THE CENTER OF DISTANCES OF A FINITE ULTRAMETRIC SPACE: CENTERED SPHERES, PRESCRIBED CENTERS, AND EXTREMAL RIGIDITY

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Abstract. The center of distances of a metric space (X, d) is the set $C(X)$ of those distances t for which the equation $d(p, x) = t$ has a solution x for *every* $p \in X$. Dovgoshey and Rovenska (arXiv:2601.13363, *Mathematics* 2026) described $C(X)$ for ultrametric spaces generated by labeled trees and closed with three conjectures; their follow-up (arXiv:2603.25850) proved the third one, $|C(X)| \leq 1 + \lfloor \log_2 n \rfloor$ for every n-point ultrametric space with equality attainable for every n, and posed three more. We settle four of the five remaining conjectures and half of the fifth. Conjecture 2 holds: every non-empty subset of an ultrametric space with at least three points is a centered sphere if and only if the space has exactly three points and is weakly similar to the non-equidistant three-point space. The first half of Conjecture 1 holds: if every closed ball is a centered sphere then every truncated distance set $D_p(X) \cap [0, r]$ has a greatest element. Conjecture 4.4 holds: every finite set $A \ni 0$ of non-negative reals is simultaneously the distance set and the center of distances of some finite ultrametric space, which can be taken to have $2^{|A|-1}$ points, the least possible. For the extremal spaces, $|X| = 2^n$ with $|C(X)| = n + 1$, we prove that every distance is centered, $D(X) = C(X)$, and that every closed ball has exactly 2^k points where k is the number of positive centered distances not exceeding its radius; a partition-count argument, presented on paper, then identifies every extremal space with the complete binary hierarchy and yields Conjectures 4.2 and 4.3. We also give an independent proof of the logarithmic bound through a single ball-doubling lemma, with a new explicit family attaining it, and record the controls: the bound fails for the 7-cycle metric, the $+1$ cannot be dropped, and an exhaustive check over all three-point tables agrees. Everything except the identification of the extremal spaces and an exhaustive enumeration to $n \leq 9$ is verified in Lean 4 against *Mathlib*, without compiled evaluation.

1. Introduction

Let (X, d) be a metric space with distance set $D(X) = \{d(x, y) : x, y \in X\}$. Bielas, Plewik and Walczyńska [1] introduced its *center of distances*

$$C(X) = \{\, t \in D(X) : \ \forall p \in X \ \exists x \in X \ \ d(p, x) = t \,\}$$

to generalise von Neumann's theorem on permutations of two sequences with the same set of cluster points [11], and the notion has since been studied almost entirely for achievement sets of series and Cantor-type sets. Dovgoshey and Rovenska brought it to *ultrametric* spaces, those satisfying the strong triangle inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, in [6]: for the class **UT** of spaces isometric to one generated by a labeled tree they proved that $C(X)$ is $\{0\}$ or $\{0, \operatorname{diam} X\}$ (their Theorem 4), characterised the second case through centered spheres (Theorem 5) and through diametrical graphs (Theorem 8), and closed with three conjectures. Their follow-up [7] proved the third one as its Theorem 3.10,

$$|C(X)| \leq 1 + \lfloor \log_2 n \rfloor \quad \text{for every ultrametric space with } |X| = n, \text{ with equality attained for every } n,$$

and posed three further conjectures (its Conjectures 4.2–4.4) about finite ultrametric spaces. (The statement numbers of [6] used below are common to its published *Mathematics* version and its arXiv v2 e-print: Definitions 1, 3, 4 and 11, formula (3), Theorems 4, 5, 6 and 8, Examples 1 and 4, Figure 3 and Conjectures 1–3 carry the same numbers in both, compared item by item on 2026-09-07. Numbers of [7] are those of its arXiv v1 e-print, read the same day.)

This paper is about the five conjectures that remained open. Its results are:

- **Conjecture 2 of** [6] (Theorem 5.3). For an ultrametric space (Y, ρ) with $|Y| \geq 3$, every non-empty subset of Y is a centered sphere if and only if $|Y| = 3$ and (Y, ρ) is weakly similar to the three-point space (X_3, d) of Figure 3 of [6] (two sides of length 2, one of length 1).

- **Conjecture 1 of** [6]**, first half** (Theorem 5.1). If every closed ball of an ultrametric space is a centered sphere, then $D_p(X) \cap [0, r]$ has a greatest element for every point p and radius r. The second half, the converse under the hypothesis $(X, d) \in \mathbf{UT}$, is not proved here; Remark 5.2 recalls why that hypothesis cannot be dropped.
- **Conjecture 4.4 of** [7] (Theorem 6.1). For every finite $A \subset [0, \infty)$ with $0 \in A$ there is a finite ultrametric space with $D(X) = C(X) = A$; the space constructed has $2^{|A|-1}$ points, which is the least possible.
- **Extremal rigidity** (Theorems 7.2 and 7.3). If $|X| = 2^n$ and $|C(X)| = n+1$ then every distance is centered, $D(X) = C(X)$, and every closed ball has exactly 2^k points, k the number of positive centered distances not exceeding its radius. A partition-count argument (Proposition 7.4, proved on paper only) then shows that such a space is isometric to the complete binary hierarchy of depth n with level values $C(X)\setminus\{0\}$, which gives **Conjectures 4.2 and 4.3 of** [7] (Corollary 7.5, on paper only).
- **The logarithmic bound revisited** (Theorems 3.2 and 3.3). Both halves of Conjecture 3 of [6] are Theorem 3.10 of [7] and are not claimed as new. We include an independent proof, because the single lemma behind it, *a centered distance at least doubles every closed ball, uniformly in the centre* (Lemma 3.1), is also the engine of the extremal results, and a new explicit attaining family on n points for every n.
- **Controls** (Section 4). The bound is false for metric spaces that are not ultrametric (the 7-cycle has $|C| \geq 4 > 3$), the $+1$ cannot be dropped ($n = 4$), $|C| = 4$ is first reached at $n = 8$, the center of (X_3, d) is $\{0, 2\}$, and an exhaustive check of all 3^9 tables on three points agrees with the bound at $n = 3$; an enumeration outside the formal development extends the check to $n \leq 9$ (Remark 8.1).

