COPRIMALITY OF CONSECUTIVE EULER UP/DOWN NUMBERS AND THE PREPERIOD AT ODD PRIME POWERS

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ABSTRACT. Let E_n be the number of alternating permutations of $\{1,\dots,n\}$, so that $\sec z+\tan z = \sum_{n\ge 0} E_n z^n/n!$. Knuth and Buckholtz proved that $(E_n \bmod q)$ is eventually periodic; write $d(q)$ for its minimal eventual period and $s(q)$ for its preperiod. Ramassamy conjectured $s(p^r) = r$ at every odd prime power; Güleç has recently disproved this with $s(5^5) = 4$, proved the companion period conjecture, and — on the strength of a search over $p^r \le 10^{100000}$ — conjectured the weaker bound $s(p^r) \ge r-2$. We observe that his divisibility criterion turns that conjecture into a statement about greatest common divisors of consecutive Euler numbers: the single integer $\gcd(E_{r-3}, E_{r-2}, E_{r-1})$ settles the index r for *every* prime p at once, with no upper bound on p whatsoever. Computing the Euler numbers exactly to index 10 000 we prove that no prime divides three consecutive Euler up/down numbers of index at most 10 000, hence that $s(p^r) \ge r-2$ for every odd prime p and every $4 \le r \le 10001$ — a region no search organised by the size of p^r can cover. We classify, over all odd primes at once, the cells with $s(p^r) \le r-2$ in that range: they are exactly $p = 43$ and $r \in \{980, 2786, 4592, 6398, 8204\}$. In particular 43^{980} is the *smallest* odd prime power with $s(p^r) \le r-2$, a minimality the source only suggests. We also determine the primes dividing two consecutive Euler up/down numbers below index 10 000 — exactly 43, 433, 1093, 5107, 13367 — and reproduce independently every computational assertion of the source's final section that names an explicit value, assertions its own declaration attributes to language models and for which no code is distributed. Every computation below is machine-checked in Lean 4; the passage from divisibility to preperiods is the source's criterion, quoted and not reproved.

1. INTRODUCTION

For $n \ge 0$ let E_n be the number of alternating permutations of $\{1,\dots,n\}$ — the *Euler up/down numbers*, sequence A000111 of the OEIS [9]; see [12] for a survey of alternating permutations. André [1] showed that

$$\sum_{n\ge 0} E_n \frac{z^n}{n!} = \sec z + \tan z, \qquad E = 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \dots \tag{1}$$

Knuth and Buckholtz [5] proved that for every $q \ge 1$ the reduced sequence $(E_n \bmod q)_{n\ge 0}$ is eventually periodic. Write $d(q)$ for its minimal eventual period and $s(q)$ for its *preperiod*, the least index from which some (equivalently, by [3, Lemma 2.2], every) eventual period holds. For an odd prime p they proved

$$d(p) = \operatorname{lcm}(p-1, 4), \qquad d(p^r) \mid p^{r-1} d(p), \qquad s(p^r) \le r \quad (r \ge 1), \tag{2}$$

and Ramassamy [11] conjectured that the last two bounds are attained: $d(p^r) = p^{r-1} d(p)$ for $r \ge 2$ and $s(p^r) = r$ for $r \ge 1$. His evidence is stated, in the e-print version of [11], as "Mathematica simulations done for all odd prime powers $q < 1000$".

Güleç [3] settles both halves, in opposite directions. A frequency expansion for the Euler sequence over $(\mathbb{Z}/p^r\mathbb{Z})[x]/(x^2+1)$ yields the period conjecture in full, and also yields $s(5^5) \le 4 < 5$, refuting the preperiod conjecture; $5^5 = 3125$ is the smallest odd prime power at which $s(p^r) \ne r$, which is why it lies just outside Ramassamy's range. The same expansion produces the criterion that the present paper runs on: for p an odd prime, $r \ge 2$ and $1 \le k \le r-1$,

$$s(p^r) \le r - k \iff p^j \mid E_{r-j} \quad \text{for every } 1 \le j \le k. \tag{3}$$

The preperiod at p^r is thus decided by the divisibility of the *first few* Euler numbers E_{r-1}, E_{r-2}, E_{r-3}, ... by increasing powers of p. On the strength of a search reported in [3, §7] over odd prime powers $p^r \le 10^{100000}$, which found no cell with $s(p^r) \le r-3$, that paper conjectures:

Conjecture 1.1 ([3, Conjecture 7.6])**.** $s(p^r) \ge r-2$ for every odd prime p and every $r \ge 2$.

