Paley tournament on 43 vertices — vertex set $\mathbb{Z}_{43}$, arc $u \to v$ whenever $v - u$ is a nonzero quadratic residue — is not 5-inducible, so $N(5) \leq 43$ with an explicit witness. It is the first explicit tournament of small size that is not 5-inducible: the explicit witnesses available before it had $\sim 6 \times 10^8$ vertices and rested on a bound for the domination number of a majority tournament of five voters [1, §7]. Their single conjecture is that the margin hierarchy never collapses.

Conjecture 1.1 ([1, Conjecture 8.1])**.** For every odd m and every odd t with $1 \leq t \leq m - 2$, $\mathcal{I}_{m,t} \subsetneq \mathcal{I}_{m,t+2}$.

They prove this only at $m = 3$, by a census: exactly 254 of the 191 536 tournaments on nine vertices are 3-inducible but not margin-1 3-inducible, and no such tournament exists on eight or fewer vertices [1, §5, Theorem G.2]. The case $m = 5$ they name as the first open one, in two concrete halves — a tournament in $\mathcal{I}_{5,5} \setminus \mathcal{I}_{5,3}$, and one in $\mathcal{I}_{5,3} \setminus \mathcal{I}_{5,1}$.

Their second open problem asks for the exact value of $N(5)$, currently $12 \le N(5) \le 38$, and singles out Paley(23): its 5-inducibility is open, and it "resisted about six weeks of SAT computation" in [2]. The authors then propose an attack, and report where it stopped. We quote [1, §8.2, Open Problem 2] in full, because it is the cell this paper settles:

> Its seven order-20 subtournaments (deletion classes of triples: $(q-2)/3$ classes for $q \equiv 2 \bmod 3$ by a Burnside count [4]) offer an extension route – induce a subtournament, then decide extendability of the frozen induction. We verified that all seven are 5-inducible, but could not determine whether the unique order-21 subtournament (unique since Aut acts transitively on unordered pairs of vertices) is.

Results. Since m-inducibility is hereditary under taking subtournaments [1, §1], a non-5-inducible order-21 subtournament would have decided Paley(23) in the negative. It is not one.

Theorem 1.2 (Theorems 3.1 and 3.3). *Let $P =$ Paley(23) and let $u \neq v$ be vertices of P. The subtournament of P on the remaining 21 vertices is the majority tournament of five voters, no arc of which is unanimous. That is, all 253 order-21 subtournaments of Paley(23) lie in $\mathcal{I}_{5,3}$, and in particular they are 5-inducible.*

The witness is a single explicit profile, printed as Table 1, together with a transport along the affine automorphisms $x \mapsto ax + b$ of Paley(23); we check that this group of order 253 acts *regularly* on the 253 unordered pairs of vertices, which is the precise form of the transitivity that [1] asserts when calling the order-21 subtournament unique (Proposition 3.2). Theorem 1.2 removes the cheapest available obstruction to the 5-inducibility of Paley(23) itself, which remains open.

Around that result we assemble the rest of the local picture, each item with an explicit certificate that the source states without one. All seven order-20 subtournaments lie in $\mathcal{I}_{5,3}$, not merely in $\mathcal{I}_{5,5}$ (Theorem 4.3); the seven representatives really do exhaust the order-20 subtournaments, by an orbit computation that verifies the Burnside count of [1] (Proposition 4.1); Paley(11) lies at the very bottom of the hierarchy, Paley(11) $\in \mathcal{I}_{5,1}$ (Theorem 6.1); and McG(Paley(11)) $= 5$ with the lower bound proved rather than cited, by four refutations replayed through a verified proof checker (Theorem 6.2). At the base of the margin hierarchy we name a tournament: an explicit 9-vertex tournament that is 3-inducible only via a unanimous arc, so that $\mathcal{I}_{3,1} \subsetneq \mathcal{I}_{3,3}$ is a proved strict inclusion with a witness one can print (Theorem 7.1).

Finally, two measurements say where the next step is not. The extension route proposed in the passage quoted above is far more rigid than it sounds: freezing an inducing profile and inserting one new vertex is an exhaustive search of size $(n + 1)^5$, and all three rungs we ran came back empty (Section 8). And a complete census of the 198 940 tournaments on five to nine vertices places every one of them in $\mathcal{I}_{5,1}$, so any witness for Conjecture 1.1 at $m = 5$ has at least ten vertices (Remark 7.2). Both are computations outside the formal development and are labelled as such.

Every theorem below is machine-checked in Lean 4; Section 9 says exactly what is delegated to compiled evaluation and what is not.

2. Preliminaries

We keep the vertex set as a list of labels rather than an initial segment of the integers, because every question here concerns induced subtournaments of one fixed tournament and a list of labels expresses "delete these vertices" with no relabelling.

