ULTRA-EUCLIDEAN FUNCTIONS INVERT THE INTEGERS: $\mathbb{Z}$ CARRIES NONE, AND TWO SETTLED CASES OF A CONJECTURE OF SEKHON

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ABSTRACT. Let R be an integral domain and $f\colon R\setminus\{0\}\to\mathbb{Z}_{\geq 0}$ a Euclidean function in the modern sense, so that submultiplicativity $f(a)\leq f(ab)$ is *not* assumed. Following Sekhon, call f ultra-Euclidean if $f(a+b)\leq\max\{f(a),f(b)\}$ whenever $a,b,a+b\neq 0$, and write $\widetilde{f}(a)=\min_{b\neq 0}f(ab)$ for Rogers' refinement. Sekhon conjectures that $\widetilde{f}$ is ultra-Euclidean whenever f is, which would force every domain carrying an ultra-Euclidean function to be a field or a polynomial ring $K[x]$; that consequence was asked independently, and earlier, by Elliott on MathOverflow and by Elliott and Epstein, and under the extra hypothesis of submultiplicativity it is known. We prove that an ultra-Euclidean function makes every nonzero integer multiple of 1 a unit. Hence a characteristic-zero domain carrying one is a $\mathbb{Q}$-algebra, and $\mathbb{Z}$ — although it is a Euclidean domain — carries no ultra-Euclidean function at all, as does no ring of integers of a number field; on all such rings the conjecture holds vacuously. In characteristic $p>0$ we prove the conjecture for every domain whose units are exactly the nonzero elements of the prime field, in the strong form that every ultra-Euclidean function there is already submultiplicative and equal to its own refinement; the case $R=\mathbb{F}_p[x]$ is included. We also settle the conjecture on every field and exhibit an ultra-Euclidean function on $\mathbb{Q}$ that is not submultiplicative, which our first theorem shows is the smallest such example in characteristic zero. The results below are machine-checked in Lean 4, with the exceptions identified in §7; none of them rests on an exhaustive search or on compiled evaluation.

1. INTRODUCTION

The objects. Throughout, R is an integral domain, that is, a commutative ring with 1 and no nonzero zero divisors, and $R^\times$ is its group of units. A *Euclidean function* on R is a map $f\colon R\setminus\{0\}\to\mathbb{Z}_{\geq 0}$ such that for all $a,b\in R$ with $b\neq 0$ there exist $q,r\in R$ with

$$a=qb+r\quad\text{and}\quad\big(r=0\ \text{ or }\ f(r)<f(b)\big).$$

This is the modern definition, [1, Definition 2.1]; it is also [8, Definition 1.2], which discusses the coexistence of the two definitions in the textbook literature. The older definition, which is the one under which the classical theory was developed, demands in addition that

$$f(a)\leq f(ab)\qquad\text{for all } a,b\in R\setminus\{0\},\tag{1}$$

and a function with that extra property is called *strongly Euclidean* [1, Definition 2.2] or *submultiplicative* [2, §8]. The two definitions describe the same class of *rings*, by a theorem of Rogers [5]: the *refinement*

$$\widetilde{f}(a)\ =\ \min_{b\in R\setminus\{0\}}f(ab)\qquad(a\in R\setminus\{0\})$$

of a Euclidean function is again Euclidean, and is strongly Euclidean ([1, Proposition 2.6], [8, Theorem 2.1], [2, §8]; it should not be confused with the *minimal* Euclidean function of the classical theory of Euclidean rings, see [9], which is a different construction). The two definitions do *not* describe the same class of functions, and it is exactly on that difference that the question treated here turns.

Sekhon [1, Definition 2.8] calls a Euclidean function f on R *ultra-Euclidean* if

$$f(a+b)\leq\max\{f(a),f(b)\}\qquad\text{for all } a,b\in R\setminus\{0\}\text{ with } a+b\neq 0,$$

the ultrametric triangle inequality; in [3] and [2, Definition 8.1] the same condition is called *non-archimedean*. The degree function on $K[x]$ is ultra-Euclidean; the absolute value on $\mathbb{Z}$ is not, since $|1+1|=2>1=\max\{|1|,|1|\}$.

The question. The closing section of [1] poses:

Conjecture ([1, §5]). If f is ultra-Euclidean on R, so is $\widetilde{f}$.

with the comment "We were not able to prove or disprove this conjecture", and derives from it the

Conjecture (Corollary) ([1, §5]). If R has an ultra-Euclidean function, then either R is a field or $R \cong K[x]$ for some field K.