Two features of [3] shape what follows. First, its Declaration on the use of generative AI states that "the computational analysis of finding counterexamples associated with [Section 7] was likewise performed by the models, under the author's direction and command", and no code accompanies the paper. Its computational assertions — the count of 461 exceptions below 10^{300}, the single example $s(43^{980}) = 978$ found between 10^{1500} and 10^{2000}, and the exhaustive search over $p^r \le 10^{100000}$ behind Conjecture 1.1 — therefore have no independently runnable justification on record. Second, the search behind Conjecture 1.1 is organised by the *size* of p^r. Any such search is, for each fixed r, a search over the primes $p \le X^{1/r}$; it can never make a statement about all primes at a given index.

The device. Rewriting (3) at $k=2$ and $k=3$ removes the prime from the data being computed. If $s(p^r) \le r-2$ then p divides both E_{r-2} and E_{r-1}, so

$$p \mid \gcd(E_{r-2}, E_{r-1}); \qquad \text{and if } s(p^r) \le r-3 \text{ then } p \mid \gcd(E_{r-3}, E_{r-2}, E_{r-1}).$$

One integer per index r therefore rules out every prime simultaneously, with no upper bound on p at all. The price is that these gcds must be formed from the exact Euler numbers, not from residues: E_{10000} has 33 699 decimal digits. That is the whole cost of the method, and it buys a region of the (p, r) plane that is a half-plane in p.

Results. Section 3 reproduces, independently and from the boustrophedon recurrence alone, every computational assertion of [3, §7] that names an explicit value: the count of 461 exceptions and $s(43^{980}) = 978$, together with the criterion itself on a table of cells and the minimality of 5^5. All are confirmed; there is no divergence to report, and the 461 exceptions are obtained twice by routes sharing no code. The one assertion of [3, §7] not reproduced here is the exhaustive search over $p^r \le 10^{100000}$, which is out of reach at that scale; Theorem 4.1 answers instead in a region that search cannot enter. This was carried out before any new claim below was formed.

Section 4 contains the new results. The first settles Conjecture 1.1 on a region unreachable by a $p^r \le X$ search.

Theorem 1.2 (Theorem 4.1)**.** *No prime divides three consecutive Euler up/down numbers of index at most* 10 000*. Consequently $s(p^r) \ge r-2$ for every odd prime p and every $4 \le r \le 10001$, with no restriction on p; with the endpoint bound $s(p^r) \ge 1$ that is Conjecture 1.1 for every $2 \le r \le 10001$.*

The region $\{(p, r) : p \text{ odd prime},\ r \le 10001\}$ is not contained in the band $p^r \le 10^{100000}$: every prime $p > 10^{10}$ with $r = 10000$ lies in the first and outside the second. Neither region contains the other — for $p = 3$ the band reaches $r \approx 209590$ — so the two verifications are incomparable, and the one below is complete in the direction the other cannot reach.

The second result is a classification rather than an example.

Theorem 1.3 (Theorem 4.2)**.** *For every odd prime p and every $2 \le r \le 10001$ one has $s(p^r) \le r-2$ if and only if $p = 43$ and $r \in \{980, 2786, 4592, 6398, 8204\}$. In particular 43^{980} is the smallest odd prime power with $s(p^r) \le r-2$.*

The minimality is a genuine strengthening: [3, §7] reports finding $s(43^{980}) = 978$ after a search that found no example up to 10^{1500}, and asserts nothing about minimality. Since $43^{980} = 10^{1600.79\ldots}$, every odd prime power below it has $r \le 3355 < 10001$, so the classification decides all of them at once. The five indices form an arithmetic progression of common difference $1806 = 43 \cdot 42 = p(p-1)$; whether that is a theorem is Question 5.1.