Fix a finite list V of distinct vertices. A *ranking* is a list r that is a permutation of V, read best first; write $i \prec_r j$ when i occurs before j in r. A *profile of m voters on V* is a list $P = (r_1, \ldots, r_m)$

of rankings of V. For $i, j \in V$ put

$$s_P(i,j) = \#\{v : i \prec_{r_v} j\}, \qquad \mathrm{maj}_P(i,j) \iff m < 2\, s_P(i,j), \qquad \mu_P(i,j) = \big|\, s_P(i,j) - s_P(j,i) \,\big|.$$

Thus maj_P is the majority relation and $\mu_P(i,j) = |2s_P(i,j) - m|$ is the margin. A tournament on V is a Boolean function T with $T(i,i)$ false and $T(i,j) \neq T(j,i)$ for distinct $i, j \in V$; we only ever use T restricted to V, and write $T \restriction V$ when the restriction needs emphasis.

Definition 2.1. *P induces T on V within margin t* if for all distinct $i, j \in V$ we have $\mathrm{maj}_P(i,j) = T(i,j)$ and $\mu_P(i,j) \le t$. We write $T \restriction V \in \mathcal{I}_{m,t}$ if some profile of m voters on V does so, and call $T \restriction V$ *m-inducible* when $T \restriction V \in \mathcal{I}_{m,m}$.

Since a voter ranking both i and j ranks exactly one of them first, $s_P(i,j) + s_P(j,i) = m$ for every profile, which gives the two basic facts we need.

Proposition 2.2. *Let P be a profile of m voters on V.*

1. *$\mu_P(i,j) \le m$ for all i, j; consequently $\mathcal{I}_{m,t} = \mathcal{I}_{m,m}$ for $t \ge m$.*
2. *If m is odd then for all distinct $i, j \in V$ exactly one of $\mathrm{maj}_P(i,j)$, $\mathrm{maj}_P(j,i)$ holds: the majority relation of an odd profile really is a tournament on V.*

Part (2) is the non-vacuity check for Definition 2.1: without the hypothesis that each r_v is a permutation of V it fails, and the definition would be about nothing. The next lemma is what turns one witness into an orbit of witnesses, and is the only nontrivial piece of general theory the paper needs.

Lemma 2.3 (transport)**.** *Let $N \in \mathbb{N}$, let V be a list of vertices all $< N$, let f be injective on $\{0, \ldots, N-1\}$, and suppose $T(f(i), f(j)) = T(i,j)$ for all $i, j \in V$. If $T \restriction V \in \mathcal{I}_{m,t}$ then $T \restriction f(V) \in \mathcal{I}_{m,t}$.*

Proof sketch. Map each voter r of a witnessing profile to $f \circ r$. Injectivity of f on the vertices in play gives $i \prec_r j \iff f(i) \prec_{f \circ r} f(j)$ by induction on r, whence $s_{f \cdot P}(f(i), f(j)) = s_P(i,j)$; the majority and margin conditions transfer, and the arc condition transfers because f preserves T. The image profile consists of permutations of $f(V)$ because f is injective. $\square$

Paley tournaments. For a prime $q \equiv 3 \pmod 4$ let $Q_q \subset \mathbb{Z}_q$ be the set of nonzero quadratic residues. The *Paley tournament* $\mathrm{Paley}(q)$ has vertex set $\mathbb{Z}_q$ and an arc $u \to v$ whenever $v - u \in Q_q$. Because $q \equiv 3 \pmod 4$, exactly one of $a, -a$ lies in Q_q for each $a \neq 0$, which is what makes $\mathrm{Paley}(q)$ a tournament. We use $q = 11$ and $q = 23$; at $q = 23$,

$$Q_{23} = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}.$$

In the formal development $\mathrm{Paley}(q)$ is built from the residues by search rather than transcribed as an arc list, and that it is a tournament is verified at $q = 11$ and $q = 23$; as a negative control, $\mathrm{Paley}(9)$ is verified *not* to be a tournament, since $9 \equiv 1 \pmod 4$ makes -1 a residue and gives the pair $\{0, 1\}$ two arcs. The quadratic-residue construction is therefore not vacuously a tournament for every modulus, and the two positive checks are saying something.

Write $A_q = \{x \mapsto ax + b : a \in Q_q,\ b \in \mathbb{Z}_q\}$, a group of order $q(q-1)/2$ acting on $\mathbb{Z}_q$. Every element of A_q is an automorphism of $\mathrm{Paley}(q)$: translation leaves differences unchanged, and multiplication by a residue multiplies differences by a residue, the residues being closed under multiplication. At $q = 23$ we have $|A_{23}| = 11 \cdot 23 = 253$. We do not need, and do not verify, that A_q is all of $\mathrm{Aut}(\mathrm{Paley}(q))$: every statement below is about A_q, and what is checked about A_q is its order, that its elements are automorphisms of $\mathrm{Paley}(q)$, and its orbits.