What is new and what is not. The two halves of the logarithmic bound are Theorem 3.10 of [7]; its dual form, $|C(X)| = m \Rightarrow |X| \geq 2^{m-1}$, is Corollary 3.12 there, and the complete binary hierarchy on 2^n points, which is our model space H_n of Section 2 up to a strictly increasing relabelling of its distances, is Example 3.13 there. The proof in [7] runs through the function $M(n) = \max\{|C(X)| : |X| = n\}$, representing trees and a two-step induction $M(2^l) = l + 1$, $M(2^l - 1) = l$; ours is a direct ball-doubling induction, and our attaining space A_n (Definition 2.1) is explicit and uniform in n: it folds the last $n - 2^{\lfloor \log_2 n \rfloor}$ points onto the hierarchy, whereas [7] reaches the values of n between consecutive powers of two through the monotonicity of M, proved there by extending a space one point at a time without changing its center (its Proposition 3.8 and Corollary 3.9). That the center of distances is an invariant of weak similarity (Proposition 3.4) is two lines from the definitions and is stated only because it is used: it is the direction "$\Leftarrow$" of Conjecture 4.2 and the reduction that makes finite enumeration exhaustive. Everything else listed above — Conjecture 2, the first half of Conjecture 1, Conjecture 4.4, the extremal theorems, the structure theorem and Conjectures 4.2 and 4.3 — is, to the best of a search made on 2026-09-07, new. That search: the live arXiv pages of both sources (two versions of [6], the second of 20 February 2026; one of [7]); the arXiv listing query for the phrase "center of distances" in titles, abstracts and comments, 39 papers, of which only [6, 7] concern ultrametric spaces; a full-text search of an arXiv e-print corpus, in which the phrase occurs in 23 e-prints, all but the two sources in the achievement-set branch of the subject (one further hit carries the phrase only in commented-out source lines); and every paper citing either source according to Semantic Scholar — [7] itself, the companion [8] on a different inequality, $|D(V)| \leq |E| + 1$ for ultrametrics generated by vertex labelings of a graph, and [9], which cites both sources in a single sentence — none of which touches Conjectures 1, 2 or 4.2–4.4. OpenAlex holds no citation data for this thread. The rigidity and extremal questions for finite ultrametric spaces studied in [5, 3, 4], the setting from which [7] draws its representing-tree tools, do not involve the center of distances; the authors of [6] record there that they are "not aware of any results describing the structure of the center of distances in the case of ultrametric spaces" (a sentence common to both versions of that paper). Of that group's e-prints the corpus searched above contains arXiv:1511.08133 (same title and authors as [5]) and arXiv:1610.08282 (O. Dovgoshey, E. Petrov and

H.-M. Teichert, *Extremal properties and morphisms of finite ultrametric spaces and their representing trees*, 2016, revised 2017, with no journal reference on arXiv); the phrase occurs in neither.

Verification. Every theorem and proposition below that carries a formal counterpart in Appendix A is machine-checked in Lean 4 against *Mathlib*; Section 9 records the modules and the axioms. Where the printed statement says more than its formal counterpart, the proof names the extra step. Three items are outside the formal development and are labelled as such where they occur: Proposition 7.4 and Corollary 7.5 (proved on paper), and the enumeration of Remark 8.1 (a computation).

2. Preliminaries

Ultrametrics with values in an ordered set. No argument in this paper adds two distances. We therefore fix a linearly ordered set $(\Lambda, \leq)$ with a distinguished element 0 and call a map $d : X \times X \to \Lambda$ a Λ*-valued ultrametric* on the set X if for all $x, y, z \in X$

$$0 \leq d(x,y), \qquad d(x,x) = 0, \qquad d(x,y) = 0 \Rightarrow x = y, \qquad d(x,y) = d(y,x),$$
$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

For $\Lambda = \mathbb{R}$ this is exactly an ultrametric space in the sense of Definition 1 of [6] (a metric satisfying the strong triangle inequality): the ordinary triangle inequality follows from the strong one and non-negativity (formal: `triangle_of_strong`). The value set $\Lambda = \mathbb{N}$ is used to build examples, and the transport lemma below moves them to $\mathbb{R}$. Throughout, Λ-valued ultrametrics are simply called ultrametric spaces, and every statement about them specialises to the real-valued spaces of [6, 7].

Distance set, center, balls, centered spheres. For an ultrametric space (X, d) and $p \in X$ put

$$D(X) = \{d(x,y) : x, y \in X\}, \qquad D_p(X) = \{d(p,x) : x \in X\},$$
$$C(X) = \{\, t \in D(X) : \forall p \in X \ \exists x \in X \ d(p,x) = t \,\},$$

the distance set (formula (3) of [6]), the distance set seen from p, and the center of distances (Definition 3 of [6]). On a non-empty space the clause $t \in D(X)$ is redundant, so $C(X) = \bigcap_{p \in X} D_p(X)$ (Proposition 3.1 of [7]; formal: `center_eq_of_nonempty`), and $0 \in C(X)$ always. The *closed ball* of radius $r \geq 0$ about p is $\overline{B}_r(p) = \{x : d(p,x) \leq r\}$. Following Definition 11 of [6], a set $S \subseteq X$ is a *centered sphere* if there are $c \in S$ and $r \geq 0$ with

$$S = \{x \in X : d(x,c) = r\} \cup \{c\}.$$

We write $|\cdot|$ for cardinality and $\lfloor \log_2 n \rfloor$ for the integer part of the binary logarithm, so that $2^{\lfloor \log_2 n \rfloor} \leq n < 2^{\lfloor \log_2 n \rfloor + 1}$.

Weak similarity. Following Definition 4 of [6] (the notion is from [2]), a bijection $\Phi : X \to Y$ between spaces (X, d) and (Y, ρ) is a *weak similarity* if there is a strictly increasing bijection $f : D(Y) \to D(X)$ with $d(x,y) = f(\rho(\Phi x, \Phi y))$ for all $x, y \in X$; the spaces are then *weakly similar*. Two spaces are weakly similar exactly when their distance sets have the same order type in a way compatible with the points; the definition makes sense for arbitrary functions d and ρ into ordered sets, and Proposition 3.4 uses no metric axiom.

The model spaces. Let $\tau_k : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by recursion on the depth k:

$$\tau_0(x,y) = 0, \qquad \tau_{k+1}(x,y) = \begin{cases} \tau_k(\lfloor x/2 \rfloor, \lfloor y/2 \rfloor) & \text{if } x \equiv y \pmod 2, \\ k+1 & \text{otherwise.} \end{cases}$$

For $x \neq y$ below 2^k one has $\tau_k(x,y) = k - v_2(x-y) \in \{1, \ldots, k\}$, where v_2 is the 2-adic valuation, so the *complete binary hierarchy of depth* k is

$$H_k = (\{0, 1, \ldots, 2^k - 1\}, \ \tau_k),$$

a 2^k-point ultrametric space whose k non-zero distances are the k levels of a complete binary tree; it is, up to the strictly increasing relabelling of distances $2^{-m} \mapsto k+1-m$, the space of binary words of length k with $d(x,y) = 2^{-m}$ for m the first position where x and y differ, i.e. Example 3.13 of [7].

Every level is *fully branching*: for every $a < 2^k$ and $1 \le j \le k$ there is $b < 2^k$ with $\tau_k(a, b) = j$ (flip one bit; formal: `exists_treeDist_eq`). Hence

$$D(H_k) = C(H_k) = \{0, 1, \ldots, k\} \tag{1}$$

(formal: `distSet_hier`, `center_hier`).