The third result concerns the gcd stream in its own right, and appears not to be recorded anywhere.

Theorem 1.4 (Theorem 4.3)**.** *The primes dividing two consecutive Euler up/down numbers of index below* 10 000 *are exactly* 43*,* 433*,* 1093*,* 5107 *and* 13367*.*

Section 5 collects what the computation suggests and does not prove. Section 6 states precisely what has been machine-checked and where the finite searches enter.

Priority. The refutation of the preperiod conjecture is not first published in [3]. The same author deposited it on Zenodo on 2026-08-16, eleven days before the arXiv announcement of 2026-08-27, as version 1 of the record with concept DOI `10.5281/zenodo.21960726` [4], under the title *Modular Periodicity of the Euler Up/Down Numbers at Odd Prime Powers: A Proof of the Period Conjecture and a Counterexample to the Preperiod Conjecture*. Its Theorem 7.4 is the counterexample $s(5^5) \neq 5$, and its §8 reports an exhaustive search over all odd prime powers $p^r \leq 999\,999\,999$ in which $s(p^r) = r$ fails exactly at 5^5 and 5^9, the two preperiods being reported as $s(5^5) = 4$ and $s(5^9) = 8$. Priority for the counterexample and for the value $s(5^5) = 4$ of Proposition 3.3 therefore sits with that deposit. What version 1 does *not* contain is the criterion (3), the count of 461 exceptions, or $s(43^{980}) = 978$: those appear first in version 2 of the same record, of 2026-08-26, one day before the arXiv announcement. Both versions and the arXiv paper are by the same author, so there is no independent second source for any of the values reproduced in Section 3.

Section and result numbers cited from [3] are those of the arXiv version 1 source as it compiles: the computational section is §7, carrying Lemmas 7.1–7.3, Theorem 7.5 and Conjecture 7.6, while the counterexample is Theorem 6.4 and the period conjecture is Theorem 5.9. They may of course move in a later version or in print.

2. Preliminaries

2.1. **The boustrophedon recurrence.** Every computation below takes as its only input the Entringer triangle [2, 8]

$$T(0,0) = 1, \qquad T(n,0) = 0 \ (n \geq 1), \qquad T(n,k) = T(n,k-1) + T(n-1,n-k), \tag{4}$$

whose diagonal is the Euler sequence, $E_n = T(n,n)$. Row $n+1$ is the sequence of partial sums of row n read backwards, so one pass over the triangle produces $E_0, \ldots, E_N$ in $\Theta(N^2)$ additions. Nothing from the algebra of [3] — Hurwitz series, the frequency expansion — is used anywhere in this paper: (1) and (4) are all that is assumed about E. The same recurrence reduced modulo m produces $(E_n \bmod m)_{n \leq N}$ without ever forming a large integer, and the two agree: this is the elementary compatibility of (4) with reduction.

2.2. **Periods and preperiods.** Let $a = (a_n)_{n \geq 0}$ be a sequence. A positive integer P is an *eventual period* of a if $a_{n+P} = a_n$ for all n beyond some threshold t; the set of admissible thresholds for P is

$$\mathcal{T}_{a,P} = \{t \geq 0 : \ \forall n \geq t, \ a_{n+P} = a_n\}.$$

By [3, Lemmas 2.1 and 2.2] the eventual periods of an eventually periodic a are exactly the multiples of the least one $d(a)$, and $\mathcal{T}_{a,P} = \mathcal{T}_{a,d(a)}$ for every eventual period P; so the preperiod $s(a) = \min \mathcal{T}_{a,d(a)}$ may be computed from any eventual period at all. For $q \geq 1$ we write $d(q)$ and $s(q)$ for these quantities applied to $(E_n \bmod q)_{n \geq 0}$.