Finally, for a tournament T on n vertices with $C = \binom{n}{2}$ arcs, $\mathrm{MAS}(T)$ denotes the maximum, over linear orders of the vertices, of the number of arcs of T that are forward in that order, and $\mathrm{slack}_5(T) = 5\,\mathrm{MAS}(T) - 3C$ [1, §1]. These appear only in remarks.

r_1	21	2	22	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r_2	15	8	16	9	2	17	10	18	3	11	19	4	12	20	13	5	21	14	6	22	7
r_3	12	16	18	22	5	11	17	7	13	19	2	6	8	14	20	3	9	15	21	4	10
r_4	21	14	7	11	4	20	13	17	10	3	16	6	19	9	22	12	2	15	5	18	8
r_5	20	6	15	19	10	5	14	9	18	13	4	22	8	17	3	12	21	7	16	2	11

TABLE 1. Five voters inducing Paley(23) on $V_{21} = \mathbb{Z}_{23} \setminus \{0, 1\}$, best first. Each of the 210 arcs is ranked forward by three or four of the five voters; none by all five.

3. THE ORDER-21 SUBTOURNAMENTS OF Paley(23)

Let $P = \text{Paley}(23)$ and let $V_{21} = \mathbb{Z}_{23} \setminus \{0, 1\}$, a set of 21 vertices spanning $\binom{21}{2} = 210$ arcs.

Theorem 3.1. *The five rankings of Table 1 induce* Paley(23) *on* V_{21} *with every margin at most* 3*. Hence* $\text{Paley}(23) \restriction V_{21} \in \mathcal{I}_{5,3}$*, and in particular the order-21 subtournament of* Paley(23) *is 5-inducible.*

Proof. Each of the five lists is a permutation of V_{21}, and for each of the 210 pairs $\{i, j\} \subset V_{21}$ one evaluates $s_P(i, j)$ and compares with the residue condition $j - i \in Q_{23}$. This is a finite computation over 210 pairs and five voters, discharged by evaluation in the proof kernel against the definition of Paley(23) from the quadratic residues; every pair comes out with $s_P \in \{3, 4\}$ on the forward side. □

The profile of Table 1 was found by the SAT solver CaDiCaL 2.1.2 [3] (the build shipped with the Lean toolchain) on the straightforward encoding of 5-inducibility — one Boolean $x_{v,a,b}$ per voter v and pair $a < b$, two transitivity clauses per voter and triple, and, per arc, the ten 3-subsets of the five agreement literals — giving 1050 variables and 15 404 clauses (including four clauses for the voter-ordering symmetry break of Section 8), solved in 4 min 37 s on one core. Nothing about that search enters the proof: only the five lists do, and they are re-checked against the definition. It is worth noting that the search asked only for 5-inducibility; that the profile returned has no unanimous arc, and hence certifies the stronger membership in $\mathcal{I}_{5,3}$, was observed afterwards.

We now push this single witness across the affine group A_{23}. The source justifies the phrase "the unique order-21 subtournament" by transitivity of $\text{Aut}(P)$ on unordered pairs; we verify the sharper statement.

Proposition 3.2. *The group* A_{23} *of* 253 *affine maps* $x \mapsto ax + b$*,* $a \in Q_{23}$*, acts regularly on the* $\binom{23}{2} = 253$ *unordered pairs of vertices of* Paley(23)*: every pair* $\{u, v\}$ *is the image of* $\{0, 1\}$ *under exactly one element of* A_{23}*.*

Proof. Both $|A_{23}| = 253$ and the count are finite computations: for each of the 253 pairs $\{u, v\}$ one counts the elements (a, b) of A_{23} with $\{b, a+b\} = \{u, v\}$, and the 253 counts are all equal to 1. Since A_{23} consists of automorphisms of Paley(23), the 253 subtournaments obtained by deleting a pair of vertices are pairwise isomorphic. □

Theorem 3.3. *For all* $u \neq v$ *in* $\mathbb{Z}_{23}$*, the subtournament of* Paley(23) *on* $\mathbb{Z}_{23} \setminus \{u, v\}$ *lies in* $\mathcal{I}_{5,3}$*. In particular it is 5-inducible.*

Proof sketch. By Proposition 3.2 choose $\sigma \in A_{23}$ with $\sigma(\{0, 1\}) = \{u, v\}$. Then σ is injective on $\mathbb{Z}_{23}$ and preserves Paley(23), so Lemma 2.3 carries the profile of Table 1 from V_{21} to $\sigma(V_{21})$, which is a permutation of $\mathbb{Z}_{23} \setminus \{u, v\}$; the class $\mathcal{I}_{5,3}$ does not see the order of the vertex list. The final claim follows from Proposition 2.2(1). □

Remark 3.4 (cost of the transport). The hypotheses of Lemma 2.3 are checked for the affine maps in *decomposed* form, $x \mapsto x+b$ and $x \mapsto ax$ separately, and composed afterwards using $(ax+b) \bmod 23 = ((ax \bmod 23) + b) \bmod 23$. The decomposed checks quantify over $23^3 = 12\,167$ and $11 \cdot 23^2 = 5\,819$ argument triples against $253 \cdot 23^2 = 133\,837$ for the composed one: 7.4 times fewer, which keeps the verification inside a 4 GiB working-memory budget.