The derivation is short and uses the main theorem of [1]: if the Conjecture holds and f is ultra-Euclidean, then $\widetilde{f}$ is ultra-Euclidean, and it is strongly Euclidean by Rogers, so it is *uniquely* Euclidean by [1, Corollary 4.2] — meaning that the q and r above are unique — and then R is a field or a polynomial ring by [1, Theorem 4.4], which is Sekhon's improvement, freed of hypothesis (1), of a theorem of Rhai [7] and Jodeit [6].

The conjectured Corollary is not only Sekhon's. It was posed as Question 1 of the MathOverflow question [3] by Elliott on 15 January 2025, fifteen months before the first version of [1] appeared, in the explicit form: *if A is a Euclidean domain that admits a nonarchimedean Euclidean function, then must A be either a field or isomorphic $k[x]$ for some field k?*, together with the equally explicit warning that "we do not impose the requirement that a Euclidean function σ on A be submultiplicative". It appears a third time, as Problem 8.6 of [2], in the form of a request for a counterexample: *does there exist a Euclidean domain that admits a non-archimedean generalized Euclidean function and yet is neither a field nor isomorphic to $k[x]$ for some field k?* (In [2] a "Euclidean function" is by definition submultiplicative and a "generalized Euclidean function" is a Euclidean function in the sense used here.)

With hypothesis (1) added, the answer is known and positive: Question 2 of [3], the submultiplicative version, was answered by [4], and the resulting statement is printed with credit as [2, Proposition 8.5]: *an integral domain A admits a non-archimedean Euclidean function on A if and only if $A = k$ or $A \cong k[x]$, where $k = A^\times \cup \{0\}$ is a field.* Under the still stronger hypothesis $\sigma(ab) = \sigma(a) + \sigma(b)$ the statement goes back to an exercise in Jacobson's *Basic Algebra I*, and is recorded as [2, Proposition 8.4]. So the whole difficulty of the Conjecture, and of Question 1, is the removal of (1), and every published positive result assumes it.

What is settled here. Our starting point is that the ultrametric inequality by itself — with no submultiplicativity — already controls the prime subring.

Theorem 1.1 (Theorem 3.2)**.** *Let f be ultra-Euclidean on R and let $n \geq 1$ be an integer such that $k \cdot 1_R \neq 0$ for every $1 \leq k \leq n$. Then $n \cdot 1_R \in R^\times$.*

In characteristic zero this says that a domain carrying an ultra-Euclidean function contains $\mathbb{Q}$. The nearest published statement is [2, Proposition 8.4], which lists the containment of the prime subring of A in $A^\times \cup \{0\}$ among conditions on a submultiplicative Euclidean function σ that are equivalent to σ being both non-archimedean and multiplicative, $\sigma(ab) = \sigma(a) + \sigma(b)$; Theorem 3.2 reaches the same containment from the ultrametric inequality alone. Two consequences follow at once. First, a characteristic-zero domain in which 2 is not invertible carries no ultra-Euclidean function whatsoever (Theorem 3.5); in particular $\mathbb{Z}$ carries none, although $\mathbb{Z}$ *is* a Euclidean domain, and neither does the ring of integers of any number field. On all of those rings Question 1 has a positive answer and the Conjecture holds vacuously. This is stronger than [2, Example 8.2], which records that $\mathbb{Z}$ admits no non-archimedean Euclidean function, because there the word "Euclidean function" carries (1). Second, any characteristic-zero example separating the ultra-Euclidean condition from (1) must be a $\mathbb{Q}$-algebra, so the example we give on $\mathbb{Q}$ (Proposition 5.5) is as small as such an example can be.

In positive characteristic the same lemma settles the Conjecture for the domains with the smallest possible unit group.

Theorem 1.2 (Theorem 4.3)**.** *Let R have characteristic $p > 0$ and suppose every unit of R has a reciprocal of the form $j \cdot 1_R$ with $j \in \mathbb{N}$. Then every ultra-Euclidean function on R is strongly Euclidean; consequently $\widetilde{f} = f$ and the Conjecture holds for R.*

The hypothesis says exactly that $R^\times$ is the multiplicative group of the prime field (Remark 4.4), and the conclusion is much stronger than the Conjecture asks: there is no room between f and $\widetilde{f}$ at all. The instance $R = \mathbb{F}_p[x]$ is Corollary 4.5, a ring on which the Conjecture is settled that is neither a field nor one of the rings that carry no ultra-Euclidean function at all. We also settle the Conjecture on every field (Theorem 4.1), where the refinement is constant.

Section 5 collects the separating examples, including Sekhon's four-element-field example in the general form it takes over an arbitrary field of characteristic 2, and Section 6 isolates what stands in the way of the general Conjecture: one needs a *single* multiplier good for both a and b, and the set of good multipliers for a fixed element consists of units and is closed under addition. No claim in this paper rests on a computation: there is no exhaustive search, no finite data table and no appeal to compiled evaluation anywhere, and §7 says what the machine checks.