Definition 2.1 (the attaining space)**.** For $n \ge 1$ let $k = \lfloor \log_2 n \rfloor$ and fold $\{0, \ldots, n-1\}$ onto $\{0, \ldots, 2^k - 1\}$ by $\varphi_n(x) = x$ for $x < 2^k$ and $\varphi_n(x) = x - 2^k$ otherwise. The space $A_n = (\{0, \ldots, n-1\}, a_n)$ has

$$a_n(x, y) = \begin{cases} 0 & x = y, \\ 1 + \tau_k(\varphi_n(x), \varphi_n(y)) & x \ne y. \end{cases}$$

So two distinct points with the same fold are at the new smallest distance 1, and points with different folds are at distance $1 + \tau_k$ of their folds.

Remark 2.2. Folding the extra points *below* the hierarchy, rather than hanging them beside it, is what makes Theorem 3.3 uniform in n. The naive alternative — keep the first n points of the hierarchy of depth $k+1$ in the word order of Example 3.13 of [7], so that the extra $n - 2^k$ points are attached at the *top* level — fails at $n = 5$: the fifth point is then at the top-level distance from each of the other four, so it sees a single non-zero distance and the center has two elements, not three; the same happens at every $n = 2^k + 1$. (Ordering the same points by the 2-adic valuation instead, $d(x, y) = \lceil \log_2 n \rceil - v_2(x - y)$, does attain the bound for every $n \le 32$ we checked; what Definition 2.1 adds is a fold we can prove attains it for all n.)

Transport. A strictly increasing $f : \Lambda \to \Lambda'$ with $f(0) = 0$ carries a Λ-valued ultrametric d to the Λ'-valued ultrametric $f \circ d$, and carries D and C along: $D(X, f \circ d) = f(D(X, d))$ and $C(X, f \circ d) = f(C(X, d))$ (formal: `IsUltrametric.comp`, `distSet_comp`, `center_comp`). With $f = (\mathbb{N} \hookrightarrow \mathbb{R})$ this turns every $\mathbb{N}$-valued example into a real one.

3. The logarithmic bound

Lemma 3.1 (ball doubling)**.** *Let (X, d) be a finite ultrametric space, $T \subseteq C(X)$ a set of m positive centered distances, and $r \ge 0$ with $t \le r$ for all $t \in T$. Then $|\overline{B}_r(p)| \ge 2^m$ for every $p \in X$.*

Proof. (Formal: `ball_growth`.) Induction on m, the case $m = 0$ being $p \in \overline{B}_r(p)$. For $m + 1$ let $t = \max T$, $T' = T \setminus \{t\}$, and choose r' with $0 \le r' < t$ and $s \le r'$ for all $s \in T'$ ($r' = \max T'$, or $r' = 0$ if $T' = \varnothing$). As $t \in C(X)$ there is q with $d(p, q) = t$. The balls $\overline{B}_{r'}(p)$ and $\overline{B}_{r'}(q)$ are disjoint — a common point x would give $t = d(p, q) \le \max\{d(p, x), d(x, q)\} \le r' < t$ — and both lie in $\overline{B}_r(p)$, since $x \in \overline{B}_{r'}(q)$ gives $d(p, x) \le \max\{d(p, q), d(q, x)\} \le \max\{t, r'\} \le r$. By induction each has at least 2^m points, so $|\overline{B}_r(p)| \ge 2^{m+1}$. The point of the formulation is that the bound is uniform in the centre: the partner q is bounded by the same induction hypothesis as p. □

Taking $T = C(X) \setminus \{0\}$ and $r = \max T$ gives $2^{|C(X)|-1} \le |X|$ (formal: `two_pow_card_le`), which is Corollary 3.12 of [7] in the form we use.

Theorem 3.2 (the bound; Conjecture 3 of [6], first half, proved as Theorem 3.10 of [7])**.** *Let (X, d) be an ultrametric space with $|X| = n \ge 1$. Then $|C(X)| \le 1 + \lfloor \log_2 n \rfloor$.*

Proof. (Formal: `ncard_center_le` for Λ-valued spaces with $\lfloor \log_2 n \rfloor$ read as $\max\{k : 2^k \le n\}$, and `ncard_center_le_logb` for real-valued spaces with $\lfloor \log_2 n \rfloor$ read as the natural-number floor of the real binary logarithm; the two readings agree.) $C(X)$ is finite, $0 \in C(X)$, and $2^{|C(X) \setminus \{0\}|} \le n$ by Lemma 3.1, so $|C(X)| - 1 \le \lfloor \log_2 n \rfloor$. □

Theorem 3.3 (attainment; Conjecture 3 of [6], second half, proved as Theorem 3.10 of [7])**.** *For every $n \ge 1$ the space A_n of Definition 2.1 is an n-point ultrametric space with $|C(A_n)| = 1 + \lfloor \log_2 n \rfloor$. Consequently, for every $n \ge 1$ there is a real-valued ultrametric space on n points whose center of distances has exactly $1 + \lfloor \log_2 n \rfloor$ elements.*

Proof. (Formal: `attDist_isUltrametric` and `ncard_center_attDist` for the $\mathbb{N}$-valued space, and `exists_ultrametric_ncard_center_eq` for the real-valued consequence, which is obtained by transport along $\mathbb{N} \hookrightarrow \mathbb{R}$.) Write $k = \lfloor \log_2 n \rfloor$. The strong triangle inequality for a_n follows from that of τ_k, and $a_n(x,y) = 0$ only for $x = y$ because $a_n \geq 1$ off the diagonal. Every level $j \in \{1,\dots,k\}$ of the hierarchy is fully branching over $\{0,\dots,2^k - 1\}$ and every fold $\varphi_n(p)$ lies there, so for each p there is $b < 2^k \leq n$ with $\tau_k(\varphi_n(p), b) = j$, i.e. $a_n(p,b) = 1+j$ (note $\varphi_n(b) = b$ and $b \neq p$). Hence $\{0\} \cup \{2,\dots,k+1\} \subseteq C(A_n)$ (formal: `mem_center_attDist`), which is $k+1$ elements, and Theorem 3.2 closes the equality. The extra distance 1 is never centered: some fold class is a singleton because $n < 2^{k+1}$. □

Proposition 3.4 (the center is an invariant of weak similarity)**.** *Let $d : X \times X \to \Lambda$ and $\rho : Y \times Y \to \Lambda'$ be arbitrary functions into linearly ordered sets with zero, and let $\Phi : X \to Y$ be a weak similarity with comparison function f. Then $C(X,d) = f(C(Y,\rho))$; in particular $|C(X)| = |C(Y)|$ whenever the spaces are weakly similar.*

Proof. (Formal: `IsWeakSimilarity.center_eq` for the set equality, `ncard_center_eq_of_weaklySimilar` for the cardinalities, which is the statement pinned to this proposition; no metric axiom is assumed.) If $t \in C(X)$ then $t = f(s)$ for a unique $s \in D(Y)$, and for $q = \Phi p$ a solution x of $d(p,x) = t$ gives $\rho(q, \Phi x) = s$ by injectivity of f on $D(Y)$; conversely $\rho(\Phi p, y) = s$ with $y = \Phi x$ gives $d(p,x) = f(s)$. Surjectivity of Φ covers every q. □

So $|C(X)|$ depends only on the order type of the distance set with its incidence, and an n-point ultrametric space may always be replaced by a weakly similar one with distances in $\{0, 1, \dots, n-1\}$: an n-point ultrametric space has at most $n-1$ distinct non-zero distances, its chain of ball partitions refining strictly at most $n-1$ times. (This reduction is a paper step; it is what Remark 8.1 relies on.)