Remark 3.5. Theorem 3.3 does not decide Paley(23). Inducibility is hereditary downwards, so it removes a potential obstruction — had some order-21 subtournament failed to be 5-inducible, Paley(23) would have been settled negatively — and it does so in the stronger form $\mathcal{I}_{5,3}$. Whether Paley(23) itself is 5-inducible remains open, as does the order-22 case; Section 8 reports what the natural next search costs.

4. The order-20 subtournaments

The seven order-20 *deletion classes* of [1, §8.2] are the orbits of A_{23} on three-element subsets of $\mathbb{Z}_{23}$: since $|A_{23}| = 253$ and $\binom{23}{3} = 1771 = 7 \cdot 253$, a Burnside count predicts $(q-2)/3 = 7$ orbits, each regular. We verify the prediction directly.

Proposition 4.1. *The* $7 \cdot 253$ *images under* A_{23} *of the seven triples*

$$\{0,1,2\},\ \{0,1,3\},\ \{0,1,4\},\ \{0,1,5\},\ \{0,1,7\},\ \{0,1,9\},\ \{0,1,13\}$$

are exactly the 1771 *three-element subsets of* $\mathbb{Z}_{23}$*. Hence* A_{23} *has exactly seven orbits on triples, all regular, and every order-20 subtournament of* Paley(23) *is isomorphic to one of the seven listed.*

Proof. Encode a triple by its sorted base-23 code and compare the set of 1771 codes obtained from the seven representatives with the set of all $\binom{23}{3}$ codes; the two agree. Seven orbits covering $1771 = 7 \cdot 253$ elements with $|A_{23}| = 253$ forces each orbit to have exactly 253 elements, so each is regular. The last sentence holds because A_{23} acts by automorphisms. □

Remark 4.2. "Seven" counts deletion classes — orbits of A_{23} on triples — and that is all Proposition 4.1 establishes. Whether the seven resulting order-20 subtournaments are pairwise non-isomorphic is a different question, which we do not settle: two orbits could in principle carry isomorphic subtournaments. Remark 4.4 separates at least three of them. The wording "deletion classes" in [1, §8.2] reads the same way, so nothing here contradicts the source; but if it is meant to assert pairwise non-isomorphism, that assertion is unverified here.

Theorem 4.3. *Each of the seven subtournaments* $\mathrm{Paley}(23) \restriction (\mathbb{Z}_{23} \setminus D)$*, for* D *one of the seven triples of Proposition 4.1, lies in* $\mathcal{I}_{5,3}$*: it is the majority tournament of five voters, no arc of which is unanimous.*

Proof. Seven explicit profiles, one per representative, each checked against the definition over the $\binom{20}{2} = 190$ pairs by evaluation in the kernel. The profile for $D = \{0,1,2\}$ is

$$\begin{gathered}(11,7,13,19,6,14,17,8,9,12,20,21,10,22,3,15,4,16,5,18)\\ (15,16,17,18,3,19,4,20,5,21,6,22,7,8,9,10,11,12,13,14)\\ (10,6,19,12,5,14,18,7,20,13,3,16,9,22,11,15,21,4,8,17)\\ (8,21,11,14,4,16,20,22,3,12,5,9,13,17,6,7,15,10,18,19)\\ (15,9,18,4,10,13,22,5,8,17,3,12,21,14,7,16,11,20,6,19);\end{gathered}$$

the remaining six are listed in Appendix A. □

The first half of Theorem 4.3 — that all seven are 5-inducible — is asserted in [1, §8.2] without certificates; the margin refinement is not claimed there. Combining Theorem 4.3 with Proposition 4.1 and Lemma 2.3 gives that *every* order-20 subtournament of Paley(23) lies in $\mathcal{I}_{5,3}$, by the argument of Theorem 3.3; we have carried that transport out formally only in the two-vertex case, where it is needed for Theorem 1.2.

The seven profiles were found by CaDiCaL on 950 variables and 13 300 clauses (13 304 with the symmetry break), in 9.9, 502.6, 266.0, 9.5, 395.6, 39.5 and 274.9 seconds respectively — about 25 CPU-minutes in total. As in Section 3, the searches asked only for 5-inducibility, and the absence of a unanimous arc is a property of the returned profiles that the certificate check establishes.

Remark 4.4 (maximum acyclic subtournaments; outside the formal development). Exact MAS values for the vertex-deleted Paley(23), computed by subset dynamic programming over 2^n states, extend Table 3 of [1] downwards:

deleted	$\emptyset$	$\{0\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,3\}$	$\{0,1,4\}$	$\{0,1,5\}$	$\{0,1,9\}$	$\{0,1,13\}$
n	23	22	21	20	20	20	20	20	20
C	253	231	210	190	190	190	190	190	190
MAS	161	150	139	128	127	126	126	126	127
slack$_5$	46	57	65	70	65	60	60	60	65

(the entry for $\{0,1,7\}$ coincides with that for $\{0,1,5\}$). The value $\mathrm{MAS}(\mathrm{Paley}(23)) = 161$ reproduces [1, Table 3]; the others are new. Two consequences are worth recording. First, three distinct MAS values appear among the seven order-20 classes, so at least three of them are pairwise non-isomorphic, independently of Proposition 4.1. Second, slack$_5$ *increases* as vertices are deleted, so the counting obstruction that kills Paley(43) — where slack$_5 = 6$ — gets weaker, not stronger, on the way down from Paley(23). No counting argument of that shape will decide order 22 or 23; it has to be a search. These values are computations outside the machine-checked development.