2. Preliminaries

We fix an integral domain R and recall the notions in the form used below.

Definition 2.1. A function $f\colon R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ is *Euclidean* on R if for all $a, b \in R$ with $b \neq 0$ there exist $q, r \in R$ with $a = qb + r$ and ($r = 0$ or $f(r) < f(b)$). It is *strongly Euclidean* if it is Euclidean and moreover $f(a) \leq f(ab)$ for all $a, b \in R \setminus \{0\}$, and *ultra-Euclidean* if it is Euclidean and moreover $f(a + b) \leq \max\{f(a), f(b)\}$ for all $a, b \in R \setminus \{0\}$ with $a + b \neq 0$; neither of the two extra conditions implies the other. The *refinement* of a Euclidean f is $\widetilde{f}(a) = \min_{b \in R\setminus\{0\}} f(ab)$, the minimum of a nonempty set of non-negative integers.

Since $a = a \cdot 1$, we have $\widetilde{f} \leq f$ pointwise, and the minimum defining $\widetilde{f}(a)$ is attained: there is a nonzero c with $f(ac) = \widetilde{f}(a)$. Note that f is never evaluated at 0, and that $\widetilde{f}(1) = \min\{f(x) : x \in R \setminus \{0\}\}$ is the least value of f.

Proposition 2.2 (Rogers [5]; [1, Proposition 2.6])**.** *If f is Euclidean on R, then $\widetilde{f}$ is Euclidean and strongly Euclidean on R.*

Proof sketch. Given a and $b \neq 0$, choose $c \neq 0$ with $f(bc) = \widetilde{f}(b)$ and divide a by bc under f: from $a = q(bc) + r$ one gets $a = (cq)b + r$, and if $r \neq 0$ then $\widetilde{f}(r) \leq f(r) < f(bc) = \widetilde{f}(b)$. For the second assertion, choose $c \neq 0$ with $f(abc) = \widetilde{f}(ab)$; since abc is a nonzero multiple of a, $\widetilde{f}(a) \leq f(abc) = \widetilde{f}(ab)$. □

Corollary 2.3 ([1, Corollary 2.7])**.** *A Euclidean function f is strongly Euclidean if and only if $\widetilde{f}(a) = f(a)$ for every $a \in R \setminus \{0\}$. Consequently, if f is both strongly Euclidean and ultra-Euclidean, then $\widetilde{f}$ is ultra-Euclidean.*

Proof. If $\widetilde{f} = f$ then f is strongly Euclidean by Proposition 2.2. Conversely $\widetilde{f}(a) \leq f(a)$ always, and $f(a) \leq f(ab)$ for all $b \neq 0$ gives $f(a) \leq \widetilde{f}(a)$. The second sentence is then immediate. □

The next two lemmas are the tools used throughout; both are applications of the division property, and the first needs no submultiplicativity.

Lemma 2.4. *Let f be Euclidean on R and let $b \in R \setminus \{0\}$ satisfy $f(b) \leq f(x)$ for every $x \in R \setminus \{0\}$. Then $b \in R^\times$. In particular, every $b \in R \setminus \{0\}$ with $f(b) = \widetilde{f}(1)$ is a unit.*

Proof. Divide 1 by b: there are q, r with $1 = qb + r$ and ($r = 0$ or $f(r) < f(b)$). The second alternative contradicts the minimality of $f(b)$, so $r = 0$ and $qb = 1$. □

For strongly Euclidean functions the same minimum is attained *precisely* at the units, which is [1, Lemma 2.4]; the sharper statement we need is the following lemma of Sekhon.

Lemma 2.5 ([1, Lemma 2.3])**.** *Let f be strongly Euclidean on R and $a, b \in R \setminus \{0\}$. Then $f(a) = f(ab)$ if and only if $b \in R^\times$.*

Proof sketch. Only the forward implication is used below. Divide a by ab: $a = q(ab) + r$ with $r = 0$ or $f(r) < f(ab) = f(a)$, and $r = a(1 - bq)$. If $1 - bq = 0$ then b is a unit. Otherwise $r \neq 0$, so $f(r) < f(a)$, while $f(a) \leq f\big(a(1 - bq)\big) = f(r)$ by strong Euclideanness: a contradiction. The converse is the computation $f(ab) \leq f(abb^{-1}) = f(a) \leq f(ab)$. □

3. Ultra-Euclidean functions invert the integers

For $n \in \mathbb{N}$ we write $n \cdot x$ for the n-fold sum $x + \cdots + x$ in R, so that $n \cdot x = (n \cdot 1_R)x$. The whole of this section rests on one observation: the ultrametric inequality, iterated, says that adding an element to itself cannot make f grow.