Two consequences make each individual value a finite computation, and they are the only published facts the computations of Section 3 lean on. Given the Knuth–Buckholtz bounds (2), put $D := p^{r-1} \operatorname{lcm}(p-1, 4)$. Then $d(p^r)$ is the least divisor of D that is an eventual period, and $s(p^r)$ is the least $t \leq r$ with $t \in \mathcal{T}_{a,d(p^r)}$; both are decided by finitely many comparisons inside the window $0 \leq n \leq r + 2D + 2$. We record them in this shape, as hypotheses rather than as background, so that a lower bound on a preperiod is unconditional — a single index $n \geq t$ with $a_{n+P} \neq a_n$ refutes $t \in \mathcal{T}_{a,P}$ outright — while an upper bound is stated as an implication from the threshold r of (2).

2.3. **The criterion.** We use the following results of [3] as published; nothing in this paper reproves them.

Theorem 2.1 ([3, Lemmas 7.1 and 7.2])**.** *Let p be an odd prime, $r \geq 2$ and $1 \leq k \leq r-1$. Then* (3) *holds: $s(p^r) \leq r-k$ if and only if $p^j \mid E_{r-j}$ for every $1 \leq j \leq k$.*

Theorem 2.2 ([3, Lemma 7.3 and the paragraph following it])**.** *$s(p) = 1$ for every odd prime p; consequently $s(p^r) \geq 1$ for every $r \geq 1$.*

Theorem 2.2 disposes of Conjecture 1.1 for $r \leq 3$, where $r-2 \leq 1$. All the work is at $r \geq 4$, and there Theorem 2.1 takes the following form.

Lemma 2.3. *Let p be an odd prime and $r \geq 2$.*

(1) *If $s(p^r) \leq r-2$ then $p \mid \gcd(E_{r-2}, E_{r-1})$.*
(2) *If $r \geq 4$ and $s(p^r) \leq r-3$ then $p \mid \gcd(E_{r-3}, E_{r-2}, E_{r-1})$.*

In particular, if $\gcd(E_{r-3}, E_{r-2}, E_{r-1}) = 1$ then $s(p^r) \geq r-2$ for every *odd prime p simultaneously.*

Proof. Take $k = 2$, respectively $k = 3$, in Theorem 2.1 and weaken $p^j \mid E_{r-j}$ to $p \mid E_{r-j}$; a common divisor of the listed Euler numbers divides their gcd. For the last sentence, $p \mid 1$ is impossible. □

The implications in Lemma 2.3 are one-way, and deliberately so: what is cheap is the gcd, and the gcd alone already decides the question negatively for all p at once. Where a gcd is not 1 the full criterion is applied to the finitely many primes that survive, which is how Theorem 4.2 is proved.

3. The published computations, reproduced

The computational assertions of [3, §7] that name explicit values are recomputed here from (4), and each is confirmed. We record them because everything in Section 4 is read through Theorem 2.1, so an independent test of that criterion against the definition of eventual periodicity is a precondition, not a courtesy.

Proposition 3.1. *Let $\mathcal{C}$ be the 24 cells (p, r) with $r \geq 2$ and $D = p^{r-1}\operatorname{lcm}(p-1, 4) \leq 3000$. On every cell of $\mathcal{C}$, the preperiod computed from the definition of eventual periodicity agrees with the value predicted by the criterion of Theorem 2.1.*

Proposition 3.2. *On every cell of $\mathcal{C}$, D is an eventual period of $(E_n \bmod p^r)$ from index r, and the minimal eventual period is D itself; that is, $d(p^r) = p^{r-1}\operatorname{lcm}(p-1, 4)$ there. Moreover no proper divisor of 2500 is an eventual period of $(E_n \bmod 5^5)$ from index 5.*

Proposition 3.2 is Ramassamy's period conjecture at these cells, proved in general as [3, Theorem 5.9]; nothing here is new, and the second sentence is the sharpness statement for the one cell used below. Its proof is the observation that every proper divisor of 2500 divides 1250 or 500, together with the refutation of those two.

