

TWO OPERATIONS, ONE SEMIGROUP: RECTANGULAR DOPPELSEMIGROUPS AS PAIRS OF INFLATIONS OF A BAND, AND ABELIAN DIMONOIDS AS ABELIAN g -DIMONOIDS

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. A *doppelsemigroup* is a set carrying two associative operations $\dashv$ and $\vdash$ that satisfy the two interassociativity laws $(x \dashv y) \vdash z = x \dashv (y \vdash z)$ and $(x \vdash y) \dashv z = x \vdash (y \dashv z)$; it is *rectangular* when both operations obey $x \cdot y \cdot z = x \cdot z$, in which idempotency is *not* required. Gavrylkiv, correcting the published count of three-element doppelsemigroups from 75 to 77 because “the nontrivial rectangular doppelsemigroups were overlooked”, tabulates the number $\text{rds}(n)$ of rectangular doppelsemigroups of order n up to isomorphism for $n \leq 6$ and asks, as his Problem 1, for the values at $n \geq 7$. We prove a structure theorem: a pair of operations on a set is a rectangular doppelsemigroup precisely when both operations are inflations of one and the same associative rectangular operation, along two arbitrary and mutually unrelated maps fixing its products. The interassociativity laws are thus *free* on top of a shared band, and the band is intrinsic — the two operations have the same idempotents, the same image, and agree there. Four consequences follow: the rectangular interassociates of a rectangular semigroup are exactly the inflations of its own band; a rectangular doppelsemigroup whose two operations differ has at least three elements, one such exists at every $n \geq 3$, and the smallest member of that family is on the nose the structure the 2019 classification overlooked, so the published erratum is explained rather than exhibited; a second dichotomy sits beside the published “strong $\iff$ trivial”; and, using the structure theorem to generate candidates and a compiled certificate checker to verify them, $\text{rds}(7) \geq 571$, $\text{rds}(8) \geq 2193$ and $\text{rds}(9) \geq 8594$ — the first three values Problem 1 asks for, each as that many pairwise non-isomorphic algebras. The same machinery reproduces every published value of rds from $n = 3$ on, and the rectangular-semigroup counts of OEIS A390162 at $n = 3, \dots, 9$.

A second freeness theorem concerns Loday’s dimonoids and the g -dimonoids, or generalized dimonoids, which drop Loday’s axiom (L_2) . Two tables of Gavrylkiv print one and the same column of six integers, 1, 4, 17, 103, 791, 10870, for the abelian dimonoids and for the abelian g -dimonoids, and neither paper remarks on it. We show that the agreement is forced at every order: under the abelian law $x \dashv y = y \vdash x$ each of (L_1) , (L_2) , (L_3) unwinds, modulo associativity alone, to the single identity $(xy)z = (xz)y$, so the abelian dimonoids and the abelian g -dimonoids are literally the same algebras and $\text{adm}(n) = \text{agdm}(n)$ for every n — two of the eight columns the two papers leave open are one column. The dimonoid half of that criterion is a 2013 lemma of A. V. Zhuchok and is credited, not claimed; what is new is that it survives the deletion of (L_2) . This is an identity between counting problems and not a new value: the first unpublished order $n = 7$ is priced below at roughly twenty CPU-days for the enumeration used here and is not attempted. The published g -dimonoid tables at orders 2 and 3 and six published order-4 values are also re-derived from the axioms, the abelian one of them by a single computation serving both varieties. Every theorem below is machine-checked in Lean 4.

1. INTRODUCTION

The objects. A *doppelsemigroup* is a triple $(D, \dashv, \vdash)$ in which $\dashv$ and $\vdash$ are associative binary operations on the set D subject to the two *interassociativity laws*

- (1) $(x \dashv y) \vdash z = x \dashv (y \vdash z),$
- (2) $(x \vdash y) \dashv z = x \vdash (y \dashv z),$

for all $x, y, z \in D$. No unit, bar-unit or zero is required. Equivalently, $(D, \dashv)$ and $(D, \vdash)$ are two semigroup structures on one set that are *interassociative*: the two ways of bracketing a mixed product agree. Interassociativity is older than the doppelsemigroup vocabulary — [2] sketches its history — and the constructions of Boyd, Gould and Nelson [9] are the ones Section 4 connects to. The

investigation of doppelsemigroups was initiated by A. V. Zhuchok [17] and continued by him and his coauthors, for instance in [18]; [1, Introduction] gives that history and traces the notion to Loday’s dimonoids [12]. The small-order counting problem this paper takes up is Gavrylkiv’s [1].

Loday’s *dimonoids* [12] are the second family of objects here: a dimonoid is a triple $(D, \dashv, \vdash)$ with both operations associative and

$$\begin{aligned} (3) \quad & (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (4) \quad & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (5) \quad & (x \dashv y) \vdash z = x \vdash (y \vdash z), \end{aligned}$$

where a bar-unit is not an axiom. Deleting (4) gives the *generalized dimonoids*, or *g-dimonoids*: both operations associative, (3) and (5). The notion is attributed by [4] to Movsisyan, Davidov and Safaryan [8]. The labels $(D_1), (D_2), (D_3)$ are the sources’ for (3)–(5); they are renamed here because the doppelsemigroup source uses $(D_1), (D_2)$ for (1),(2). Note that (4) *is* (2): it is the only law the two varieties share, and they are incomparable (Proposition 7.5(iii)).