Lemma 3.1. *Let f be ultra-Euclidean on R, let $x \in R \setminus \{0\}$ and let $n \geq 1$ be an integer such that $k \cdot x \neq 0$ for every $1 \leq k \leq n$. Then $f(n \cdot x) \leq f(x)$.*

Proof. Induction on n. For $n = 1$ there is nothing to prove. If $f(m \cdot x) \leq f(x)$ and $m \cdot x$, x and $(m + 1) \cdot x = m \cdot x + x$ are all nonzero, then

$$f\big((m + 1) \cdot x\big) \;\leq\; \max\big\{f(m \cdot x),\, f(x)\big\} \;\leq\; f(x).$$ □

Theorem 3.2. *Let f be ultra-Euclidean on R and let $n \geq 1$ be an integer such that $k \cdot 1_R \neq 0$ for every $1 \leq k \leq n$. Then $n \cdot 1_R \in R^\times$.*

Proof. Let $m = \widetilde{f}(1)$ be the least value of f, attained at some $u \in R \setminus \{0\}$; by Lemma 2.4, $u \in R^\times$. Since R is a domain and $n \cdot 1_R \neq 0$, the element $n \cdot u = (n \cdot 1_R)u$ is nonzero, and so is $k \cdot u$ for every $1 \leq k \leq n$. Lemma 3.1 applied to $x = u$ gives $f(n \cdot u) \leq f(u) = m$, and m is the least value of f, so $f(n \cdot u) = m$. By Lemma 2.4 again, $n \cdot u$ is a unit; as u is a unit, so is $n \cdot 1_R$. □

Remark 3.3. The hypothesis on n is not a technicality but the exact content of the theorem: in characteristic $p > 0$ it forces $n < p$, and there $n \cdot 1_R$ is a nonzero element of the prime field and is a unit for trivial reasons. All of the strength is in characteristic zero, where the hypothesis is vacuous.

Corollary 3.4. *If R has characteristic zero and carries an ultra-Euclidean function, then $n \cdot 1_R$ is a unit for every integer $n \geq 1$; equivalently, R is a $\mathbb{Q}$-algebra.*

Proof. In characteristic zero $k \cdot 1_R \neq 0$ for every $k \geq 1$, so Theorem 3.2 applies to every n; that is the first assertion. The prime subring $\mathbb{Z} \cdot 1_R$ then lies in $R^\times \cup \{0\}$, so by the universal property of the localisation the embedding $\mathbb{Z} \to R$ extends (uniquely) to $\mathbb{Q}$, which is the second. □

The containment of the prime subring in $R^\times \cup \{0\}$ is condition (c) of [2, Proposition 8.4], in the form $\{a : \sigma(a) \leq \sigma(1)\} = A^\times \cup \{0\}$; there it is one of a list of conditions on a submultiplicative Euclidean function σ equivalent to σ being non-archimedean *and* multiplicative. Theorem 3.2 obtains it from the ultrametric inequality alone.

Theorem 3.5. *Let R have characteristic zero and suppose 2 is not a unit of R. Then R carries no ultra-Euclidean function at all; in particular the Conjecture holds vacuously on R, and Question 1 of* [3] *has a positive answer for R. This applies to $R = \mathbb{Z}$, which is a Euclidean domain.*

Proof. If f were ultra-Euclidean on R, Corollary 3.4 with $n = 2$ would make 2 a unit. In $\mathbb{Z}$ the element 2 is not a unit. □

Corollary 3.6. *Let K be a number field and $\mathcal{O}_K$ its ring of integers. Then $\mathcal{O}_K$ carries no ultra-Euclidean function. The same holds for every ring of quadratic integers.*

Proof. $\mathcal{O}_K$ has characteristic zero, and $N_{K/\mathbb{Q}}(2) = 2^{[K:\mathbb{Q}]} \notin \{1, -1\}$ while units of $\mathcal{O}_K$ have norm ± 1; so 2 is not a unit and Theorem 3.5 applies. □

Theorem 3.5 should be read against [2, Example 8.2], which states that "the Euclidean domain $\mathbb{Z}$ does not admit a non-archimedean Euclidean function". In the vocabulary of [2] a Euclidean function is submultiplicative by definition, so that example is the statement for functions satisfying (1); Theorem 3.5 drops that hypothesis, which is precisely the hypothesis that Problem 8.6 of the same paper, and Question 1 of [3], are about.

4. Two settled cases of the Conjecture

Fields. On a field every function is Euclidean, since one may always take $q = ab^{-1}$ and $r = 0$; and every nonzero a has $aK = K$, so the refinement forgets a entirely.