Proposition 3.3. *$s(5^5) = 4$. The lower bound $s(5^5) > 3$ is unconditional: $E_3 \equiv 2$ and $E_{2503} \equiv 2877 \pmod{3125}$, so 2500 is not a period from index 3. The upper bound is the single congruence $E_{2504} \equiv E_4 \pmod{3125}$, which moves the Knuth–Buckholtz threshold $r = 5$ down to 4.*

This is the counterexample of [3, Theorem 6.4], the exact value being [3, Theorem 7.5]; both are already in version 1 of [4], and the value is theirs. What is added is that it is recomputed from the definition of eventual periodicity rather than from the frequency expansion, so that Proposition 3.1 is a real test of (3) and not a re-run of it. Two implementations sharing no code agree on $s(5^5) = 4$, on $d(5^5) = 2500$, and on $s(p^r) = r$ for all 26 odd prime powers $p^r < 3125$ with $r \geq 2$; with $s(p) = 1 = r$ at $r = 1$ that is the minimality of 5^5.

Proposition 3.4. *There are exactly 444 odd prime powers $p^r \leq 10^{300}$ with $p \leq 10^6$ and $s(p^r) \neq r$, and none of them has $s(p^r) \leq r-2$. There are a further 17 explicit cells with $p > 10^6$, each verified to satisfy $p^r \leq 10^{300}$, $p \mid E_{r-1}$ and $p^2 \nmid E_{r-2}$. Together that is $444 + 17 = 461$ exceptions, every one with $s(p^r) = r-1$ exactly.*

The 17 cells with $p > 10^6$ are, as pairs (p, r):

$$(1567103, 23),\ (2294797, 24),\ (2717447, 29),\ (2796203, 46),\ (2947939, 44),\ (4241723, 15),$$
$$(8429689, 35),\ (15669721, 31),\ (42737921, 33),\ (73026287, 37),\ (77980901, 29),$$
$$(87224971, 19),\ (228135437, 17),\ (305065927, 32),\ (354957173, 21),\ (1001259881, 30),$$
$$(1427513357, 23).$$

Both numbers of [3, §7] — the count 461 and the assertion that the preperiod was $r - 1$ in every case — are thus reproduced exactly. The count was obtained twice, by routes sharing no code: a boustrophedon sweep modulo p^2 over all odd primes $p \le 10^7$, and complete factorisation of E_{r-1} for $r \le 46$, which suffices because $p > 10^7$ and $p^r \le 10^{300}$ force $r \le 42$. The two routes agree set for set on the 104 cells in their overlap.

Remark 3.5. The completeness of the count in Proposition 3.4 is machine-checked only for $p \le 10^6$, which is 444 of the 461 cells. The 17 cells above 10^6 are each verified individually, but the claim that there are no others rests on the two computations named in the previous paragraph, neither of which is carried out inside the proof assistant: the sweep to 10^7 is a C program, and the factorisations were produced by `GNU factor` 9.11. Closing that gap needs primality certificates for factors of up to 38 decimal digits.

Proposition 3.6. *$s(43^{980}) = 978$: one has $43 \mid E_{979}$, $43^2 \mid E_{978}$ and $43^3 \nmid E_{977}$.*

This is the value reported in [3, §7], and in version 2 of [4] but not in version 1; it is recomputed here from the boustrophedon recurrence modulo 43^4. Its *minimality* is a separate claim, and is Theorem 4.2 below.

Proposition 3.7 (Controls)**.** *$s(3^7) = 7$ and $s(5^4) = 4$, both equal to r; and 2500 is an eventual period of $(E_n \bmod 5^5)$ from index 4 but is* not *an eventual period of $(E_n \bmod 5^6)$ from index 5.*

Proposition 3.7 is not decoration. The first two cells refute the possibility that the machinery reports a preperiod drop wherever it is pointed — 5^4 sits one power below the counterexample and shows the criterion's first test $p \mid E_{r-1}$ genuinely failing, since $5 \nmid E_3 = 2$. The third refutes the possibility that a period at one modulus survives at the next power, which would make Proposition 3.2 vacuous.

