NON-UNIMODAL h^*-VECTORS OF VERY AMPLE POLYTOPES: THE LASOŃ–MICHAŁEK FAMILY IN CLOSED FORM

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ABSTRACT. Lasoń and Michałek constructed, for every $k \geq 2$ and $a \geq 0$, a very ample lattice polytope $\mathcal{P}_{k,a}$ of dimension $2k-1$ as a lattice segmental fibration over the edge polytope of the even cycle C_{2k}. Hofscheier, Kurylenko and Nill recently observed that $\mathcal{P}_{15,332} \times \mathcal{P}_{15,332}$ has a non-unimodal h^*-vector, answering a question of Ferroni and Higashitani and one of Balletti; their example was found with a language model and their proof is the display of three 22-digit integers computed with software the paper does not name. We compute the Ehrhart polynomial and the h^*-polynomial of $\mathcal{P}_{k,a}$ in closed form for all k and a, show that putting an arbitrary integer height vector c over the cycle changes nothing — the Ehrhart polynomial depends only on the alternating sum of the heights — and use the resulting arithmetic to settle four points the source leaves open. Throughout, a *violation* at index i means $h^*_{i-1} > h^*_i < h^*_{i+1}$, which is the failure of unimodality the source exhibits; every negative statement below asserts the absence of one, which is formally weaker than unimodality. For $a \leq 4000$ the h^*-vector of $\mathcal{P}_{15,a} \times \mathcal{P}_{15,a}$ has a violation exactly for $a \in \{332, \dots, 338\}$, always at index 26; the minimality of $k = 15$, which the source asserts without proof, holds over the ranges searched; there are non-unimodal products of two very ample polytopes that are not isomorphic and not even equidimensional, namely $\mathcal{P}_{14,559} \times \mathcal{P}_{16,205}$, with factors of dimensions 27 and 31; and, for the source's closing question, no violation in the ranges searched sits above $d/2$ — every one sits at least 3 below it, the same distance as in all three published examples. All statements are machine-checked in Lean 4.

1. Introduction

Let $P \subset \mathbb{R}^n$ be a lattice polytope of dimension d, that is, the convex hull of finitely many points of $\mathbb{Z}^n$. Its Ehrhart polynomial is the unique polynomial E_P with $E_P(m) = |mP \cap \mathbb{Z}^n|$ for every integer $m \geq 1$, and its h^*-polynomial is the numerator of the Ehrhart series,

$$\sum_{m \geq 0} E_P(m)\, t^m \;=\; \frac{h^*_P(t)}{(1-t)^{d+1}}, \qquad h^*_P(t) = \sum_{i=0}^{d} h^*_i t^i.$$

The vector $(h^*_0, \dots, h^*_d)$ is *unimodal* if $h^*_0 \leq \cdots \leq h^*_j \geq \cdots \geq h^*_d$ for some j. P is *IDP* if for every $m \geq 2$ each lattice point of mP is a sum of m lattice points of P, and *very ample* if that holds for all m beyond some bound. Whether every IDP polytope has a unimodal h^*-vector is the main open problem of Ehrhart theory ([3, Conjecture 1.1], [11, Question 1.1]); it is untouched by what follows.

Ferroni and Higashitani [3, Question 3.9(a)], who attribute the question to Michałek, asked whether the hypothesis can be weakened to very ampleness, and Balletti [1, Question 5.1(b)] asked the product form: are there very ample P_1, P_2 with $h^*_{P_1 \times P_2}$ not unimodal? Balletti added that "it is reasonable to believe that the answer to the second question is also negative". Both questions were answered affirmatively by Hofscheier, Kurylenko and Nill [4]; part (b) of the Ferroni–Higashitani question, on log-concavity for IDP polytopes, was answered by the same three authors in [5]. Their example is $Q \times Q$ with $Q = \mathcal{P}_{15,332}$, a member of a family of very ample, non-normal polytopes constructed by Lasoń and Michałek [6, §3.1]: for $k \geq 2$, $a \geq 0$ and $u_i = e_i + e_{i+1} \in \mathbb{Z}^{2k}$ with indices cyclic modulo $2k$,

$$\mathcal{P}_{k,a} \;:=\; \operatorname{conv}\big((u_1, a),\, (u_1, a+1),\, (u_i, 0),\, (u_i, 1) \;:\; 2 \leq i \leq 2k\big) \;\subset\; \mathbb{R}^{2k} \times \mathbb{R}. \tag{1}$$

The whole proof of [4, Theorem 1.3] is the display of the three integers

$$h^*_{25} = 1631347108606059869973, \qquad h^*_{26} = 1627940480253228812568,$$
$$h^*_{27} = 1628167830322084291128$$

of $Q \times Q$, computed with software the paper does not name; the example itself is attributed to a language model — "The example was found using ChatGPT 5.6 Sol." [4, Acknowledgment]. Nothing else about the family is computed there: h^*_Q is stated with the remark that it "can be easily computed with any suitable software package", the assertion that $\mathcal{P}_{15,332}$ is minimal for the construction carries no proof and no reported search, and the remark that unimodality also fails for $a \in \{332, \dots, 338\}$ is an inclusion with no index and no claim of exhaustiveness. The paper closes with

Question 1.1 ([4, Question 1.4])**.** Is it possible to find a d-dimensional very ample polytope whose h^*-vector is not-unimodal, and there is an index $i > d/2$ where the unimodality is violated?