Theorem 4.1. *Let K be a field and $f\colon K \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ any function. Then $\widetilde{f}$ is constant on $K \setminus \{0\}$, and $\widetilde{f}$ is ultra-Euclidean. In particular the Conjecture holds on every field, and the hypothesis that f be ultra-Euclidean is not needed there.*

Proof. For $a \neq 0$, the sets $\{ab : b \neq 0\}$ and $\{b : b \neq 0\}$ coincide, so $\widetilde{f}(a) = \widetilde{f}(1)$. Since f is Euclidean, $\widetilde{f}$ is Euclidean by Proposition 2.2, and a constant function satisfies the ultrametric inequality. □

Multiplier sets, and characteristic p. For f Euclidean and $a \in R \setminus \{0\}$ put

$$G_f(a) \;=\; \{\, w \in R \setminus \{0\} \;:\; f(aw) = \widetilde{f}(a) \,\},$$

the set of multipliers that realise the refinement at a. It is nonempty, because the minimum is attained.

Lemma 4.2. *Let f be Euclidean on R and $a \in R \setminus \{0\}$. Every $w \in G_f(a)$ is a unit of R.*

Proof. From $\widetilde{f}(aw) \leq f(aw) = \widetilde{f}(a)$ and $\widetilde{f}(a) \leq \widetilde{f}(aw)$ (Proposition 2.2) we get $\widetilde{f}(a) = \widetilde{f}(aw)$, and $\widetilde{f}$ is strongly Euclidean, so Lemma 2.5 applied to $\widetilde{f}$ gives $w \in R^\times$. □

So f and $\widetilde{f}$ always agree on *some* associate of every element. The Conjecture would follow if one could choose the associate uniformly; the following theorem is the case in which the choice $w = 1$ always works.

Theorem 4.3. *Let R have characteristic $p > 0$ and suppose that every unit u of R admits an integer $j \geq 0$ with $(j \cdot 1_R)u = 1$. Then every ultra-Euclidean function f on R is strongly Euclidean. Consequently $\widetilde{f} = f$, and $\widetilde{f}$ is ultra-Euclidean: the Conjecture holds on R.*

Proof. By Corollary 2.3 it suffices to prove $\widetilde{f}(a) = f(a)$ for every $a \in R \setminus \{0\}$. Choose $c \in G_f(a)$, so $f(ac) = \widetilde{f}(a)$; by Lemma 4.2, c is a unit, so the hypothesis gives j with $(j \cdot 1_R)c = 1$. Since $p \cdot 1_R = 0$, the element $j \cdot 1_R$ depends only on j modulo p; writing $j' = j \bmod p$ we still have $(j' \cdot 1_R)c = 1$, and $j' \neq 0$ because $0 \cdot c = 0 \neq 1$, so $1 \leq j' < p$. For every $1 \leq k \leq j'$ we have $k \cdot 1_R \neq 0$, because $p \nmid k$, and hence $k \cdot (ac) \neq 0$. Lemma 3.1 applied to $x = ac$ and $n = j'$ gives

$$f\big(j' \cdot (ac)\big) \;\leq\; f(ac).$$

But $j' \cdot (ac) = a\big((j' \cdot 1_R)c\big) = a$, so $f(a) \leq f(ac) = \widetilde{f}(a) \leq f(a)$ and therefore $\widetilde{f}(a) = f(a)$. Corollary 2.3 now gives both remaining assertions. □

Remark 4.4. In characteristic p the prime subring is the field $\mathbb{F}_p = \{j \cdot 1_R : 0 \leq j < p\}$, whose nonzero elements are units of R. The hypothesis of Theorem 4.3 says that the inverse of every unit lies in $\mathbb{F}_p$, hence that every unit lies in $\mathbb{F}_p$; conversely, if $R^\times = \mathbb{F}_p \setminus \{0\}$ then every unit has an inverse of the form $j \cdot 1_R$. So the hypothesis is exactly the condition $R^\times = \mathbb{F}_p \setminus \{0\}$: the domain has as few units as a domain of characteristic p can have.

Corollary 4.5. *Let p be a prime. Every ultra-Euclidean function f on $\mathbb{F}_p[x]$ is strongly Euclidean and satisfies $\widetilde{f} = f$; in particular $\widetilde{f}$ is ultra-Euclidean, so the Conjecture holds on $\mathbb{F}_p[x]$.*

Proof. The units of $\mathbb{F}_p[x]$ are the nonzero constants, and every element of $\mathbb{F}_p$ is of the form $j \cdot 1$ with $j \in \mathbb{N}$; so the hypothesis of Theorem 4.3 holds, with p the characteristic. □

As far as we are aware, $\mathbb{F}_p[x]$ is the first ring on which the Conjecture is settled that is neither a field nor one of the rings of Theorem 3.5, where it holds because there is nothing to refine. Note that it is not covered by [2, Proposition 8.5]: that proposition concerns rings admitting a submultiplicative non-archimedean function, whereas Corollary 4.5 quantifies over *all* ultra-Euclidean functions on $\mathbb{F}_p[x]$ and asserts that each is automatically submultiplicative.