4. One gcd rules out every prime

Fix $N = 10000$ and let $g_2(n) = \gcd(E_n, E_{n+1})$ and $g_3(n) = \gcd(E_n, E_{n+1}, E_{n+2})$. One pass of (4) over the exact integers produces $E_0, \dots, E_N$ and both streams. The results of this section are read off that single pass.

Theorem 4.1. *$g_3(n) = 1$ for every $n \le 9998$. Equivalently, no prime divides three consecutive Euler up/down numbers of index at most 10 000. Consequently*

$$s(p^r) \ge r - 2 \qquad \text{for every odd prime } p \text{ and every } 4 \le r \le 10001,$$

with no upper bound on p; together with Theorem 2.2, which covers $r \le 3$, Conjecture 1.1 holds for every odd prime p and every $2 \le r \le 10001$.

Proof sketch. The first sentence is an exhaustive computation: the 9999 values $g_3(0), \dots, g_3(9998)$ are formed from the exact Euler numbers and every one of them is checked to be 1. This is the only search behind the theorem, and it is complete over its stated range. A prime dividing E_n, E_{n+1} and E_{n+2} would divide $g_3(n) = 1$; this is the second sentence. For the third, let p be an odd prime and $4 \le r \le 10001$ and suppose $s(p^r) \le r - 3$. By Lemma 2.3(2), $p \mid g_3(r - 3)$ with $r - 3 \le 9998$, which is impossible. Hence $s(p^r) \ge r - 2$. □

The passage from the divisibility statement to the preperiod statement is exactly one application of the published criterion (3); the computation itself knows nothing about preperiods. Note also what the search is *not*: it is not a search over primes. For each index r a single integer is formed and tested, and no prime is ever enumerated, so the conclusion is uniform in p. This is why the region covered is incomparable with a band $p^r \le X$, as recorded in Section 1.

Theorem 4.2. *Let p be an odd prime and $2 \le r \le 10001$. Then $s(p^r) \le r-2$ if and only if $p = 43$ and $r \in \{980, 2786, 4592, 6398, 8204\}$. Consequently 43^{980} is the smallest odd prime power p^r with $s(p^r) \le r-2$, over all odd primes and all r whatsoever.*

Proof sketch. The finite search is the same single pass: for every $n \le 9999$ the value $g_2(n)$ is formed, the indices with $g_2(n) \ne 1$ are collected, and among those the indices with $g_2(n)^2 \mid E_n$ are retained. The retained list is exactly

$$(978, 43),\ (2784, 43),\ (4590, 43),\ (6396, 43),\ (8202, 43),$$

each pair being $(n, g_2(n))$. The five values of $r = n+2$ are those in the statement.

For the "only if" direction let $s(p^r) \le r-2$ with $2 \le r \le 10001$, and put $n = r-2 \le 9999$. By Theorem 2.1, $p \mid E_{n+1}$ and $p^2 \mid E_n$, so $p \mid g_2(n)$ and $g_2(n) \ne 1$. By Theorem 4.3 below, whose search is the same pass, $g_2(n)$ is one of the five primes $43, 433, 1093, 5107, 13367$; since p and $g_2(n)$ are both prime and $p \mid g_2(n)$, in fact $p = g_2(n)$, so $g_2(n)^2 \mid E_n$ and (n, p) lies in the retained list. The "if" direction is the same list read forwards: each retained pair gives $43 \mid E_{n+1}$ and $43^2 \mid E_n$, hence $s(43^{n+2}) \le n$ by Theorem 2.1.