Results. The starting point is that the whole family can be computed by hand. Section 3 derives, from the fibration rather than from software,

$$E_{\mathcal{P}_{k,a}}(m) = (m+1)\binom{m+2k-1}{2k-1} - (m+1-a)\binom{m+k-1}{2k-1},$$
$$h^*_{\mathcal{P}_{k,a}}(t) = 1 + 2k\big(t + \cdots + t^{k-1}\big) + (a+k-1)\,t^k,$$

which the literature does not contain (Theorem 3.1). Lasoń and Michałek compute the Hilbert basis and the gap vector of $\mathcal{P}_{k,a}$ and never write h^* at all; the published values we could find — h^* at $(k,a) = (2,16)$ [3, Theorem 3.8] and at $(15,332)$ [4, Proposition 1.2(3)], and the $k=2$ Ehrhart polynomial [2, Lemma 3.4] — all agree with it. Section 4 shows the construction is genuinely one-parameter: a segmental fibration over the edge polytope of C_{2k} with an *arbitrary* integer height vector c has the Ehrhart polynomial of $\mathcal{P}_{k,|\gamma|}$, where $\gamma = \sum_i (-1)^{i-1} c_i$ (Theorem 4.1). Only the pairing of c with the generator of the kernel of the incidence map of C_{2k} survives.

Everything else is then integer arithmetic. Section 5 recomputes the source's h^*_Q and its three integers from the closed form. Section 6 proves that $h^*_{\mathcal{P}_{15,a}\times\mathcal{P}_{15,a}}$ has a violation — in the sense of Definition 2.1, which is the failure the source exhibits — *exactly* for $a \in \{332, \dots, 338\}$ in the range searched, always at index 26 (Theorem 6.1), and that no $k \le 14$ produces a violation at all there (Theorem 6.3), which is the dimension half of the source's unproved minimality claim. Section 7 answers Balletti's Question 5.1(b) with two very ample factors that are not isomorphic, and in one case not even equidimensional: $\mathcal{P}_{14,a} \times \mathcal{P}_{16,205}$ has a violation exactly for $a \in [554, 561]$, with factors of dimensions 27 and 31, although neither factor squared has one anywhere in the range searched (Theorem 7.1). Section 8 reports on Question 1.1: we do not answer it, and we say precisely where the answer is not. Over the ranges searched every violation sits at least 3 below $d/2$ (Theorem 8.3), and 3 is attained — by the source's own polytope, and by both of Balletti's published examples in dimensions 52 and 54.

Read Section 8 as a measurement, not as a theorem about all very ample polytopes. What is proved there is a positive lower bound on $d/2 - i$ over an explicit finite range of parameters; what is observed is that the bound is tight and that the same distance 3 recurs across two independent constructions. Whether $i \le d/2 - 3$ is a theorem is, we believe, the sharpest question this family poses.

2. Preliminaries

We use the conventions of [3] throughout. For a d-dimensional lattice polytope P the h^*-vector is nonnegative [10], $h^*_0 = 1$, $h^*_1 = |P \cap \mathbb{Z}^n| - (d+1)$, and $h^*_P(1)$ is the normalised volume $d!\operatorname{vol}(P)$. The *codegree* of P is $\operatorname{codeg} P = \min\{t \ge 1 : \operatorname{int}(tP) \cap \mathbb{Z}^n \ne \emptyset\}$, and $\deg h^*_P = d + 1 - \operatorname{codeg} P$.

Definition 2.1. Let $h = (h_0, \dots, h_d)$ be a sequence of integers. We say h has a *violation at index* i, for $1 \le i \le d-1$, if $h_{i-1} > h_i < h_{i+1}$.

If h has a violation at i then h is not unimodal: for any $j \geq i$ the chain $h_0 \leq \cdots \leq h_j$ fails at $h_{i-1} > h_i$, and for any $j < i$ the chain $h_j \geq \cdots \geq h_d$ fails at $h_i < h_{i+1}$. The converse is false in general — $(3,1,1,3)$ is not unimodal and has no violation — so "no violation" is formally weaker than "unimodal". Question 1.1 is phrased in terms of an index at which unimodality is violated, so Definition 2.1 is the reading under which it is searched here, and we adopt it as a standing convention: below, every *positive* statement exhibits a violation, hence genuine non-unimodality, while every *negative* statement asserts the absence of a violation and is *not* a claim of unimodality. The one place where unimodality itself is asserted is Corollary 3.5, which is proved by hand.

Cartesian products satisfy $E_{P\times P'}(m) = E_P(m)E_{P'}(m)$; a product of very ample polytopes is very ample [3, Proposition 2.2(ii)], and a product is IDP if and only if both factors are [6, Lemma 5(3)], so non-IDP-ness is preserved too. Writing d, d' for the dimensions, $\dim(P \times P') = d + d'$ and $h^*_{P\times P'}(1) = \binom{d+d'}{d} h^*_P(1) h^*_{P'}(1)$.

For a finite simple graph G on vertex set $[n]$, the *edge polytope* is $\mathcal{P}(G) = \operatorname{conv}\{e_i + e_j : \{i,j\} \in E(G)\} \subset \mathbb{R}^n$ [9].

Definition 2.2 ([6, Definition 2])**.** A projection $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ restricted to a lattice polytope $P \subset \mathbb{R}\times\mathbb{R}^n$ is a *lattice segmental fibration* if the preimage $f^{-1}(x)$ of every $x \in f(P)\cap\mathbb{Z}^n$ is a lattice segment of positive length.

Two facts of [6] are used as stated there. First, [6, Theorem 3]: a polytope admitting a lattice segmental fibration onto a unimodular polytope is very ample. Second, [6, Proposition 6], attributed there to Ohsugi, Herzog and Hibi [8, Example 3.6(b)]: for a connected graph G, $\mathcal{P}(G)$ is unimodular if and only if G contains no two disjoint odd cycles. Since C_{2k} contains no odd cycle whatsoever, $\mathcal{P}(C_{2k})$ is unimodular, and the polytope $\mathcal{P}_{k,a}$ of (1) — whose fibre over each of the $2k$ lattice points u_i of $\mathcal{P}(C_{2k})$ is a lattice segment of length 1 — is very ample *for every $k \geq 2$ and every $a \geq 0$*, with no hypothesis relating k and a.[1] By [6, Theorem 12] the gap vector of $\mathcal{P}_{k,a}$ has i-th entry $(a-i-1)\binom{i+k-1}{2k-1}$ for $k \leq i \leq a-2$ and 0 otherwise, so $\mathcal{P}_{k,a}$ is IDP if and only if $a \leq k+1$. Finally, $\dim \mathcal{P}_{k,a} = 2k-1$, and the $4k$ points listed in (1) are its vertices; at $k = 15$ that count is [4, Proposition 1.2(1)].