4. Controls

Proposition 4.1 (the hypothesis is not decoration)**.** *Let c_7 be the graph metric of the 7-cycle, $c_7(x,y) = \min\{(x-y) \bmod 7,\ (y-x) \bmod 7\}$ on $\{0,\dots,6\}$. Then c_7 is a metric, c_7 is not an ultrametric, and $|C(\{0,\dots,6\}, c_7)| > 3 = 1 + \lfloor \log_2 7 \rfloor$.*

Proof. (Formal: `cyc7_isMetric` and `cyc7_not_ultrametric`, both decided by evaluation inside the kernel without any axiom; `cyc7_center_gt`.) The four values $0,1,2,3$ are all centered because the cycle is vertex-transitive. In fact $C = \{0,1,2,3\}$ exactly, a direct computation that is not part of the formal statement. So the logarithmic bound is a statement about ultrametric spaces and fails for metric spaces. □

Proposition 4.2 (the $+1$, and $n = 8$)**.** *$|C(A_4)| > \lfloor \log_2 4 \rfloor = 2$, so the bound $|C(X)| \leq \lfloor \log_2 n \rfloor$ without the $+1$ is false; and $|C(A_8)| = 4$, so four centered distances are reached on eight points.*

Proof. (Formal: `not_ncard_center_le_log`, `attDist_two_pow_eight`.) Both are Theorem 3.3 at $n = 4$ and $n = 8$. The case $n = 8$ is the 2-adic computation that motivated Conjecture 3 in [6], and it refutes every bound of the form $|C(X)| \leq 3$. □

Proposition 4.3 (the three-point space of Figure 3 of [6])**.** *Let $X_3 = \{x_1, x_2, x_3\}$ with $d(x_1, x_3) = 1$ and $d(x_1, x_2) = d(x_2, x_3) = 2$. Then $C(X_3) = \{0, 2\}$; in particular $|C(X_3)| = 2 = 1 + \lfloor \log_2 3 \rfloor$.*

Proof. (Formal: `center_X3N`, for the $\mathbb{N}$-valued table with $x_1, x_2, x_3 = 0, 1, 2$.) The value 1 is realised only from x_1 and x_3, not from x_2; 2 is realised from every point. The source does not print this center; it is the check that the formal definition of C computes the paper's $C(X)$ on the paper's own example. □

Proposition 4.4 (exhaustive check at $n = 3$)**.** *Let $d : \{0,1,2\} \times \{0,1,2\} \to \{0,1,2\}$ be any table with $d(x,x) = 0$, $d(x,y) = 0 \Rightarrow x = y$, $d(x,y) = d(y,x)$ and $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ (values ordered $0 < 1 < 2$). Then at most two values $t \in \{0,1,2\}$ satisfy $\forall p\, \exists x\ d(p,x) = t$, and some such table has exactly two.*

Proof. (Formal: `exhaustive_three`, `exhaustive_three_attained`, decided in the kernel over all $3^9 = 19683$ tables.) Five of the tables are ultrametrics, and all five have exactly two centered values. Since a three-point ultrametric space has at most two distinct non-zero distances (its triangles are isosceles with the two longer sides equal) and $|C|$ is invariant under weak similarity (Proposition 3.4), this covers every three-point ultrametric space; that reduction is a paper step. □

5. Centered spheres: Conjectures 1 and 2 of [6]

Conjecture 1 of [6] is the closed-ball counterpart of its Theorem 6, which concerns open balls and the sets $D_p(X) \cap [0, r)$. We prove its first half.

Theorem 5.1 (Conjecture 1 of [6], first half)**.** *Let (X, d) be an ultrametric space in which every closed ball $\overline{B}_r(c)$, $c \in X$, $r \geq 0$, is a centered sphere. Then for every $p \in X$ and every $r \geq 0$ the set $D_p(X) \cap [0, r]$ has a greatest element.*

Proof. (Formal: `exists_greatest_distFrom`; no finiteness is assumed.) Fix p and $r \geq 0$. By hypothesis $\overline{B}_r(p) = \{x : d(x, c) = s\} \cup \{c\}$ for some $c \in \overline{B}_r(p)$ and $s \geq 0$. Since $p \in \overline{B}_r(p)$, either $p = c$ or $d(p, c) = s$; in both cases $d(p, c) \leq s$. For $x \in \overline{B}_r(p)$, $x \neq c$, the strong triangle inequality gives $d(p, x) \leq \max\{d(p, c), d(c, x)\} = s$, so s is an upper bound of $D_p(X) \cap [0, r]$. If the sphere $\{x : d(x, c) = s\}$ is non-empty, s is attained — at c if $p \neq c$, at any sphere point if $p = c$ — and $s \leq r$ because that point lies in $\overline{B}_r(p)$; so s is the greatest element. If the sphere is empty then $\overline{B}_r(p) = \{c\} = \{p\}$ and the greatest element is 0. □

Remark 5.2 (the second half, and why **UT** is load-bearing). The second half of Conjecture 1 asserts the converse for $(X, d) \in \mathbf{UT}$. It is not proved here, and the hypothesis cannot be removed: a closed ball with at least two points is a centered sphere $\{x : d(x, c) = s\} \cup \{c\}$ only if c is isolated in that ball, hence in X (closed balls of an ultrametric space are open); in the p-adic space $(\mathbb{Q}, d_p)$ of Example 1 of [6] no point is isolated, while every $D_q(\mathbb{Q}) \cap [0, r] = \{0\} \cup \{p^\gamma \leq r\}$ has a greatest element. This is Example 4 of [6], which makes the same point for open balls, transposed to closed balls. Formalising the class **UT** (labeled trees, the max-label ultrametric and the isometry classification of [6], Section 2) is a separate development and was not attempted. This remark is outside the formal development.

For the second conjecture we use the order type of (X_3, d): an ultrametric space (Y, ρ) is of *three-point type* if $Y = \{a, b, e\}$ with a, b, e distinct and $\rho(a, b) < \rho(a, e) = \rho(b, e)$.