For minimality, suppose $p^r < 43^{980}$ and $s(p^r) \le r-2$. Then $r \ge 2$, since $s \ge 0$ makes $s(p) \le -1$ impossible. As $p \ge 3$ and $43^{980} = 10^{1600.79\ldots}$ we get $r \le 3355 < 10001$, so the classification applies and forces $p^r \ge 43^{980}$, a contradiction. □

Theorem 4.3. *The primes p for which there exists n with $n+1 \le 10000$, $p \mid E_n$ and $p \mid E_{n+1}$ are exactly*

$$43, \quad 433, \quad 1093, \quad 5107, \quad 13367.$$

Proof sketch. Exhaustively, the distinct values taken by $g_2(n)$ above 1 for $n \le 9999$ are the five listed integers, and each of them is prime. If $p \mid E_n$ and $p \mid E_{n+1}$ then $p \mid g_2(n)$ and $g_2(n) \ne 1$, so $g_2(n)$ is one of the five and $p = g_2(n)$ by primality. Conversely each of the five occurs as some $g_2(n)$ and so divides two consecutive Euler numbers. □

The list is startlingly short: 10 000 gcds produce five distinct prime values. We have not found the set $\{p : \exists n,\ p \mid E_n \text{ and } p \mid E_{n+1}\}$ recorded anywhere, either as a sequence or in the literature on the arithmetic of A000111.

Proposition 4.4 (Controls)**.** $\gcd(E_{12}, E_{13}) = 43$*;* $43^2 \mid E_{978}$ *and* $43 \mid E_{979}$*; and* $43^3 \nmid E_{977}$.

These three facts are what stop Theorems 4.1 and 4.2 from being vacuous or from being the shadow of something weaker. The first shows that consecutive Euler numbers are *not* always coprime, so Theorem 4.1 is a statement about triples and not a corollary of pairwise coprimality. The second exhibits the hypothesis of Theorem 4.2 as realised, so its list is not empty for want of an example. The third shows that the criterion's next test genuinely fails at 43^{980}, so the preperiod there is 978 exactly and not smaller — which is also Proposition 3.6.

Remark 4.5. Four of the five primes of Theorem 4.3 are Euler-irregular, i.e. divide E_{2k} for some $2k < p-1$ (sequence A120337 [10], whose b-file lists all 1000 such primes below 22 728): 43, 433, 5107 and 13367 are in that list; 1093 is not. And 1093 is one of the two known Wieferich primes. Neither property is the criterion, and four out of five is no evidence for one: the Euler-irregular primes are conjectured to have density $1 - e^{-1/2} = 0.3934\ldots$ among all primes [10]. Computation outside the formal development, reported here as a measurement only: 461 is Euler-irregular, yet $(E_n \bmod 461)$ has no two consecutive zeros for $n < 1854$. "Divides two consecutive E_n" is a strictly stronger condition than either, and we have no structural explanation for it.

5. Questions

An arithmetic progression. The five indices of Theorem 4.2 are $r = 980 + 1806k$ for $0 \le k \le 4$, with

$$1806 = 43 \cdot 42 = p(p-1).$$

This is not an accident of the range: it is the shape one expects from the period of $(E_n \bmod p^2)$, which by (2) and [3, Theorem 5.9] is $p \cdot \operatorname{lcm}(p-1, 4) = 3612$ for $p = 43$ — but 1806 is half of that, so a parity argument is needed as well as a periodicity argument, and we have not supplied one.

Question 5.1. Is it true that, for a fixed odd prime p, the set $\{r : s(p^r) \le r - 2\}$ is (eventually) a union of arithmetic progressions of common difference dividing $p(p-1)$? Concretely: does $s(43^{980+1806k}) = 978 + 1806k$ hold for every $k \ge 0$?

Question 5.1 costs no computation to attack; it asks for a proof that the divisibility pattern $43 \mid E_{r-1}$, $43^2 \mid E_{r-2}$ propagates along the period of the Euler sequence modulo 43^2. It is the cheapest open problem this paper produces.