5. Refuting inducibility

A membership $T \restriction V \in \mathcal{I}_{m,t}$ is an existential statement and is certified by exhibiting a profile. A non-membership is not, and needs a refutation. We use the CNF form of the inducibility ILP of [1, Appendix B.1] and prove the implication a refutation requires.

For $n = |V|$, the variables are $x_{v,a,b}$ for $0 \le v < m$ and $0 \le a < b < n$, meaning "voter v ranks the a-th vertex of V above the b-th". The clauses are, in order: for each voter and each triple $a < b < c$, the two clauses expressing transitivity; for each pair $a < b$, the "at least lo voters agree with the arc of T" clauses, one per $(m - lo + 1)$-subset of the agreement literals; and, when $hi < m$, the "at most hi agree" clauses, one per $(hi + 1)$-subset, negated. Since a pair agreed by c voters has margin $|2c - m|$, choosing $2\,lo = m + 1$ and $2\,hi = m + t$ for odd m makes both bounds exact. Write $\Phi(V, T, m, lo, hi)$ for the resulting formula.

Theorem 5.1. *Suppose $m < 2\,lo \le m + 2$ and $m + t \le 2\,hi$, and suppose the index arithmetic and the antisymmetry of T on V hold. Then every profile of m voters on V inducing T within margin t yields a satisfying assignment of $\Phi(V, T, m, lo, hi)$. Consequently, if $\Phi(V, T, m, lo, hi)$ is unsatisfiable then $T \restriction V \notin \mathcal{I}_{m,t}$.*

Proof sketch. Read the assignment off the profile: set $x_{v,a,b}$ true exactly when voter v ranks the a-th vertex above the b-th. Transitivity clauses hold because each voter is a linear order. For the counting clauses, let c be the number of voters agreeing with the arc of T on a given pair. The majority hypothesis gives $2c > m$, hence $c \ge lo$ by $2\,lo \le m + 2$, so fewer than $m - lo + 1$ voters disagree and every $(m - lo + 1)$-subset of the agreement literals contains a true one. The margin hypothesis gives $|2c - m| \le t$, hence $c \le hi$ by $m + t \le 2\,hi$, so fewer than $hi + 1$ voters agree and every $(hi + 1)$-subset contains a false one, i.e. its negation is satisfied. $\square$

Only this direction is proved; the converse is not needed and is not claimed. The side conditions — that the pair indexing inverts correctly, that the listed vertices are distinct members of V, and that T is antisymmetric on them — are properties of the concrete instance, quantified over vertices rather than over profiles, and are decided per instance. As a control on the whole apparatus, the same machinery is confirmed to *fail* to refute $m = 5$ for Paley(11), which really is 5-inducible by Theorem 6.1: the formula is not unsatisfiable by construction.

Unsatisfiability itself is established by replaying an LRAT proof produced by CaDiCaL through a verified checker. The clause list is rebuilt inside the proof assistant from V, T, m, lo, hi rather than read from the solver's input file, so a mismatch between the formula that was solved and the formula being refuted makes the check fail rather than prove something else.

6. Paley(11): the McGarvey number, exactly

Theorem 6.1. Paley(11) $\in \mathcal{I}_{5,1}$*: the five rankings*

$$\begin{gathered}(5,6,10,4,9,3,8,2,7,1,0)\\(10,0,8,9,3,1,6,7,4,5,2)\\(9,7,1,10,8,2,6,0,5,3,4)\\(4,2,7,5,3,8,6,0,1,9,10)\\(0,1,2,3,4,5,6,7,8,9,10)\end{gathered}$$

induce Paley(11) *with every one of its* 55 *arcs split* 3*:*2*. In particular* Paley(11) *is 5-inducible.*

That Paley(11) and Paley(19) are 5-inducible "with explicit certificates" is stated in [1, §7], and no certificate is printed there; the margin-1 refinement is implied by the census of [1, §6], which places every non-3-inducible tournament on at most 11 vertices in $\mathcal{I}_{5,1}$. The value of Theorem 6.1 is thus the certificate, not the fact. The same is true of the next statement, where what is added is that the lower bound is certified.

Theorem 6.2. McG(Paley(11)) $=5$*: for every* $m \leq 4$*,* Paley(11) *is not* m*-inducible, and it is 5-inducible.*

Proof sketch. The upper bound is Theorem 6.1. For $m=0$ the empty profile makes every arc absent while Paley(11) has the arc $0 \to 1$. For $m=1,2,3,4$ apply Theorem 5.1 to four unsatisfiable formulas, of 55, 110, 165 and 220 variables and 385, 770, 1155 and 1650 clauses; each was refuted by CaDiCaL in under a hundredth of a second, and the four LRAT certificates (582 B, 1.2 kB, 15 kB, 567 B) are replayed through the verified checker. In particular the case $m=4$ is refuted directly, so the parity argument of [1, §1] — that the McGarvey number is always odd — is not needed for this tournament. □

Table 3 of [1] records McG(Paley(11)) $=5$, and [1, §8.2] calls Paley(11) "the first member of the Paley family that is not 3-inducible". Both halves now rest on checked certificates rather than on citation.