Theorem 5.3 (Conjecture 2 of [6])**.** *Let (Y, ρ) be a real-valued ultrametric space with $|Y| \geq 3$ (Y may be infinite). The following are equivalent:*

(i) *every non-empty subset of Y is a centered sphere in (Y, ρ);*
(ii) *Y has exactly three points and (Y, ρ) is weakly similar to (X_3, d).*

Proof. (Formal: `all_centeredSphere_iff`, with (ii) rendered literally as a weak similarity $Y \to X_3$ in the sense of Definition 4 of [6]; the four steps below are `threePointType_of_all_centeredSphere`, `weaklySimilar_X3_of_threePointType`, `threePointType_of_weaklySimilar_X3` and `isCenteredSphere_of_threePointType`.)

(i)⇒(ii). Y itself is a centered sphere: $\rho(y, c) = r$ for all $y \neq c$, some $c \in Y$, $r \geq 0$; as $|Y| \geq 2$, $r > 0$. Next $Y \setminus \{c\}$ is a centered sphere: $\rho(y, c') = r'$ for all $y \in Y \setminus \{c, c'\}$, some $c' \neq c$; since $\rho(c, c') = r$ and $c \notin Y \setminus \{c\}$, $r' \neq r$. Pick $y_2 \notin \{c, c'\}$. Then $r' = \rho(y_2, c') \leq \max\{\rho(y_2, c), \rho(c, c')\} = r$, so $r' < r$. Now suppose $Y \setminus \{c\}$ had a third point $z \notin \{c', y_2\}$. Then $\rho(z, y_2) \leq \max\{\rho(z, c'), \rho(c', y_2)\} = r' < r$, and the set $T = \{c, z, y_2\}$ is a centered sphere with some centre $v \in T$ and radius t. If $v = c$ then $t = \rho(z, c) = r$, so $\rho(c', c) = r = t$ forces $c' \in T$, impossible. If $v = z$ then $t = \rho(c, z) = r$ and also $t = \rho(y_2, z) < r$, impossible; $v = y_2$ is symmetric. Hence $Y = \{c, c', y_2\}$, and with $a = c'$, $b = y_2$, $e = c$ we have $\rho(a, e) = \rho(b, e) = r > r' = \rho(a, b)$: three-point type. The weak similarity is then explicit: $a, b, e \mapsto x_1, x_3, x_2$ with $f(0) = 0$, $f(1) = \rho(a, b)$, $f(2) = \rho(a, e)$, a strictly increasing bijection $\{0, 1, 2\} \to D(Y)$.

(ii)⇒(i). A weak similarity to X_3 pulls the order type back, so $Y = \{a, b, e\}$ with $\rho(a, b) < \rho(a, e) = \rho(b, e)$. Each of the seven non-empty subsets is a centered sphere: singletons with radius 0; $\{a, b\}$

centred at a with radius $\rho(a,b)$; $\{a,e\}$ at a and $\{b,e\}$ at b with radius $\rho(a,e)$; and Y at e with radius $\rho(a,e)$. The strict inequality $\rho(a,b) < \rho(a,e)$ is what keeps the third point off each sphere. □

6. Prescribed centers: Conjecture 4.4 of [7]

Theorem 6.1 (Conjecture 4.4 of [7]). *Let $A \subset [0,\infty)$ be finite with $0 \in A$. Then there is a finite real-valued ultrametric space (X,d) with $D(X) = C(X) = A$.*

Proof. (Formal: `exists_ultrametric_distSet_eq_center_eq`; the statement asserts existence on some $\{0,\dots,m-1\}$ with $m \geq 1$, and the point count below is read off the construction, not from the formal statement.) Let $|A| = k+1$ and take the complete binary hierarchy H_k on 2^k points, with $D(H_k) = C(H_k) = \{0,\dots,k\}$ by (1). Let $\ell : \mathbb{N} \to \mathbb{R}$ send $i \leq k$ to the i-th smallest element of A and continue strictly increasingly beyond k; then ℓ is strictly increasing with $\ell(0) = 0$, so $\ell \circ \tau_k$ is an ultrametric on 2^k points with distance set and center $\ell(\{0,\dots,k\}) = A$ by the transport lemma. □

The space has $2^{|A|-1}$ points, and no space with $|C(X)| = |A|$ has fewer (Theorem 3.2, or Corollary 3.12 of [7]). The hierarchy itself is Example 3.13 of [7], one page before the conjecture; what the conjecture needs is the observation that a strictly increasing relabelling of the levels carries both D and C along.

7. Extremal spaces

Call a finite ultrametric space *extremal* if $|X| = 2^n$ and $|C(X)| = n+1$, the maximum allowed by Theorem 3.2. The hierarchy H_n is extremal, and so is A_{2^n} (formal: `attDist_extremal`), so the class is non-empty for every n. Two refinements of Lemma 3.1 drive the results.

Lemma 7.1 (relative and mixed doubling). *Let (X,d) be a finite ultrametric space, $s \geq 0$, and suppose every ball $\overline{B}_s(z)$, $z \in X$, has at least m points. Let $T \subseteq C(X)$ consist of j centered distances in $(s,r]$, $r \geq s$.*

(i) *Every ball $\overline{B}_r(p)$ has at least $2^j m$ points.*
(ii) *If moreover $|\overline{B}_s(p_0)| \geq M$ for one distinguished point p_0, then $|\overline{B}_r(p_0)| \geq M + (2^j - 1)m$.*

Proof. (Formal: `ball_growth_rel`, `ball_growth_mixed`.) The induction of Lemma 3.1, started from radius s instead of from single points. In (ii) the partner balls that appear in the induction only ever need the uniform bound (i), so the better bound M at p_0 survives the whole climb. □

Theorem 7.2 (every distance of an extremal space is centered). *Let (X,d) be a finite ultrametric space with $|X| = 2^n$ and $|C(X)| = n+1$. Then $D(X) = C(X)$.*

Proof. (Formal: `distSet_eq_center_of_extremal`; its non-vacuity on A_{2^n} is `distSet_attDist_two_pow`.) Suppose $u = d(x,y)$ is not centered; then $u > 0$. Let i be the number of centered distances in $(0,u)$ and s the largest of them ($s = 0$ if $i = 0$). By Lemma 3.1 every ball of radius s, and every ball of radius u, has at least 2^i points. The balls $\overline{B}_s(x)$ and $\overline{B}_s(y)$ are disjoint, their centres being $u > s$ apart, and both lie in $\overline{B}_u(x)$, so $|\overline{B}_u(x)| \geq 2^{i+1}$. The remaining $n-i$ centered distances lie above u (none equals u); climbing them with Lemma 7.1(ii) from radius u, with $m = 2^i$ and $M = 2^{i+1}$, gives a ball with at least

$$2^{i+1} + (2^{n-i} - 1)\,2^i = 2^n + 2^i > 2^n = |X|$$

points, a contradiction. □

Theorem 7.3 (ball sizes of an extremal space). *Let (X,d) be a finite ultrametric space with $|X| = 2^n$ and $|C(X)| = n+1$. Then for every $p \in X$ and every $r \geq 0$,*

$$|\overline{B}_r(p)| = 2^{k(r)}, \qquad k(r) = |\{t \in C(X) : 0 < t \leq r\}|.$$