It will be convenient to allow an arbitrary height over every edge. For $c \in \mathbb{Z}^{2k}$ put

$$\mathcal{P}(k,c) := \operatorname{conv}\big\{(u_i, c_i),\ (u_i, c_i+1)\ :\ 1 \leq i \leq 2k\big\} \subset \mathbb{R}^{2k}\times\mathbb{R}, \tag{2}$$

so that $\mathcal{P}_{k,a} = \mathcal{P}(k,(a,0,\ldots,0))$. Every $\mathcal{P}(k,c)$ is again a lattice segmental fibration over $\mathcal{P}(C_{2k})$ with all fibres of lattice length 1, hence very ample.

3. The Ehrhart polynomial and the h^*-vector in closed form

Theorem 3.1. *Let $k \geq 2$ and $a \geq 0$. Then for every integer $m \geq 0$*

$$E_{\mathcal{P}_{k,a}}(m) \;=\; (m+1)\binom{m+2k-1}{2k-1} - (m+1-a)\binom{m+k-1}{2k-1},$$

and, with $d = \dim \mathcal{P}_{k,a} = 2k-1$,

$$h^*_{\mathcal{P}_{k,a}}(t) \;=\; 1 + 2k\big(t + t^2 + \cdots + t^{k-1}\big) + (a+k-1)\,t^k.$$

In particular $\deg h^*_{\mathcal{P}_{k,a}} = k$, $\operatorname{codeg} \mathcal{P}_{k,a} = k$, *and the normalised volume of $\mathcal{P}_{k,a}$ is $2k^2 - k + a$.*

The proof occupies the rest of this section. Fix $k \geq 2$ and a height vector $c \in \mathbb{Z}^{2k}$; we work with $\mathcal{P}(k,c)$ of (2) at once, since Section 4 needs the general case, and specialise to $c = (a,0,\ldots,0)$ at the end.

[1] Two typographical slips in [6, §3.1], recorded because they can mislead a reader reconstructing the family: $\mathcal{P}_{k,a}$ is said to lie in $\mathbb{Z}^k \times \mathbb{Z}$, where the next line correctly has $\mathbb{Z}^{2k}\times\mathbb{Z}$; and $A_{k,a}$ is called a set of $k-a-1$ points where it has $a-k-1$. Neither affects anything used here.

Lemma 3.2. *For every integer $m \ge 0$,*

$$m\,\mathcal{P}(k,c) = \Big\{(x,h)\ :\ \exists\, s \in \mathbb{R}^{2k}_{\ge 0},\ \textstyle\sum_i s_i = m,\ \sum_i s_i u_i = x,\ \sum_i c_i s_i \le h \le \sum_i c_i s_i + m\Big\}.$$

Proof. A point of $m\mathcal{P}(k,c)$ is $\sum_i \big(\lambda_i(u_i,c_i) + \mu_i(u_i,c_i+1)\big)$ with $\lambda,\mu \ge 0$ and $\sum_i(\lambda_i+\mu_i) = m$. Put $s_i = \lambda_i + \mu_i$ and $\nu = \sum_i \mu_i$; then $x = \sum_i s_i u_i$, $h = \sum_i c_i s_i + \nu$, and ν ranges over $[0,m]$ independently of s. ☐

Lemma 3.3. *Let $x \in \mathbb{Z}^{2k}$ and let $S(x) = \{s \in \mathbb{R}^{2k}_{\ge 0} : \sum_i s_i u_i = x\}$. Then $S(x)$ is either empty or a segment*

$$s_j(\tau) = A_j(x) + (-1)^{j-1}\tau, \qquad \tau \in [\alpha(x),\beta(x)],$$

where $A_1 = 0$, $A_j = x_j - A_{j-1}$, $\alpha = \max_{j \text{ odd}}(-A_j)$ and $\beta = \min_{j \text{ even}} A_j$ are integers; $s_1 = \tau$; and every $s \in S(x)$ has $\sum_i s_i = \frac{1}{2}\sum_j x_j$.

Proof. Since $u_i = e_i + e_{i+1}$ with cyclic indices, the equation $\sum_i s_i u_i = x$ reads $s_{j-1} + s_j = x_j$ for every j modulo $2k$. Because the cycle has even length, this linear system has a one-dimensional solution set: setting $s_1 = \tau$ and propagating gives $s_j = A_j + (-1)^{j-1}\tau$ with A_j as stated, and the equations close up consistently. Nonnegativity of s_j reads $\tau \ge -A_j$ for odd j and $\tau \le A_j$ for even j, which is the interval $[\alpha,\beta]$; both endpoints are integers when x is. Summing all equations gives $\sum_j x_j = 2\sum_i s_i$. ☐

Lemma 3.4. *Let $x \in m\mathcal{P}(C_{2k}) \cap \mathbb{Z}^{2k}$ and put $\gamma = \sum_i (-1)^{i-1} c_i$. The fibre of $m\mathcal{P}(k,c)$ over x contains exactly $|\gamma|\,(\beta(x)-\alpha(x)) + m + 1$ lattice points.*

Proof. By Lemmas 3.2 and 3.3 the fibre is the union, over $\tau \in [\alpha,\beta]$, of the intervals $[\Gamma(\tau), \Gamma(\tau)+m]$ where

$$\Gamma(\tau) = \sum_i c_i s_i(\tau) = \Gamma_0(x) + \gamma\tau, \qquad \Gamma_0(x) = \sum_i c_i A_i(x).$$

As τ runs continuously over $[\alpha,\beta]$, $\Gamma(\tau)$ runs continuously over an interval of length $|\gamma|(\beta-\alpha)$, so the union is a single closed interval of length $|\gamma|(\beta-\alpha)+m$. Its endpoints are integers, because $A_i(x)$, α, β and c are. Hence it contains $|\gamma|(\beta-\alpha)+m+1$ integers. ☐

Note what Lemma 3.4 does *not* depend on: the constant $\Gamma_0(x)$ translates the fibre interval but cannot change how many integers it holds. This is the whole content of Section 4.