5. Examples and controls

We record the examples that keep the hypotheses honest. The first two are Sekhon's [1, §2].

Proposition 5.1. *The absolute value $|\cdot|$ is a Euclidean and strongly Euclidean function on $\mathbb{Z}$, and it is not ultra-Euclidean, since $|1+1| = 2 > 1 = \max\{|1|, |1|\}$. Hence a Euclidean, and even a strongly Euclidean, function need not be ultra-Euclidean.*

Proposition 5.2. *For any field K, the degree function is Euclidean, strongly Euclidean and ultra-Euclidean on $K[x]$, and its refinement is ultra-Euclidean. In particular ultra-Euclidean functions exist on rings that are not fields, so the hypothesis of Theorem 3.2 is not vacuous; taking $K = \mathbb{F}_p$ makes the same point for Theorem 4.3.*

Proof. Division with remainder gives the first assertion, $\deg(ab) = \deg a + \deg b$ the second, $\deg(a+b) \le \max\{\deg a, \deg b\}$ the third; the last follows from Corollary 2.3. □

Proposition 5.3. *2 is a unit of $\mathbb{Q}[x]$.*

This is a consistency check rather than a result: $\mathbb{Q}[x]$ has characteristic zero, is not a field, and carries an ultra-Euclidean function by Proposition 5.2, so Corollary 3.4 forces 2 to be invertible there — and it is. The same corollary applied to $\mathbb{Z}$, where 2 is not invertible, is Theorem 3.5.

The next example is Sekhon's separation of the ultra-Euclidean condition from (1), stated over an arbitrary field of characteristic 2 rather than over $\mathbb{F}_4$.

Proposition 5.4. *Let K be a field of characteristic 2 and let $f\colon K \setminus \{0\} \to \mathbb{Z}_{\ge 0}$ be given by $f(1) = 0$ and $f(x) = 1$ for $x \neq 0, 1$. Then f is ultra-Euclidean; and if K contains an element $\alpha \neq 0, 1$, then f is not strongly Euclidean, because $f(\alpha) = 1 > 0 = f(1) = f(\alpha \cdot \alpha^{-1})$. Taking $K = \mathbb{F}_4 = \{0, 1, \alpha, \beta\}$ recovers the example of* [1, §3]*, where $f(1) = 0$ and $f(\alpha) = f(\beta) = 1$.*

Proof. Every function on a field is Euclidean. If a, b and $a+b$ are nonzero then $a \neq b$ because $\operatorname{char} K = 2$, so a and b are not both equal to 1 and $\max\{f(a), f(b)\} = 1 \ge f(a+b)$. □

By Corollary 3.4, a separation of this kind in characteristic zero can only live on a $\mathbb{Q}$-algebra. The smallest one exists.

Proposition 5.5. *Define $q\colon \mathbb{Q} \setminus \{0\} \to \mathbb{Z}_{\ge 0}$ by $q(x) = 0$ if $x \in \mathbb{Z}$ and $q(x) = 1$ otherwise. Then q is ultra-Euclidean on $\mathbb{Q}$ and is not strongly Euclidean. Since any characteristic-zero domain carrying an ultra-Euclidean function contains $\mathbb{Q}$, no smaller characteristic-zero example is possible.*

Proof. Every function on a field is Euclidean. If $q(a) = q(b) = 0$ then $a, b \in \mathbb{Z}$, so $a + b \in \mathbb{Z}$ and $q(a+b) = 0$; otherwise $\max\{q(a), q(b)\} = 1 \ge q(a+b)$. Finally $q(\frac{1}{2}) = 1 > 0 = q(\frac{1}{2} \cdot 2)$, so (1) fails. □

Remark 5.6. Propositions 5.4 and 5.5 live on fields, where Theorem 4.1 settles the Conjecture. They show that it is not settled there by the trivial route of Corollary 2.3 — neither f equals its own refinement — but for the different reason that the refinement is constant.