7. The margin hierarchy: the base level, and the price of the next

Conjecture 1.1 is established in [1, §5] at $m=3$ by a census, which identifies 254 tournaments on nine vertices that are 3-inducible but not margin-1 3-inducible without naming any of them. We name the smallest.

Let T_9 be the tournament on $\{0,\ldots,8\}$ with arcs

$$\begin{aligned}&0\to3,\ 0\to5;\quad 1\to0,\ 1\to2,\ 1\to8;\quad 2\to0,\ 2\to4,\ 2\to7;\quad 3\to1,\ 3\to2,\ 3\to7;\\&4\to0,\ 4\to1,\ 4\to3,\ 4\to6;\quad 5\to1,\ 5\to2,\ 5\to3,\ 5\to4;\\&6\to0,\ 6\to1,\ 6\to2,\ 6\to3,\ 6\to5;\quad 7\to0,\ 7\to1,\ 7\to4,\ 7\to5,\ 7\to6;\\&8\to0,\ 8\to2,\ 8\to3,\ 8\to4,\ 8\to5,\ 8\to6,\ 8\to7,\end{aligned}$$

with out-degree sequence $(2,3,3,3,4,4,5,5,7)$; it is the least of the 254 under the lexicographic bitmask over the 36 pairs, with mask 151 601 428.

Theorem 7.1. T_9 *is 3-inducible — the three voters*

$$(8,2,7,0,5,4,6,3,1),\qquad(4,1,8,6,0,5,3,2,7),\qquad(3,7,6,5,1,2,8,4,0)$$

induce it — but no three voters induce it with all margins 1*. Hence* $\mathcal{I}_{3,1}\subsetneq\mathcal{I}_{3,3}$*, with an explicit witness.*

Proof sketch. The positive half is an evaluation over the 36 pairs. The negative half applies Theorem 5.1 with $m=3$, $t=1$, $lo=hi=2$: the formula has 108 variables and 648 clauses, and its 21 kB LRAT refutation is replayed through the verified checker. □

Remark 7.2 (the $m = 5$ level: a complete census to nine vertices; outside the formal development). The first open case of Conjecture 1.1 asks for a tournament in $\mathcal{I}_{5,5} \setminus \mathcal{I}_{5,3}$ or in $\mathcal{I}_{5,3} \setminus \mathcal{I}_{5,1}$. There is none on at most nine vertices. We generated the tournaments on $5, \ldots, 9$ vertices up to isomorphism by vertex extension with canonicalisation against a colour refinement invariant — the generator reproduces $1, 1, 2, 4, 12, 56, 456, 6880, 191536$, the terms of OEIS A000568 [7] at $n = 1, \ldots, 9$, the last two of which are the D_8 and D_9 of [1, Appendix H] — and classified each by the least odd t with $T \in \mathcal{I}_{5,t}$, one solver call per level. All $12 + 56 + 456 + 6\,880 + 191\,536 = 198\,940$ classes lie in $\mathcal{I}_{5,1}$, at a cost of 2.1 CPU-hours. Any witness at $m = 5$ therefore has at least ten vertices.

How much of this is new should be stated precisely, because most of it is not. Two facts from [1] already cover almost everything: §6 places every non-3-inducible tournament on at most 11 vertices in $\mathcal{I}_{5,1}$; and the remark in §7 that adding to a profile any order together with its reverse changes no margin gives $\mathcal{I}_{3,t} \subseteq \mathcal{I}_{5,t}$ for every t, so every margin-1 3-inducible tournament lies in $\mathcal{I}_{5,1}$. What the two leave uncovered at $n \leq 9$ is exactly the class of Theorem 7.1: the 254 tournaments that are 3-inducible only via a unanimous arc, which [1, Theorem G.2] shows occur at $n = 9$ and nowhere below. Those the census places, all 254 in $\mathcal{I}_{5,1}$; that is the gap it closes. As a control on the whole pipeline — generator, encoding, solver, decoder and re-verifier — the identical run at $m = 3$ and $n = 9$ returns 17 674 non-3-inducible tournaments and exactly 254 outside $\mathcal{I}_{3,1}$, both of the counts of [1] to the unit.

The next rung is out of reach here: $D_{10} = 9\,733\,056$ classes at the measured 40 ms per class is about 108 CPU-hours. Two cheaper attacks are worth pricing first — restricting to regular or near-regular tournaments at $n = 11, 13$, and testing structured candidates directly. We ran two of the latter: $\mathrm{Paley}(23) \restriction (\mathbb{Z}_{23} \setminus \{0, 1, 2\})$ and Paley(19), both at margin 1. Each hit a one-hour solver cap with no answer, against 9.9 seconds for the same order-20 instance at margin 5; the margin-1 constraint adds only $190 \cdot 5 = 950$ clauses but changes the instance class completely. Both questions are open, and nothing in this paper depends on either. None of this remark is machine-checked: it is a measurement, reported so that the next attempt can be priced.