Proof of Theorem 3.1. Take $c = (a,0,\dots,0)$, so $\gamma = a \ge 0$. Write $L(m) = |m\mathcal{P}(C_{2k}) \cap \mathbb{Z}^{2k}|$. Summing Lemma 3.4 over the base,

$$E_{\mathcal{P}_{k,a}}(m) = \sum_x \big[a(\beta-\alpha) + m + 1\big] = a\sum_x(\beta-\alpha+1) + (m+1-a)L(m). \tag{3}$$

By Lemma 3.3 the integer points of $S(x)$ are exactly the $s(\tau)$ with $\tau \in \mathbb{Z}\cap[\alpha,\beta]$, and every $s \in \mathbb{Z}^{2k}_{\ge 0}$ with $\sum_i s_i = m$ arises this way for exactly one x. Hence

$$\sum_x(\beta-\alpha+1) = \big|\{s \in \mathbb{Z}^{2k}_{\ge 0} : \textstyle\sum_i s_i = m\}\big| = \binom{m+2k-1}{2k-1}.$$

For $L(m)$: each x has exactly one integer preimage with $\tau = \alpha$, namely the one with $s_j = 0$ for some odd j; conversely any $s \in \mathbb{Z}^{2k}_{\ge 0}$ with $\sum_i s_i = m$ and $s_j = 0$ for some odd j satisfies $\tau = \alpha$ for its own x. Therefore $L(m)$ is the number of $s \in \mathbb{Z}^{2k}_{\ge 0}$ with $\sum_i s_i = m$ minus the number of those with $s_j \ge 1$ for each of the k odd indices j:

$$L(m) = \binom{m+2k-1}{2k-1} - \binom{m+k-1}{2k-1}. \tag{4}$$

Substituting into (3) gives the stated $E_{\mathcal{P}_{k,a}}$.

For the h^*-polynomial, write $d = 2k-1$ and use $(m+1)\binom{m+d}{d} = (d+1)\binom{m+d+1}{d+1} - d\binom{m+d}{d}$, which follows from $(d+1)\binom{m+d+1}{d+1} = (m+d+1)\binom{m+d}{d}$. With $\sum_m \binom{m+e}{e} t^m = (1-t)^{-e-1}$,

$$\sum_{m\geq 0}(m+1)\binom{m+2k-1}{2k-1}t^m = \frac{2k}{(1-t)^{2k+1}} - \frac{2k-1}{(1-t)^{2k}} = \frac{1+(2k-1)t}{(1-t)^{2k+1}}.$$

Since $\binom{m+k-1}{2k-1} = 0$ for $m < k$, substituting $m = n+k$ gives

$$\sum_{m\geq 0}(m+1-a)\binom{m+k-1}{2k-1}t^m = t^k\left[\frac{1+(2k-1)t}{(1-t)^{2k+1}} + \frac{k-a}{(1-t)^{2k}}\right].$$

Subtracting and using $1-t^k = (1-t)(1+t+\cdots+t^{k-1})$,

$$\sum_{m\geq 0} E_{\mathcal{P}_{k,a}}(m)t^m = \frac{(1+t+\cdots+t^{k-1})(1+(2k-1)t)+(a-k)t^k}{(1-t)^{2k}},$$

whose numerator expands to $1+2k(t+\cdots+t^{k-1})+(a+k-1)t^k$. As $(1-t)^{2k} = (1-t)^{d+1}$, this is $h^*_{\mathcal{P}_{k,a}}$, and in particular $\dim \mathcal{P}_{k,a} = 2k-1$. The consequences are immediate: $\deg h^* = k$ because $a+k-1 \geq k-1 > 0$, hence $\operatorname{codeg} = d+1-k = k$; and $h^*(1) = 1+2k(k-1)+(a+k-1) = 2k^2-k+a$. $\square$

Two of the ingredients are themselves published, and the derivation reproduces rather than assumes them. Identity (4) says $h^*_{\mathcal{P}(C_{2k})}(t) = 1+t+\cdots+t^{k-1}$ on a polytope of dimension $2k-2$ and normalised volume k, which is [7, Fact 4.1(2)]. At $k=2$, Theorem 3.1 gives $E_{\mathcal{P}_{2,a}}(m) = (\frac{a}{6}+1)m^3+3m^2+(3-\frac{a}{6})m+1$, which is [2, Lemma 3.4].

Corollary 3.5. *$h^*_{\mathcal{P}_{k,a}}$ is unimodal for every $k \geq 2$ and $a \geq 0$.*

Proof. The vector is $(1, 2k, \dots, 2k, a+k-1, 0, \dots, 0)$. If $a \geq k+1$ it is non-decreasing up to index k and then drops to 0; if $a \leq k$ it rises at index 1 and is non-increasing from there. $\square$

Corollary 3.5 covers in particular every non-IDP member of the family, i.e. every $a \geq k+2$. So in all of what follows the failure of unimodality is created by the Cartesian product and never by a factor.

Proposition 3.6. *$h^*_{\mathcal{P}_{2,16}}(t) = 1+4t+17t^2$, and for every $0 \leq m \leq 300$ the value $E_{\mathcal{P}_{15,332}}(m)$ of Theorem 3.1 agrees with $\binom{m+29}{29} + \sum_{i=1}^{14} 30\binom{m+29-i}{29} + 346\binom{m+14}{29}$.*

The first assertion identifies $\mathcal{P}_{2,16}$ with the three-dimensional very ample polytope whose h^*-polynomial $17x^2+4x+1$ fails to be log-concave [3, Theorem 3.8]. The second is the agreement of Theorem 3.1 with the Ehrhart formula printed in the proof of [4, Theorem 1.3]. Both are exhaustive integer computations. Independently of the proof above, the two identities of Theorem 3.1 have been machine-verified for all $2 \leq k \leq 20$ and $0 \leq a \leq 200$, that is for 3 819 polytopes of dimension up to 39, in the h^*-form and, for $m \leq 60$, in the Ehrhart form $E(m) = \sum_r h^*_r\binom{m+d-r}{d}$.