Proof. (Formal: `ball_card_of_extremal`.) "$\geq$" is Lemma 3.1 at radius r with $T = \{t \in C(X) : 0 < t \leq r\}$. For "$\leq$", let $h = n - k(r)$ be the number of centered distances above r and $M = |\overline{B}_r(p)|$; every ball of radius r has at least $2^{k(r)}$ points, so Lemma 7.1(ii) climbed over those h distances gives $2^n = |X| \geq M + (2^h - 1)2^{k(r)} = M + 2^n - 2^{k(r)}$, i.e. $M \leq 2^{k(r)}$. □

With r running through $C(X) = \{0 = t_0 < t_1 < \cdots < t_n\}$, Theorems 7.2 and 7.3 say that the balls of an extremal space form a complete binary tree of depth n: $|\overline{B}_{t_i}(p)| = 2^i$ for every p, and no distance falls strictly between consecutive levels. Reading an isometry off this tree is the remaining step, which we carry out on paper.

Proposition 7.4 (extremal rigidity; proved on paper only, not formalised)**.** *Let (X, d) be a finite ultrametric space with $|X| = 2^n$ and $|C(X)| = n+1$, and write $C(X) = \{0 = t_0 < t_1 < \cdots < t_n\}$. Then $D(X) = C(X)$, and (X, d) is isometric to the model space $M(t_1, \ldots, t_n) = (\{0,1\}^n, \delta)$ with $\delta(x, y) = t_j$ for $j = \max\{i : x_i \neq y_i\}$ and $\delta(x, x) = 0$.*

Proof. For $r \geq 0$ let $P_{\leq r}$ and $P_{<r}$ be the partitions of X into the classes of the equivalence relations $d(x, y) \leq r$ and $d(x, y) < r$; these are equivalences because d is an ultrametric. Two counts. First, $t_i \in C(X)$ with $t_i > 0$ means that every point has a partner at distance exactly t_i, so every $\leq t_i$-class splits into at least two $< t_i$-classes: $|P_{<t_i}| \geq 2|P_{\leq t_i}|$. Second, $t_i < t_{i+1}$ gives $d \leq t_i \Rightarrow d < t_{i+1}$, so $P_{\leq t_i}$ refines $P_{<t_{i+1}}$ and $|P_{<t_{i+1}}| \leq |P_{\leq t_i}|$. Chaining, with $|P_{<t_1}| \leq |X|$ and $|P_{\leq t_n}| \geq 1$,

$$2^n = |X| \geq |P_{<t_1}| \geq 2|P_{\leq t_1}| \geq 2|P_{<t_2}| \geq 4|P_{\leq t_2}| \geq \cdots \geq 2^n |P_{\leq t_n}| \geq 2^n,$$

so every inequality is an equality. Consequences: $|P_{<t_1}| = |X|$, so the $< t_1$-classes are singletons and no distance lies in $(0, t_1)$; $|P_{<t_{i+1}}| = |P_{\leq t_i}|$ with $P_{\leq t_i}$ refining $P_{<t_{i+1}}$ forces $P_{\leq t_i} = P_{<t_{i+1}}$, so no distance lies in (t_i, t_{i+1}); and $|P_{\leq t_n}| = 1$, so no distance exceeds t_n. Hence $D(X) = C(X)$, which re-proves Theorem 7.2 by a different route. Moreover $|P_{\leq t_{i-1}}| = |P_{<t_i}| = 2|P_{\leq t_i}|$ while each $\leq t_i$-class contains at least two $\leq t_{i-1}$-classes, so each contains exactly two. Choosing, for every $\leq t_i$-class, an ordering of its two $\leq t_{i-1}$-subclasses and recording the choice as the bit x_i defines a bijection $X \to \{0,1\}^n$ under which $d(x, y) = t_j$ for the least j with x, y in a common $\leq t_j$-class, i.e. $j = \max\{i : x_i \neq y_i\}$. That is $M(t_1, \ldots, t_n)$. ☐

Corollary 7.5 (Conjectures 4.2 and 4.3 of [7]; proved on paper only)**.** *Let $n \in \mathbb{N}$ and let (X, d), (Y, ρ) be ultrametric spaces with $|X| = |Y| = 2^n$ and $|C(X)| = n + 1$. Then*

(i) *$|C(Y)| = |C(X)|$ if and only if (X, d) and (Y, ρ) are weakly similar;*
(ii) *$C(X) = C(Y)$ if and only if (X, d) and (Y, ρ) are isometric.*

Proof. (i) "⇐" is Proposition 3.4. "⇒": if $|C(Y)| = n+1$ then Y is extremal too, so $X \cong M(t_1, \ldots, t_n)$ and $Y \cong M(s_1, \ldots, s_n)$ by Proposition 7.4; the identity of $\{0,1\}^n$ with the strictly increasing bijection $s_i \mapsto t_i$, $0 \mapsto 0$, between $D(Y) = \{0, s_1, \ldots, s_n\}$ and $D(X) = \{0, t_1, \ldots, t_n\}$ is a weak similarity. (ii) Isometric spaces have equal centers (as [7] notes before its Conjecture 4.3). Conversely, if $C(X) = C(Y)$ then $|C(Y)| = n + 1$, both spaces are extremal, and both are isometric to $M(t_1, \ldots, t_n)$ for the *same* values. ☐

Of this section, Theorems 7.2 and 7.3 are formalised; Proposition 7.4 and Corollary 7.5 are not. The missing formal step is the construction of the bijection $X \to \{0,1\}^n$ by recursion on the ball hierarchy; its analytic input — that the hierarchy *is* a complete binary tree — is exactly the content of the two formalised theorems.

8. An exhaustive enumeration

Remark 8.1 (a computation outside the formal development). By the reduction after Proposition 3.4, every n-point ultrametric space is weakly similar to one with distances in $\{0, 1, \ldots, n-1\}$, and $|C|$ is a weak-similarity invariant. A backtracking enumerator over the isosceles condition (of any three distances the two largest are equal) walks every symmetric table on n labelled points with off-diagonal values in $\{1, \ldots, n-1\}$, computes C from the definition, and records the maximum of $|C|$ and the number of tables whose value set is $\{0, 1, \ldots, k\}$ without gaps (the *order types*):

n	tables	order types	$\max\lvert C\rvert$	$1+\lfloor\log_2 n\rfloor$
1	1	1	1	1
2	1	1	2	2
3	5	4	2	2
4	60	32	3	3
5	1,304	436	3	3
6	44,590	9,012	3	3
7	2,201,856	262,760	3	3
8	148,154,860	10,270,696	4	4
9	13,024,655,442	518,277,560	4	4

The order-type column agrees term by term with OEIS A005121, "number of ultradissimilarity relations on an n-set" (the chains from minimum to maximum in the partition lattice) [10], which is the independent check that the enumerator enumerates ultrametric order types; the maximum column is $1+\lfloor\log_2 n\rfloor$, OEIS A070939, throughout, and $n=8$ is the first value where the maximum is 4. Every row was reproduced for this paper from a fresh compilation of the enumerator, 40 seconds on one core for $n \le 8$ and 76 minutes for $n=9$; $n=10$ (about 1.4×10^{12} tables) was not attempted. None of this is load-bearing: Theorems 3.2 and 3.3 cover every n, and the $n=3$ row is the kernel-checked Proposition 4.4.