Preperiod $r - 3$. No cell with $s(p^r) \le r - 3$ is known. Confirming or breaking Conjecture 1.1 requires a prime dividing three consecutive Euler numbers, and Theorem 4.1 says there is none below index 10 000. Theorem 4.3 suggests why a search is hard: even *two* consecutive divisions already force p into an extremely thin set, so a targeted search — for primes whose zero set modulo p contains three consecutive indices — looks more promising than any sweep.

Remark 5.2 (Measurement, outside the formal development). The exact gcd stream has been carried to index 20 000 by a C program using GMP, with the following outcome, which is reported here as a measurement and is *not* part of the machine-checked development: no prime divides three consecutive Euler up/down numbers of index at most 20 000; the cells with $s(p^r) \le r - 2$ and $r \le 20001$ are exactly $43^{980+1806k}$ for $0 \le k \le 10$; and one further prime, 30241, joins the list of Theorem 4.3 at $n = 19979$, its only occurrence below index 20 000. The six primes first occur at $n = 12$, 214, 1091, 4870, 5329, 19979 respectively, and each of the five primes of Theorem 4.3 recurs, in the index n, with period exactly $p - 1$ throughout that range; for 30241 the range is too short for a recurrence to be seen at all. Extending the stream is priced by the recurrence: computing $E_0, \dots, E_N$ exactly costs $\Theta(N^3 \log N)$ bit operations, measured at 45 minutes and 1.7 GB of memory at $N = 20000$, so $N = 40000$ costs about six processor-hours. Since the *output* is only $\Theta(N^2 \log N)$ bits, a power-series route for $\sec + \tan$ over $\mathbb{Z}$ — Newton iteration with Kronecker-substitution products — should reach $N = 10^5$ far more cheaply; we have not implemented it.

Below the surface. Finally, the classification of Theorem 4.2 says that in the whole range only one prime, 43, ever reaches $s(p^r) \le r - 2$. Whether the set of primes admitting such a cell is finite, and whether the set of Theorem 4.3 is finite, are both open; we have not formulated a heuristic that predicts the counts we observe.

6. Verification

Every statement in this paper labelled as a theorem or a proposition has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [6, 7], with two exceptions of different kinds. Theorems 2.1 and 2.2 are quoted from [3]; they are neither reproved nor formalised here. And what the formal development contains is in every case the divisibility or congruence content of the statement, not the sentence about preperiods that is read off it: for Theorems 4.1, 4.2 and 4.3 it is that $\gcd(E_n, E_{n+1}, E_{n+2}) = 1$ for $n \le 9998$, that the pairs (n, p) with $n + 1 \le 10000$, p prime, $p \mid E_{n+1}$ and $p^2 \mid E_n$ are exactly the five listed, and that the values $\gcd(E_n, E_{n+1}) > 1$ for $n \le 9999$ are exactly the five listed primes. The passage from there to a preperiod, and likewise the minimality argument of Theorem 4.2 — that every $p^r < 43^{980}$ has $r \le 3355$ — is written out in the proofs and is on paper. The development is free of `sorry`, and the repository reference is to be added at submission. The Euler sequence is defined there twice — exactly from (4), and reduced modulo

m — and the identification of the two, $E_n \bmod m$ for the reduced stream and $[E_0, \ldots, E_N]$ for the exact one, is proved with no axioms beyond `propext` and `Quot.sound`; that identification is what lets the cheap modular sweeps and the expensive exact gcd stream be read as statements about the same sequence. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches, each recording one evaluation axiom: one pass of (4) to index 10 000 over the exact integers for Theorems 4.1, 4.2 and 4.3; one pass modulo p^r per cell for Propositions 3.1, 3.2, 3.3 and 3.7; one pass modulo p^2 per prime for the sweep of Proposition 3.4; and one pass modulo 43^4 for Propositions 3.6 and 4.4. The scope of that delegation is stated exactly in Remark 3.5 for the one claim whose completeness is not machine-checked throughout, and in Remark 5.2 for the extension to index 20 000, which is a measurement and carries no formal status. Every numerical value quoted was first produced by an independent implementation in C — for Proposition 3.4 by two such implementations, agreeing — and the formal statement was then written against it.

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