4. The height vector collapses to one parameter

Theorem 4.1. *Let $k \geq 2$ and $c \in \mathbb{Z}^{2k}$, and put $\gamma = \sum_{i=1}^{2k}(-1)^{i-1}c_i$. Then $\mathcal{P}(k,c)$ is a very ample lattice polytope of dimension $2k-1$ and*

$$E_{\mathcal{P}(k,c)} = E_{\mathcal{P}_{k,|\gamma|}}, \qquad \textit{hence} \qquad h^*_{\mathcal{P}(k,c)} = h^*_{\mathcal{P}_{k,|\gamma|}}.$$

Proof. Very ampleness and the dimension were recorded in Section 2. By Lemma 3.4 the number of lattice points over each $x \in m\mathcal{P}(C_{2k}) \cap \mathbb{Z}^{2k}$ is $|\gamma|(\beta(x)-\alpha(x))+m+1$, which is exactly the count for the height vector $(|\gamma|, 0, \dots, 0)$. Summing over x gives $E_{\mathcal{P}(k,c)}(m) = E_{\mathcal{P}_{k,|\gamma|}}(m)$ for every m, and the h^*-polynomials agree because the dimensions do. $\square$

The vector $(1, -1, 1, -1, \dots)$ spans the kernel of the incidence map of C_{2k}, and γ is the pairing of c with it; Theorem 4.1 says that this single pairing is all of c that the Ehrhart polynomial can see. The certificate is independent of Theorem 3.1: for 8 192 height vectors — 4 096 at $k = 2$ with entries in $[-3, 4]$ and $m \leq 4$, and 4 096 at $k = 3$ with entries in $[-1, 2]$ and $m \leq 3$ — the lattice-point count was recomputed directly from Lemmas 3.2–3.4, by enumerating the integer solutions s and measuring the swept interval, and compared with $E_{\mathcal{P}_{k,|\gamma|}}$; all agree. Two non-vacuity checks accompany it: the count reproduces $E_{\mathcal{P}_{3,7}}$ on the Lasoń–Michałek height vector $(7, 0, 0, 0, 0, 0)$, and it is not constant in c, since $(7, 7, 0, 0, 0, 0)$ has $\gamma = 0$ and gives the strictly smaller $E_{\mathcal{P}_{3,0}}$.

Corollary 4.2. *Every one-step lattice segmental fibration over the edge polytope of an even cycle, with all fibres of lattice length* 1*, has the* h^**-polynomial of Theorem 3.1.*

This is what makes the classifications below classifications of the whole construction rather than of a hand-picked subfamily, and it is why we say in Section 8 that an answer to Question 1.1 in this style would have to come from a different base graph.

5. The source's polytope, recomputed

Set $Q = \mathcal{P}_{15,332}$, so $\dim Q = 29$, Q has 60 vertices, and Q is very ample and not IDP ($a = 332 > k + 1 = 16$).

Proposition 5.1. $h^*_Q(t) = 1 + 30(t + \cdots + t^{14}) + 346\, t^{15}$. *Consequently Q has exactly* 60 *lattice points, all of them vertices, and* $h^*_Q(1) = 767$*; and for the product,* $h^*_1(Q \times Q) = 60^2 - 59 = 3541$ *and* $\sum_r h^*_r(Q \times Q) = \binom{58}{29} \cdot 767^2$.

Proof. The first assertion is Theorem 3.1 at $k = 15$, $a = 332$: the coefficient of t^{15} is $a + k - 1 = 346$. Then $|Q \cap \mathbb{Z}^{31}| = h^*_1 + \dim Q + 1 = 30 + 30 = 60$, which is the number of vertices $4k$, and $h^*_Q(1) = 2k^2 - k + a = 767$. For the product, $h^*_1(Q \times Q) = |Q \times Q \cap \mathbb{Z}^{62}| - 59 = 3600 - 59$ and $h^*_{Q\times Q}(1) = \binom{58}{29} h^*_Q(1)^2$. All four identities were also verified directly, by exact integer evaluation of the h^*-transform. ☐

Proposition 5.1 is [4, Proposition 1.2(3)], whose published proof is that it "can be easily computed with any suitable software package". The point of restating it is that here it is a corollary of Theorem 3.1, so the agreement is evidence about the derivation rather than a restatement of the source.

Theorem 5.2. *The* h^**-vector of the* 58*-dimensional very ample non-IDP polytope* $Q \times Q$ *satisfies*

$$\begin{aligned} h^*_{25} &= 16313471086060598699 73 \; > h^*_{26} = 1627940480253228812568, \\ h^*_{26} &= 1627940480253228812568 \; < h^*_{27} = 1628167830322084291128, \end{aligned}$$

and 26 *is the* only *index at which it has a violation.*

Proof. The three values are obtained from $h^*_r = \sum_{m=0}^{r} (-1)^{r-m} \binom{59}{r-m} E_Q(m)^2$ with E_Q taken from Theorem 3.1, not from [4]; they reproduce the printed integers digit for digit. The last assertion is an exhaustive scan of all 57 interior indices of the 59-entry vector. ☐

The inequality chain is [4, Theorem 1.3]. The statement that 26 is the unique violated index is one step sharper than the published theorem, which exhibits only the three-term chain.

6. The exact window, and the minimality of $k = 15$

Concerning the window, [4] remarks, without proof, only this: "Unimodality also fails when we consider $Q = \mathcal{P}_{15,a}$ for $a \in \{332, \dots, 338\}$." Nothing is claimed there about exhaustiveness, about the violated index, or about other values of k.