9. Verification

Theorems 3.2, 3.3, 5.1, 5.3, 6.1, 7.2, 7.3 and Propositions 3.4, 4.1, 4.2, 4.3, 4.4, together with Lemmas 3.1 and 7.1, rest on declarations formally verified in Lean 4 against *Mathlib* [12, 13] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. Where a printed statement says more than its formal counterpart the proof names the extra step: the exact center $\{0,1,2,3\}$ in Proposition 4.1, the point count $2^{|A|-1}$ after Theorem 6.1, and the reduction of three-point spaces to tables in Proposition 4.4. The development is eight modules, 1847 lines in all: definitions and faithfulness lemmas (103 lines); the doubling lemma and the bound (153); weak similarity and transport (111); the hierarchy τ_k, the attaining space A_n and both forms of attainment (255); centered spheres, closed balls, Conjecture 1 and Conjecture 2 (526); the relabelling of levels and Conjecture 4.4 (175); the two refined doubling lemmas and the extremal theorems (367); and the controls (157). The 19 declarations bound to the results above were checked with `#print axioms`: each depends only on `propext`, `Classical.choice` and `Quot.sound`, except that `center_X3N` uses only `propext` and `Quot.sound` and the two 7-cycle decisions `cyc7_isMetric`, `cyc7_not_ultrametric` depend on no axiom at all; the supporting lemmas named in the proofs above pass the same check, several of them without `Classical.choice`. There is no compiled evaluation (`native_decide`), no `sorry`, and no axiom declared by hand; the exhaustive $n=3$ check is `decide +kernel` over all 3^9 tables. The value type is an arbitrary linear order with a zero throughout, except where a statement mentions $\mathbb{R}$ (Theorems 3.3, 5.3, 6.1 and the real-valued form of Theorem 3.2). As recorded, checking the eight modules took about 17 minutes of wall time over some 25 compilations, with a peak resident set of about 7 GB per process (the *Mathlib* import baseline); the enumeration of Remark 8.1 took 40 seconds for $n \le 8$ and 76 minutes for $n=9$. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). Throughout, `X`, `Y` are types, `α`, `β` are types with `[LinearOrder α] [Zero α]` (respectively `β`), and `d : X → X → α`, `ρ : Y → Y → β`. `Set.ncard` is the cardinality of a finite set as a natural number, `Nat.log 2 n` is $\max\{k : 2^k \le n\}$, `⌊Real.logb 2 n⌋₊` the natural-number floor of the real binary logarithm, `Fin n` the type $\{0,\dots,n-1\}$, and `Fintype.card X` the number of elements of `X`.

Definitions.

```
structure IsUltrametric (d : X → X → α) : Prop where
  nonneg : ∀ x y, 0 ≤ d x y
  self : ∀ x, d x x = 0
  eq_of_dist_eq_zero : ∀ x y, d x y = 0 → x = y
  symm : ∀ x y, d x y = d y x
  strong_triangle : ∀ x y z, d x z ≤ max (d x y) (d y z)

def distSet (d : X → X → α) : Set α := {t | ∃ x y, d x y = t}

def center (d : X → X → α) : Set α :=
  {t | t ∈ distSet d ∧ ∀ p : X, ∃ x, d p x = t}

structure IsWeakSimilarity (d : X → X → α) (ρ : Y → Y → β) (Φ : X → Y) (f : β → α) :
    Prop where
  bijective : Function.Bijective Φ
  bijOn : Set.BijOn f (distSet ρ) (distSet d)
  strictMonoOn : ∀ ⦃s t : β⦄, s ∈ distSet ρ → t ∈ distSet ρ → s < t → f s < f t
  dist_eq : ∀ x y, d x y = f (ρ (Φ x) (Φ y))

def WeaklySimilar (d : X → X → α) (ρ : Y → Y → β) : Prop :=
  ∃ (Φ : X → Y) (f : β → α), IsWeakSimilarity d ρ Φ f

def IsCenteredSphere (d : X → X → α) (S : Set X) : Prop :=
  ∃ c ∈ S, ∃ r : α, 0 ≤ r ∧ S = {x | d x c = r} ∪ {c}

def closedBall (d : X → X → α) (c : X) (r : α) : Set X := {x | d c x ≤ r}

def distFrom (d : X → X → α) (p : X) : Set α := {t | ∃ x, d p x = t}

def treeDist : ℕ → ℕ → ℕ → ℕ
  | 0, _, _ => 0
  | k + 1, x, y => if x % 2 = y % 2 then treeDist k (x / 2) (y / 2) else k + 1

def hier (k : ℕ) (x y : Fin (2 ^ k)) : ℕ := treeDist k x.val y.val

def fold (n : ℕ) (x : Fin n) : ℕ :=
  if x.val < 2 ^ Nat.log 2 n then x.val else x.val - 2 ^ Nat.log 2 n

def attDist (n : ℕ) (x y : Fin n) : ℕ :=
  if x = y then 0 else 1 + treeDist (Nat.log 2 n) (fold n x) (fold n y)

def X3N : Fin 3 → Fin 3 → ℕ :=
  fun i j => if i = j then 0 else if i = 1 ∨ j = 1 then 2 else 1

noncomputable def X3 : Fin 3 → Fin 3 → ℝ := fun i j => ((X3N i j : ℕ) : ℝ)

def cyc7 (x y : Fin 7) : ℕ :=
  min ((x.val + 7 - y.val) % 7) (7 - ((x.val + 7 - y.val) % 7))
```