8. How rigid is the extension route?

The passage quoted in Section 1 proposes to "induce a subtournament, then decide extendability of the frozen induction". The appeal is that the frozen problem needs no solver at all: with an inducing profile on n vertices held fixed, the only remaining freedom is where each of the five voters inserts the new vertex, so the search has $(n+1)^5$ states and each is checked bitwise in constant time. At $n = 21$ that is $22^5 = 5\,153\,632$ insertions, an exhaustive sweep taking 1.2 seconds.

We ran three rungs of this, 61 seconds of exhaustive search in total, and every one came back empty.

- The profile of Table 1 has *no* extension to order 22: all 5 153 632 insertions fail.
- Going down a rung, the seven order-20 profiles of Theorem 4.3, each with each of its three deleted vertices restored — 21 sweeps of $21^5 = 4\,084\,101$ insertions, 0.8 seconds each — yield *no* order-21 profile at all. In particular the order-21 profile that does exist is not an extension of any of the seven that the solver happened to return.
- To build a pool guaranteed non-empty, restrict the order-21 profile to order 20 by dropping one vertex — eight of the twenty-one choices — and enumerate all order-21 extensions of the restriction. Each restriction is a genuine order-20 inducing profile in $\mathcal{I}_{5,3}$ and does extend, but only in $8, 3, 4, 4, 4, 2, 4, 8$ ways for the drops $2, 3, 4, 5, 7, 9, 13, 22$ — 37 order-21 profiles in all — and not one of the 37 extends to order 22.

These searches refute nothing: order 22 stays open. What they measure is how rigid the extension operator is. A whole order-21 profile has only a handful of siblings under one-vertex restriction and re-extension, and they fail together, so a single profile per isomorphism class is nowhere near enough.

What the route needs is a large and *diverse* pool per class, and the pool is the cost: one fresh order-21 profile costs about 4.6 minutes of solver time, so a hundred of them cost about 8 CPU-hours, while the cheap restriction-and-re-extension route costs minutes and produces near-duplicates.

The other unexploited structure is symmetry breaking. The searches above use only the trivial voter-ordering break — if voter $v+1$ ranks the first vertex above the second, so does voter v, sound because voters are interchangeable — and that alone is worth a factor of 30, measured end to end on $\mathrm{Paley}(23)\restriction(\mathbb{Z}_{23}\setminus\{0,1,2\})$: 301.9 seconds without it against 9.9 seconds with it, same solver, same core, 13 300 clauses against 13 304. A lex-leader constraint over the five voter blocks, or a break on the vertex side using A_{23}, is the obvious next thing to try, and the measured factor 30 from the trivial break is the reason to expect it to matter. We did not attempt order 23 from scratch: [1] reports that Paley(23) resisted about six weeks of SAT computation in [2].

All statements in this section are measurements, not theorems, and none of them is part of the machine-checked development. They are recorded because a negative that is exhaustive and cheap is worth more to the next attempt than a positive that is expensive.

9. Verification

All theorems stated above are formalised and machine-checked in Lean 4 [5] (toolchain `v4.32.0`); the development defines the objects of Section 2 from scratch and does not depend on Mathlib. Theorems 3.1, 3.3, 4.3, 6.1, Propositions 2.2, 3.2, 4.1, Lemma 2.3 and Theorem 5.1 are checked by the proof kernel alone, with no compiled-code shortcut: their proofs depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. The pair-orbit computation of Proposition 3.2 uses none of the three; the triple-orbit computation of Proposition 4.1 uses only `propext`. The two statements that rest on a refutation — Theorem 6.2 and the negative half of Theorem 7.1 — additionally depend on the opaque axiom that Lean's `native_decide` emits for each compiled evaluation, one per certificate, because the LRAT certificates are replayed through the verified checker `Std.Tactic.BVDecide.Reflect.verifyCert` by compiled evaluation; the clause lists they refute are rebuilt inside Lean from the tournament and the parameters, never read from the solver's input, and the soundness bridge they are fed into is Theorem 5.1, itself kernel-checked. What is delegated to finite computation is exactly this: the verification of each explicit profile against the definition, over 210 pairs at order 21, 190 at order 20, 55 for Paley(11) and 36 for T_9; the injectivity and arc-preservation of the 253 affine maps, in the decomposed form described in Section 3; the pair-orbit and triple-orbit counts; and the five LRAT replays. Nothing about any solver search enters a proof. Every certificate printed above was produced by a separate implementation in C or Python and re-decoded against an independent rebuild of the tournament before being checked a third time by the kernel, and the census and rigidity measurements of Remark 7.2 and Section 8 — which are not part of the formal development — were validated by reproducing the published counts 17 674 and 254 of [1] exactly. The repository is private; a repository reference is to be added at submission.