Theorem 6.1. *Let* $0 \leq a \leq 4000$*. The* h^**-vector of* $\mathcal{P}_{15,a} \times \mathcal{P}_{15,a}$ *has a violation if and only if* $332 \leq a \leq 338$*, and for every such a the set of violated indices is exactly* $\{26\}$.

Proof. Exhaustive evaluation of the 4 001 vectors $h^*_{\mathcal{P}_{15,a}\times\mathcal{P}_{15,a}}$, each of length 59, from Theorem 3.1 in exact integer arithmetic, followed by a scan of the interior indices of each. Both endpoints are sharp: $a = 331$ and $a = 339$ give no violation, and no second window occurs below 4000. □

Remark 6.2. The bound 4000 can be removed by an argument that we have carried out in exact integer arithmetic but have not formally verified, and which we therefore record as a computation rather than as a theorem. Writing $E_{\mathcal{P}_{k,a}}(m) = U_k(m) + a\,V_k(m)$ with $U_k(m) = (m+1)\big[\binom{m+2k-1}{2k-1} - \binom{m+k-1}{2k-1}\big]$ and $V_k(m) = \binom{m+k-1}{2k-1}$, each h^*_r of $\mathcal{P}_{k,a}\times\mathcal{P}_{k,a}$ is a quadratic in a with integer coefficients, so each of the two conditions $h^*_{r-1} - h^*_r > 0$ and $h^*_{r+1} - h^*_r > 0$ is an integer quadratic inequality, solvable exactly. Doing so shows that at $k = 15$ the window is $[332, 338]$ for *all* $a \geq 0$.

Theorem 6.3. *For every $2 \leq k \leq 14$ and every $0 \leq a \leq 2200$, the h^*-vector of $\mathcal{P}_{k,a}\times\mathcal{P}_{k,a}$ has no violation. Moreover $\mathcal{P}_{14,a}\times\mathcal{P}_{14,a}$ has no violation for any $a \leq 4000$, while the set of $a \leq 4000$ for which $\mathcal{P}_{16,a}\times\mathcal{P}_{16,a}$ has one has exactly 25 elements, the least of which is 370.*

Proof. Exhaustive evaluation as in Theorem 6.1, over 13×2201 parameter pairs for the first assertion and 4 001 values of a for each of $k = 14, 16$. □

Combined with Theorem 5.2, this is the dimension half of the assertion of [4] that $\mathcal{P}_{15,332}$ "produces the minimal example given by the above construction in terms of dimension and volume", for which no proof and no search is reported there: no $k \leq 14$ produces a violation in the range searched, and $k = 15$ does, so 58 is the least dimension at which the square construction produces one. The bound 2200 is not a truncation of the interesting region. By the same exact analysis as in Remark 6.2, the set of violating a is empty for every $a \geq 0$ when $k \leq 14$, and lies below $a = 1842$ for $15 \leq k \leq 26$; the windows grow like $a \approx 1.4k^2$. The $k = 16$ statement exhibits the next dimension, $d = 62$: there the violating $a \leq 4000$ are 25 in number and start at 370. Those two certified facts do not by themselves force the set to be an interval; an exact computation in the style of Remark 6.2 shows that it is, namely $[370, 394]$, and we record the contiguity as a computation rather than as part of the theorem.

7. Two different very ample factors

[4] only ever multiplies a polytope by itself, and its remark on $a \in \{332, \dots, 338\}$ covers only the diagonal. Balletti's Question 5.1(b) asks for very ample P_1 and P_2 with $h^*_{P_1\times P_2}$ non-unimodal and does not require $P_1 = P_2$.

Theorem 7.1. *Let $0 \leq a \leq 4000$.*

(1) *$h^*_{\mathcal{P}_{15,a}\times\mathcal{P}_{15,332}}$ has a violation if and only if $332 \leq a \leq 344$.*
(2) *$h^*_{\mathcal{P}_{14,a}\times\mathcal{P}_{16,205}}$ has a violation if and only if $554 \leq a \leq 561$.*

In particular $\mathcal{P}_{14,559}\times\mathcal{P}_{16,205}$ is a 58-dimensional very ample non-IDP polytope whose h^*-vector has a violation exactly at index 26, and whose two factors are very ample polytopes of different *dimensions, 27 and 31.*

Proof. Exhaustive evaluation of 4 001 product h^*-vectors in each case, as in Theorem 6.1, together with the scan of the interior indices of $h^*_{\mathcal{P}_{14,559}\times\mathcal{P}_{16,205}}$. □

Part (1) already extends the source's diagonal window by six values of a. Part (2) is the one that answers Balletti's question with genuinely different factors: by Theorem 6.3, $\mathcal{P}_{14,a}\times\mathcal{P}_{14,a}$ never has a violation in the range searched, and the $a \leq 4000$ for which $\mathcal{P}_{16,a}\times\mathcal{P}_{16,a}$ has one are 25 in number and none is below 370, so 205 is not among them. The failure is a property of the pair, not of either factor.

Remark 7.2. The same exact method as in Remark 6.2 — for fixed (k, k', a') every h^*_r of $\mathcal{P}_{k,a}\times\mathcal{P}_{k',a'}$ is *linear* in a — classifies the whole 58-dimensional stratum of the construction. We record the outcome as a computation, not as a theorem: sweeping every pair with $k \leq k'$, $k + k' \leq 30$ and every $a' \leq 4000$, exactly in a, the least dimension carrying a violation is 58 for the whole product family and not merely for squares; at $d = 58$ exactly two pairs occur, $(k, k') = (15, 15)$ with $a' \in [237, 480]$ and $(14, 16)$ with $a' \in [155, 223]$; the violated index is always 26; and the total is $2\,089 + 650 = 2\,739$ ordered parameter pairs (a, a'), i.e. 1 698 polytopes up to exchanging the two factors, of which [4] exhibits 7. By Corollary 4.2 this is a classification of every one-step segmental fibration over an even-cycle edge polytope in that dimension.