The bound and its attainment (Theorems 3.2, 3.3; Lemma 3.1).

```
theorem ball_growth [Fintype X] (hd : IsUltrametric d) :
    ∀ (m : ℕ) (T : Finset α), T.card = m → (∀ t ∈ T, t ∈ center d) →
```

```
    (∀ t ∈ T, 0 < t) →
    ∀ (p : X) (r : α), 0 ≤ r → (∀ t ∈ T, t ≤ r) →
      2 ^ m ≤ ({x | d p x ≤ r} : Finset X).card

theorem ncard_center_le [Fintype X] [Nonempty X] (hd : IsUltrametric d) :
    (center d).ncard ≤ 1 + Nat.log 2 (Fintype.card X)

theorem ncard_center_le_logb {X : Type*} [Fintype X] [Nonempty X] {d : X → X → ℝ}
    (hd : IsUltrametric d) (n : ℕ) (hn : Fintype.card X = n) :
    (center d).ncard ≤ 1 + ⌊Real.logb 2 n⌋₊

theorem attDist_isUltrametric (hn : n ≠ 0) : IsUltrametric (attDist n)

theorem ncard_center_attDist (hn : n ≠ 0) :
    (center (attDist n)).ncard = 1 + Nat.log 2 n

theorem exists_ultrametric_ncard_center_eq (n : ℕ) (hn : n ≠ 0) :
    ∃ d : Fin n → Fin n → ℝ, IsUltrametric d ∧
      (center d).ncard = 1 + ⌊Real.logb 2 n⌋₊
```

Invariance (Proposition 3.4).

```
theorem ncard_center_eq_of_weaklySimilar {d : X → X → α} {ρ : Y → Y → β}
    (h : WeaklySimilar d ρ) : (center d).ncard = (center ρ).ncard
```

Controls (Propositions 4.1–4.4).

```
theorem cyc7_isMetric :
    (∀ x, cyc7 x x = 0) ∧ (∀ x y, cyc7 x y = 0 → x = y) ∧
      (∀ x y, cyc7 x y = cyc7 y x) ∧ (∀ x y z, cyc7 x z ≤ cyc7 x y + cyc7 y z)

theorem cyc7_not_ultrametric :
    ¬ (∀ x y z : Fin 7, cyc7 x z ≤ max (cyc7 x y) (cyc7 y z))

theorem cyc7_center_gt : 1 + Nat.log 2 7 < (center cyc7).ncard

theorem not_ncard_center_le_log : ¬ ((center (attDist 4)).ncard ≤ Nat.log 2 4)

theorem attDist_two_pow_eight : (center (attDist 8)).ncard = 4

theorem center_X3N : center X3N = ({0, 2} : Set ℕ)

theorem exhaustive_three :
    ∀ d : Fin 3 → Fin 3 → Fin 3,
      ((∀ x, d x x = 0) ∧ (∀ x y, d x y = 0 → x = y) ∧ (∀ x y, d x y = d y x) ∧
        (∀ x y z, d x z ≤ max (d x y) (d y z))) →
      (Finset.univ.filter (fun t : Fin 3 => ∀ p, ∃ x, d p x = t)).card ≤ 2

theorem exhaustive_three_attained :
    ∃ d : Fin 3 → Fin 3 → Fin 3,
      ((∀ x, d x x = 0) ∧ (∀ x y, d x y = 0 → x = y) ∧ (∀ x y, d x y = d y x) ∧
        (∀ x y z, d x z ≤ max (d x y) (d y z))) ∧
      (Finset.univ.filter (fun t : Fin 3 => ∀ p, ∃ x, d p x = t)).card = 2
```

Centered spheres (Theorems 5.1, 5.3).

```
theorem exists_greatest_distFrom {d : X → X → α} (hd : IsUltrametric d)
    (hball : ∀ (c : X) (r : α), 0 ≤ r → IsCenteredSphere d (closedBall d c r))
    (p : X) (r : α) (hr : 0 ≤ r) :
    ∃ m, (m ∈ distFrom d p ∧ m ≤ r) ∧ ∀ t, t ∈ distFrom d p → t ≤ r → t ≤ m

theorem all_centeredSphere_iff {Y : Type*} {ρ : Y → Y → ℝ} (hρ : IsUltrametric ρ)
    (h3 : 3 ≤ Cardinal.mk Y) :
    (∀ S : Set Y, S.Nonempty → IsCenteredSphere ρ S) ↔
      (Nat.card Y = 3 ∧ WeaklySimilar ρ X3)
```

Prescribed centers (Theorem 6.1).

```
theorem exists_ultrametric_distSet_eq_center_eq (A : Finset ℝ)
    (hA : ∀ t ∈ A, (0 : ℝ) ≤ t) (h0 : (0 : ℝ) ∈ A) :
    ∃ (m : ℕ) (d : Fin m → Fin m → ℝ), 0 < m ∧ IsUltrametric d ∧
      distSet d = ↑A ∧ center d = ↑A
```

Extremal spaces (Theorems 7.2, 7.3; Lemma 7.1).

```
theorem ball_growth_rel [Fintype X] (hd : IsUltrametric d) (m : ℕ) (s : α)
    (hs : ∀ z : X, m ≤ ({x | d z x ≤ s} : Finset X).card) :
    ∀ (j : ℕ) (T : Finset α), T.card = j → (∀ t ∈ T, t ∈ center d) →
      (∀ t ∈ T, s < t) →
      ∀ (p : X) (r : α), s ≤ r → (∀ t ∈ T, t ≤ r) →
        2 ^ j * m ≤ ({x | d p x ≤ r} : Finset X).card

theorem ball_growth_mixed [Fintype X] (hd : IsUltrametric d) (m M : ℕ) (s : α)
    (hs : ∀ z : X, m ≤ ({x | d z x ≤ s} : Finset X).card) (p₀ : X)
    (hM : M ≤ ({x | d p₀ x ≤ s} : Finset X).card) :
    ∀ (j : ℕ) (T : Finset α), T.card = j → (∀ t ∈ T, t ∈ center d) →
      (∀ t ∈ T, s < t) →
      ∀ (r : α), s ≤ r → (∀ t ∈ T, t ≤ r) →
        M + (2 ^ j - 1) * m ≤ ({x | d p₀ x ≤ r} : Finset X).card

theorem distSet_eq_center_of_extremal [Fintype X] [Nonempty X] (hd : IsUltrametric d)
    (n : ℕ) (hcard : Fintype.card X = 2 ^ n) (hC : (center d).ncard = n + 1) :
    distSet d = center d

theorem attDist_extremal (n : ℕ) :
    Fintype.card (Fin (2 ^ n)) = 2 ^ n ∧ (center (attDist (2 ^ n))).ncard = n + 1

theorem distSet_attDist_two_pow (n : ℕ) :
    distSet (attDist (2 ^ n)) = center (attDist (2 ^ n))

theorem ball_card_of_extremal [Fintype X] [Nonempty X] (hd : IsUltrametric d) (n : ℕ)
    (hcard : Fintype.card X = 2 ^ n) (hC : (center d).ncard = n + 1) (p : X) (r : α)
    (hr : 0 ≤ r) :
    ({x | d p x ≤ r} : Finset X).card
      = 2 ^ (((finite_center (d := d)).toFinset.erase 0).filter (fun t => t ≤ r)).card
```

Here `finite_center` is the finiteness of `center d` on a finite type, so the exponent is the number of non-zero centered distances $\leq r$.

References

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