6. What is not proved

The Conjecture in general remains open, and it is worth recording precisely where the argument of Theorem 4.3 stops. Let f be ultra-Euclidean and let $a, b, a+b \in R \setminus \{0\}$. To bound $\widetilde{f}(a+b)$ by $n = \max\{\widetilde{f}(a), \widetilde{f}(b)\}$ one wants a *single* nonzero w with $f(aw) \le n$ and $f(bw) \le n$, for then

$$\widetilde{f}(a+b) \;\le\; f\big((a+b)w\big) \;=\; f(aw+bw) \;\le\; \max\{f(aw), f(bw)\} \;\le\; n.$$

Choosing $c \in G_f(a)$ and $d \in G_f(b)$ separately gives only the conclusion for the twisted pair $(a, c^{-1}db)$ — both are units by Lemma 4.2, so the twist is by a unit, but it is a twist all the same. Thus:

Remark 6.1. If $G_f(a) \cap G_f(b) \neq \emptyset$ then the ultrametric inequality for $\widetilde{f}$ holds at the pair (a, b). Theorem 4.3 is exactly the case in which 1 lies in $G_f(a)$ for every a.

The multiplier sets are also additively closed, by a two-line argument that we state here rather than in §4 because we do not use it: if f is ultra-Euclidean and $w, w' \in G_f(a)$ with $w + w' \neq 0$, then $f\big(a(w+w')\big) = f(aw + aw') \leq \max\{f(aw), f(aw')\} = \widetilde{f}(a)$, while $f\big(a(w+w')\big) \geq \widetilde{f}(a)$ by the definition of the minimum; so $w + w' \in G_f(a)$. Over $R = K[x]$ the units are the nonzero constants, so $G_f(a) \subseteq K$ by Lemma 4.2, and $G_f(a) \cup \{0\}$ is then a subset of K closed under addition — in positive characteristic, an $\mathbb{F}_p$-subspace of K. For K a prime field that subspace contains 1 as soon as it is nonzero, which is Corollary 4.5; the first open case is therefore $K[x]$ for a non-prime field K. There, disjointness of the multiplier sets is by itself no obstruction: in Sekhon's example on $\mathbb{F}_4[x]$ with $f(1) = 0$ and $f(g) = \deg g + 1$ for $g \neq 1$ [1, §4], one has $G_f(1) = \{1\}$ and $G_f(\alpha) = \{\alpha^{-1}\} = \{\beta\}$, which are disjoint, and yet $\widetilde{f}$ is ultra-Euclidean: a short computation gives $\widetilde{f} = \varphi \circ \deg$ with $\varphi(0) = 0$ and $\varphi(d) = d + 1$ for $d \geq 1$, and φ is increasing while deg is ultra-Euclidean. (That computation, like the following remark, is a hand check outside the formal development.)

Remark 6.2 (verified by hand; not part of the machine-checked development). Functions of the second kind demanded by the counterexample shape of [1, §5] — strongly Euclidean but not ultra-Euclidean — do exist on $\mathbb{F}_2[x]$. Let $g(1) = 0$, $g(x) = 1$, $g(x+1) = 2$ and $g(h) = \deg h + 1$ for $\deg h \geq 2$. One checks directly that g is Euclidean (dividing by a polynomial of degree ≥ 2 leaves a remainder of strictly smaller degree, hence of strictly smaller g-value; dividing by x or $x + 1$ leaves remainder 0 or 1, and $g(1) = 0$), that $g(h) \leq \deg h + 1$ for all h and hence that g is strongly Euclidean, and that $g(1 + x) = 2 > 1 = \max\{g(1), g(x)\}$, so g is not ultra-Euclidean. This is not a counterexample to the Conjecture: by Corollary 4.5 every ultra-Euclidean function on $\mathbb{F}_2[x]$ equals its own refinement, so no ultra-Euclidean function refines to g.

Two questions are left in a shape we find concrete. First, is there a characteristic-zero domain, other than a field and other than $K[x]$ with $\operatorname{char} K = 0$, that carries an ultra-Euclidean function? By Corollary 3.4 it must be a $\mathbb{Q}$-algebra, and a negative answer would settle Question 1 of [3] in characteristic zero. Second, in characteristic p: does the Conjecture hold on $K[x]$ for every finite field K? By the paragraph above this is a question about $\mathbb{F}_p$-subspaces of K attached to the elements of $K[x]$, and for fixed K it is finite-dimensional data.