8. Where the violation is not

We do not answer Question 1.1. This section records where the answer is not, and one positive statement about how far below $d/2$ the violations of this family sit.

First, the question is not vacuous for the polytopes at hand. It would be if the h^*-vector stopped before $d/2$.

Proposition 8.1. *$h^*_{44}(Q \times Q) = 119716$ and $h^*_{45}(Q \times Q) = 0$; hence $\deg h^*_{Q\times Q} = 44$, fifteen indices above $d/2 = 29$.*

Proof. The two values are exact evaluations of the transform. Independently, $\deg h^*_Q = 15$ by Theorem 3.1, so $\operatorname{codeg} Q = 30-15 = 15$, and $\operatorname{int}\big(t(P\times P')\big) = \operatorname{int}(tP)\times\operatorname{int}(tP')$ gives $\operatorname{codeg}(P\times P') = \max(\operatorname{codeg} P, \operatorname{codeg} P')$, whence $\operatorname{codeg}(Q \times Q) = 15$ and $\deg h^*_{Q\times Q} = 59 - 15 = 44$. ☐

Remark 8.2. The codegree formula in that proof is worth stating explicitly because the plausible alternative is false and changes the shape of the problem. Codegree is *not* additive under Cartesian products; were it, one would get $\operatorname{codeg}(Q \times Q) = 30$ and $\deg h^*_{Q\times Q} = 29 = d/2$ exactly, and then no product of two copies of a polytope of this shape could ever answer Question 1.1: a violation at $i > d/2$ needs $h^*_{i+1} > h^*_i \geq 0$ with $i + 1 > \deg h^*$, which is impossible. The correct value is 44, so fifteen indices above the midpoint carry nonzero weight and the search below is a genuine one.

Theorem 8.3. *Let i be the largest index at which $h^*_{\mathcal{P}_{k,a}\times\mathcal{P}_{k,a}}$ has a violation, when one exists, and let $d = 4k - 2$. Then $2i + 6 \leq d$ for every $2 \leq k \leq 26$ and every $0 \leq a \leq 2200$. The same bound, with $d = 2k + 2k' - 2$, holds on the two slices $\mathcal{P}_{15,a} \times \mathcal{P}_{15,332}$ and $\mathcal{P}_{14,a} \times \mathcal{P}_{16,205}$ for every $a \leq 4000$.*

Proof. Exhaustive evaluation: 25×2201 product h^*-vectors for the first assertion, of lengths up to 103, and 2×4001 for the second, with the largest violated index extracted from each. (Granting the computation of Remark 6.2, which is not formally verified, $a \leq 2200$ contains every a at which a violation occurs for k in the stated range, so over $2 \leq k \leq 26$ the first assertion covers all $a \geq 0$ rather than being a truncation.) ☐

So over the parameter ranges of Theorem 8.3 the answer to Question 1.1 is negative inside this family, with room to spare: not only is there no violation above $d/2$, no violation comes closer to $d/2$ than distance 3. The bound is tight, and attained by the source's own polytope: $d/2-i = 29-26 = 3$. None of this makes a violation above $d/2$ impossible in general, and none of it proves unimodality anywhere; it says only that no violation was found here, over the ranges stated.

The following configurations were searched, in exact integer arithmetic, outside the formally verified statements above; we report them as a measurement and not as theorems. Let R range over an explicit library of 130 IDP polytopes (cubes $[0, N]^e$, dilated simplices $N\Delta_e$, dilated cross-polytopes, other members $\mathcal{P}_{k',a'}$ at fixed parameters, and the trivial cofactor).

configurations	scope	outcome
$(\mathcal{P}_{k,a})^r \times R$, $r \in \{1, 2\}$	$2 \leq k \leq 26$, all $a \geq 0$, 14 300 cells	1 613 with a violation, none above $d/2$
$\mathcal{P}_{k,a} \times \mathcal{P}_{k',a'}$	10 349 sampled tuples, $k \leq k' \leq 22$	minimum gap $d/2 - i$ exactly 3
Q^r	$r = 1, \dots, 6$	a violation only at $r = 2$
$Q^r \times R$, $Q \times R$	272 configurations, explicit IDP R	exactly one with a violation, namely $Q \times Q$

Two features of this table are worth naming. First, every nontrivial cofactor destroys the violation. The dip is a 0.209% wiggle — $h^*_{25} - h^*_{26} = 3406628352831057405$ against $h^*_{25} \approx 1.63 \cdot 10^{21}$ — and multiplying by a segment $[0, N]$ replaces h^*_r by the positive mixture $(1+Nr)h^*_r + \big(N(d+2-r)-1\big)h^*_{r-1}$ of consecutive entries, which smooths it away. Second, enlarging k moves the violation *away* from the midpoint rather than towards it: the largest violated index sits 3 below $d/2$ for $15 \le k \le 26$, and 4, 5, 6, 7 at $k = 27, 50, 80, 100$.

Remark 8.4. The distance 3 is not special to this construction. Balletti's two published examples [1, Theorem 4.4] are a 52-dimensional product violating unimodality at h^*_{23} and a 54-dimensional square violating it at h^*_{24}, and $52/2 - 23 = 54/2 - 24 = 3$; [4] has $58/2 - 26 = 3$; and so does every one of the parameter pairs of Remark 7.2. Balletti's examples come from a different construction found by a genetic algorithm, and he records that his products, though spanning, are neither IDP nor very ample — which is exactly why [4] strengthens his result — so the coincidence spans two independent families. His factors do, however, have h^*-vectors of the same silhouette as Theorem 3.1 produces, flat with a single spike at the top, a similarity [4] also notes.

Combining Corollary 4.2 with Theorem 8.3: the even-cycle construction is one-parameter, its 58-dimensional stratum is classified, and over the ranges of Theorem 8.3 no violation in it reaches $d/2 - 2$. An affirmative answer to Question 1.1 along these lines would have to use a different base graph G — one with no two disjoint odd cycles, so that [6, Theorem 3] still applies, and with cycle rank at least 2, so that the fibre over a base point is a linear program over a solution polytope of dimension greater than one and the heights no longer collapse to a single number.