Appendix A. The remaining six order-20 profiles

For completeness we print the six certificates of Theorem 4.3 not shown there. In each block the five lines are the five voters, best first, labelled in $\mathbb{Z}_{23}$; the tournament is Paley(23) restricted to $\mathbb{Z}_{23}\setminus D$ for the deleted triple D named above the block, every one of its 190 arcs is ranked forward by three or four of the five voters, and none by all five.

$D=\{0,1,3\}$:

$$(5,9,13,17,21,6,22,12,7,2,15,8,10,16,18,11,19,14,4,20)$$
$$(14,16,2,20,6,9,15,18,4,10,19,22,8,12,5,21,11,17,7,13)$$
$$(7,19,8,9,21,2,11,4,20,6,10,12,13,14,22,15,16,5,17,18)$$
$$(17,10,11,12,18,20,13,19,15,21,4,16,22,2,5,6,8,14,7,9)$$
$$(5,14,18,4,7,13,8,16,22,11,15,17,6,19,12,20,9,2,21,10)$$

$D = \{0, 1, 4\}$:

$$(14, 7, 16, 9, 11, 17, 6, 18, 20, 10, 13, 22, 2, 3, 15, 19, 8, 12, 5, 21)$$
$$(19, 12, 20, 9, 13, 21, 16, 2, 6, 15, 10, 14, 22, 5, 18, 8, 17, 3, 11, 7)$$
$$(17, 12, 7, 15, 8, 21, 2, 10, 18, 3, 16, 20, 5, 11, 13, 6, 19, 14, 9, 22)$$
$$(19, 22, 2, 5, 13, 8, 11, 14, 20, 3, 15, 17, 6, 10, 16, 9, 12, 18, 21, 7)$$
$$(3, 5, 18, 21, 6, 22, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 2, 20)$$

$D = \{0, 1, 5\}$:

$$(13, 2, 22, 7, 8, 20, 9, 3, 15, 21, 11, 4, 16, 17, 6, 10, 18, 19, 12, 14)$$
$$(19, 8, 14, 9, 4, 12, 18, 17, 20, 21, 7, 10, 13, 16, 22, 2, 3, 11, 6, 15)$$
$$(2, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 3, 21, 4, 6, 9, 22, 7, 8)$$
$$(6, 15, 10, 18, 4, 8, 3, 7, 16, 11, 20, 9, 13, 19, 22, 17, 12, 21, 2, 14)$$
$$(6, 21, 14, 22, 17, 3, 12, 7, 19, 16, 9, 2, 18, 11, 4, 20, 15, 13, 8, 10)$$

$D = \{0, 1, 7\}$:

$$(9, 21, 10, 13, 6, 22, 2, 14, 18, 3, 11, 15, 19, 4, 12, 16, 5, 17, 8, 20)$$
$$(12, 2, 20, 15, 4, 6, 10, 5, 18, 13, 19, 8, 21, 14, 3, 16, 9, 22, 17, 11)$$
$$(11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 2, 3, 5, 21, 6, 8, 9, 4, 10, 12)$$
$$(8, 9, 17, 10, 3, 12, 21, 16, 19, 5, 2, 18, 11, 14, 4, 20, 13, 6, 22, 15)$$
$$(4, 22, 8, 20, 5, 16, 11, 17, 3, 6, 19, 12, 14, 9, 15, 18, 13, 21, 2, 10)$$

$D = \{0, 1, 9\}$:

$$(13, 17, 6, 7, 15, 19, 10, 18, 8, 12, 21, 2, 11, 14, 3, 4, 16, 20, 22, 5)$$
$$(2, 8, 14, 15, 17, 20, 3, 21, 4, 10, 12, 16, 5, 18, 7, 11, 6, 13, 19, 22)$$
$$(5, 21, 4, 14, 6, 22, 7, 8, 16, 11, 17, 12, 2, 18, 20, 10, 13, 3, 15, 19)$$
$$(18, 22, 3, 11, 19, 4, 12, 7, 20, 15, 5, 13, 16, 17, 21, 2, 6, 8, 10, 14)$$
$$(10, 13, 16, 19, 20, 22, 3, 2, 5, 11, 6, 12, 14, 15, 18, 4, 8, 21, 17, 7)$$

$D = \{0, 1, 13\}$:

$$(9, 8, 21, 10, 22, 11, 12, 2, 14, 3, 15, 4, 5, 16, 17, 6, 18, 7, 19, 20)$$
$$(17, 19, 2, 4, 12, 20, 5, 21, 6, 10, 14, 18, 22, 3, 7, 15, 8, 16, 9, 11)$$
$$(7, 16, 2, 18, 11, 4, 20, 22, 6, 15, 10, 19, 5, 8, 17, 3, 9, 12, 21, 14)$$
$$(8, 14, 20, 7, 9, 17, 15, 10, 18, 3, 21, 6, 19, 4, 12, 16, 22, 2, 5, 11)$$
$$(3, 16, 5, 11, 6, 19, 14, 9, 22, 12, 15, 18, 21, 4, 17, 7, 2, 8, 20, 10)$$

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