7. Verification

Every theorem, proposition, lemma and corollary above except Corollary 3.6 — that is, Propositions 2.2, 5.1, 5.2, 5.3, 5.4 and 5.5, Corollaries 2.3, 3.4 and 4.5, Lemmas 2.4, 2.5, 3.1 and 4.2, and Theorems 3.2, 3.5, 4.1 and 4.3 — has been machine-checked in Lean 4 [10] against Mathlib [11], in a development of four files: the definitions of Definition 2.1 with Rogers' theorem, the results of §3 and §4, the examples of §5, and one file relating the definitions to the library. The encoding is faithful in the one place where it could fail to be: a Euclidean function is represented as a total map $R \to \mathbb{N}$, but every condition quantifies only over nonzero arguments, so the value at 0 is never referenced and no hidden hypothesis at 0 is assumed. Mathlib's own `EuclideanDomain` structure is the *classical* notion — among its axioms is `mul_left_not_lt`, which transcribes (1) for its well-founded relation — and the fourth file proves that a commutative domain carrying a Euclidean function in the sense of Definition 2.1 satisfies those axioms, with the well-founded relation built from $\widetilde{f}$; the step supplying the submultiplicativity axiom is Proposition 2.2. The remarks above are commentary and are not part of the formal development; besides them, three deductions are paper arguments and are labelled as such where they occur: Corollary 3.6, which is the standard norm argument recalled in its proof; the passage in Corollary 3.4 from "every $n \cdot 1_R$ is a unit", which is the formalised statement, to the $\mathbb{Q}$-algebra structure; and the two computations on $\mathbb{F}_4[x]$ and $\mathbb{F}_2[x]$ in §6, neither of which is used in any proof. There is no `sorry` anywhere in the development, no appeal to compiled evaluation, no axiom declared by us and no finite data table: every theorem is checked by the kernel from the axioms of Lean's type theory. Exactly two steps invoke Lean's `decide` tactic, each on one arithmetic statement about a fixed integer — that $|1 + 1| > \max\{|1|, |1|\}$ in Proposition 5.1, and that 2 is not a unit of $\mathbb{Z}$ in Theorem 3.5 — and nothing larger is evaluated anywhere. The axiom closure of each theorem is the standard one (propositional extensionality, quotient soundness and, where a classical choice is made, the axiom of

choice; Lemma 3.1 uses no choice and Proposition 5.1's failure of the ultrametric inequality at $1 + 1$ uses no axioms at all). Repository reference to be added at submission.

References

[1] S. Sekhon, *A necessary and sufficient condition for uniqueness of Euclidean division*, arXiv:2604.24399v2 [math.AC], 29 April 2026. All references are to the numbering of the v2 e-print, whose environment counter is shared across definitions, lemmas, propositions, theorems and corollaries and is reset at each section: Definitions 2.1, 2.2, 2.5, 2.8 and 2.9, Lemmas 2.3 and 2.4, Proposition 2.6, Corollary 2.7, Theorem 4.1, Corollary 4.2, Theorem 4.4 and Corollary 4.5; the Conjecture and its conjectured Corollary are the two unnumbered displays of §5, the $\mathbb{F}_4$ example is the unnumbered example at the end of §3, and the $\mathbb{F}_4[x]$ example is the unnumbered example at the end of §4.

[2] J. Elliott and N. Epstein, *Additive subgroups of a module that are saturated with respect to a subset of the ring*, arXiv:2501.11575v4 [math.RA], last revised 1 August 2025. §8 ("Euclidean domains"): Definition 8.1, Example 8.2, Theorem 8.3, Propositions 8.4 and 8.5, Problem 8.6.

[3] J. Elliott, *"Nonarchimedean" Euclidean domains*, MathOverflow question 485986, 15 January 2025, `https://mathoverflow.net/q/485986`.

[4] J. Mensah, answer to [3], MathOverflow answer 486062, 16 January 2025, `https://mathoverflow.net/a/486062`. [2] credits the answer to "user527492", which was the display name of the same account (MathOverflow user 544688) at the time; it now displays as "Jeffery Mensah".

[5] K. Rogers, *The axioms for Euclidean domains*, Amer. Math. Monthly **78** (1971), no. 10, 1127–1128, `https://doi.org/10.2307/2316324`. The refinement is quoted here from [1, Proposition 2.6] and [8, Theorem 2.1].

[6] M. A. Jodeit, *Uniqueness in the division algorithm*, Amer. Math. Monthly **74** (1967), no. 7, 835–836, `https://doi.org/10.2307/2315810`.

[7] T.-S. Rhai, *A characterization of polynomial domains over a field*, Amer. Math. Monthly **69** (1962), no. 10, 984–986, `https://doi.org/10.2307/2313196`.

[8] K. Conrad, *Remarks about Euclidean domains*, expository note, whose Theorem 2.1 is Rogers' refinement; available at `https://kconrad.math.uconn.edu/blurbs/ringtheory/euclideanrk.pdf`.

[9] P. Samuel, *About Euclidean rings*, J. Algebra **19** (1971), 282–301, `https://doi.org/10.1016/0021-8693(71)90110-4`.

[10] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, 625–635.

[11] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381.