9. Controls

Every statement above is a computation on binomial coefficients, so we verify the arithmetic layer itself rather than trusting it.

Proposition 9.1. *The binomial function used throughout satisfies Pascal's rule together with* $\binom{n}{0} = 1$ *and* $\binom{0}{k+1} = 0$ *at every point of* $[0, 350]^2$. *The* h^**-transform used throughout returns* $(1, 0, 0, 0, 0, 0)$ *on the unimodular* 5*-simplex, the Eulerian numbers* $(1, 4, 1, 0)$ *and* $(1, 11, 11, 1, 0)$ *on the unit* 3*- and* 4*-cubes, and* $(1, 1)$ *on the lattice segment of length* 2. *The violation predicate fires on* $(1, 5, 3, 4, 2)$ *at index* 2 *and is silent on* $(1, 5, 7, 4, 2)$*; and* $h^*_{\mathcal{P}_{15,332}}$ *itself has no violation.*

Those three facts determine the binomial coefficient on $[0, 350]^2$, and every argument occurring anywhere in this paper lies in that square: the largest first argument is $\binom{329}{29}$, from Proposition 3.6 at $m = 300$, and the largest occurring in a sweep is $\binom{153}{51}$, from a product at $k = k' = 26$ in Theorem 8.3. A transform off by a sign, an index or a binomial argument fails all four of the textbook h^*-vectors.

Proposition 9.2. $h^*_{\mathcal{P}_{15,a} \times \mathcal{P}_{15,a}}$ *has no violation for* $a = 331$*,* $a = 339$ *and* $a = 500$*, and has one for* $a = 332$*; and for* $a = 332$ *unimodality is not violated at index* 25 *or at index* 27*, since* $h^*_{24} < h^*_{25}$ *and* $h^*_{27} > h^*_{28}$.

These are the refutations that keep the results above from being satisfiable by a misplaced quantifier: the window is not "all large a", it is not wider than $[332, 338]$ at either end, the phenomenon is not empty, and index 26 is not one of a run.

10. Verification

Every statement in this paper carrying a formal counterpart — Theorems 3.1 (over the box stated after Proposition 3.6), 4.1 (over its 8 192 height vectors), 5.2, 6.1, 6.3, 7.1 and 8.3, and Propositions 3.6, 5.1, 8.1, 9.1 and 9.2 — has been machine-checked in Lean 4 as 46 theorems in eight modules, in 15 minutes 35 seconds of wall time and under 1.6 GB of memory; the development is free of `sorry` and declares no axiom of its own. It does not use Mathlib: the arithmetic layer (binomial coefficients by the multiplicative recurrence, the h^*-transform $h^*_r = \sum_{m \le r} (-1)^{r-m} \binom{d+1}{r-m} E(m)$, the product Ehrhart values, the violation predicate) imports nothing at all, so `Classical.choice` occurs nowhere in the development. Each theorem's axiom set is `propext` together with the single native-evaluation axiom generated for it by Lean's `native_decide`, plus `Quot.sound` for the twelve of them whose statement is an array equality decided through a quotient. What is delegated to compiled evaluation is exactly the finite searches described in the proofs, each stated as a single Boolean over an explicit range so that the enumeration is visible in the statement; the analytic content — Lemmas 3.2–3.4, the generating-function computation in Theorem 3.1, Theorem 4.1 for all c, Corollaries 3.5 and 4.2, Remarks 6.2, 7.2 and the table in Section 8 — is proved or computed by hand and is not formally verified. Every numerical value was produced first by an independent implementation in Python and the Lean statement written against it; in particular the closed form of Theorem 3.1 was checked at $k = 2, 3$ against an exact-rational lattice-point count that uses only the vertex list (1) and none of the fibre argument. Repository reference to be added at submission.

References

[1] G. Balletti, *A genetic algorithm to search the space of Ehrhart h^*-vectors*, Exp. Math. 34 (2025), no. 4, 824–835; arXiv:2309.16848v2.

[2] M. Beck, J. Delgado, J. Gubeladze and M. Michałek, *Very ample and Koszul segmental fibrations*, J. Algebraic Combin. 42 (2015), no. 1, 165–182; arXiv:1307.7422v4.

[3] L. Ferroni and A. Higashitani, *Examples and counterexamples in Ehrhart theory*, EMS Surv. Math. Sci., published online 17 September 2024, `doi:10.4171/EMSS/86`; arXiv:2307.10852v4.

[4] J. Hofscheier, V. Kurylenko and B. Nill, *A very ample lattice polytope with a non-unimodal h^*-vector*, arXiv:2608.21507v1 (2026).

[5] J. Hofscheier, V. Kurylenko and B. Nill, *Examples of IDP lattice polytopes with non-log-concave h^*-vector*, arXiv:2505.18896v1 (2025).

[6] M. Lasoń and M. Michałek, *Non-normal very ample polytopes — constructions and examples*, Exp. Math. 26 (2017), no. 2, 130–137; arXiv:1406.4070v2.

[7] T. Matsui, *Ehrhart series for connected simple graphs*, arXiv:1102.4804v2; published as Graphs Combin. 29 (2013), no. 3, 617–635.

[8] H. Ohsugi, J. Herzog and T. Hibi, *Combinatorial pure subrings*, Osaka J. Math. 37 (2000), no. 3, 745–757.

[9] H. Ohsugi and T. Hibi, *Normal polytopes arising from finite graphs*, J. Algebra 207 (1998), no. 2, 409–426.

[10] R. P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. 6 (1980) 333–342.

[11] J. Schepers and L. Van Langenhoven, *Unimodality questions for integrally closed lattice polytopes*, Ann. Comb. 17 (2013), no. 3, 571–589; arXiv:1110.3724v1.