Following the source’s terminology, a semigroup $(S, \cdot)$ is *rectangular* when every element of S is a middle identity,

$$(6) \quad x \cdot y \cdot z = x \cdot z \quad \text{for all } x, y, z \in S,$$

and a doppelsemigroup, dimonoid or *g-dimonoid* is *rectangular* when both of its semigroup reducts are. It is essential that (6) does *not* include idempotency: a rectangular semigroup in this sense is an inflation of a rectangular band, not a band, and the null semigroup is rectangular. Reading “rectangular” as “rectangular band” collapses everything the first half of this paper is about, and it is precisely the distinction on which the published erratum recalled below turns. Such a structure is *trivial* when $\dashv = \vdash$, in which case it is just a semigroup; *commutative* when both reducts are commutative; *abelian* when $x \dashv y = y \vdash x$ for all x, y , that is, when $\vdash$ is the opposite operation of $\dashv$; and a doppelsemigroup is *strong* when $x \dashv (y \vdash z) = x \vdash (y \dashv z)$ for all x, y, z .

Two structures on a finite set are identified when a bijection of the carrier carries the first operation to the first and the second to the second; anti-isomorphism and duality are *not* quotiented out. We write $\text{rds}(n)$ for the number of isomorphism classes of rectangular doppelsemigroups on an n -element set, and $\text{rs}(n)$ for the number of isomorphism classes of rectangular semigroups of order n — equivalently, of *trivial* rectangular doppelsemigroups. For dimonoids and *g-dimonoids* we follow the sources and write $\text{dm}, \text{cdm}, \text{adm}, \text{rdm}$ and $\text{gdm}, \text{cgdm}, \text{agdm}, \text{rgdm}$ for the numbers of isomorphism classes of all, commutative, abelian and rectangular members of each variety.

The sources, and what they leave open. Gavrylkiv and Rendziak [2] classified the doppelsemigroups of order 3 and reported 75 of them. Gavrylkiv [1, §6] corrects this to 77, opening that section with “In carrying out the classification, up to isomorphism, of all doppelsemigroups of order 3 in [8], the nontrivial rectangular doppelsemigroups were overlooked” (the source’s own reference [8] being [2] here), and exhibits one missing structure together with its dual. The same paper tabulates

n	0	1	2	3	4	5	6
$\text{rds}(n)$	1	1	3	7	20	51	171
$\text{rs}(n)$	1	1	3	5	10	14	27

as its Table 7, and closes with

Problem 1.1 ([1, Problem 1]). Determine the numbers $\text{ds}(n)$, $\text{sds}(n)$ for $n \geq 6$ and $\text{cds}(n)$, $\text{ads}(n)$, $\text{rds}(n)$ for $n \geq 7$.

Everything [1] states about the rectangular class itself is: that rectangularity is preserved by the opposite, dual and cross-dual constructions; that a rectangular doppelsemigroup is strong if and only if it is trivial; that — quoted from the unpublished manuscript discussed at the end of this introduction — the only commutative one of each order is the null one; and the two order-3 Cayley tables of the erratum. Its tool for that class is the representation of a rectangular semigroup that [1] attributes to

Two companion papers of the same author count dimonoids [3] and g -dimonoids [4] in the same style, tabulating

and closing with one problem each.

Problem 1.3 ([4, closing problem]). Determine the numbers $\text{gdm}(n)$ for $n \geq 6$ and $\text{cgdm}(n)$, $\text{agdm}(n)$, $\text{rgdm}(n)$ for $n \geq 7$.

What is settled here. The two halves of this paper prove two freeness theorems, and in both the conclusion is that a pair of operations is carried by a single semigroup.

Section 4 draws the consequences. Read in the vocabulary of interassociativity, the structure theorem determines the rectangular interassociates of a rectangular semigroup exactly (Theorem 4.1), and gives in three lines the rectangular half of the characterisation of the semigroups that coincide with all their interassociates which [2] quotes from [9, Lemma 5.5] (Corollary 4.2). It reproves [1]’s “strong $\iff$ trivial” without the Nagy–Nagy representation (Theorem 4.3), and it produces a second, independent dichotomy of the same shape (Theorem 4.4), whose two halves are shown to be separately insufficient. Finally it locates the erratum: a rectangular doppelsemigroup with $\neg \neq \vdash$ needs at least three elements, at every $n \geq 3$ there is one, and its $n = 3$ member *is* the structure [2] dropped (Theorem 4.6). So the correction 75 $\rightarrow$ 77 happens at the smallest order at which it can happen, for a reason.

Section 5 turns to Problem 1.1. The structure theorem makes the class constructible rather than merely testable: instead of scanning the n^{2n^2} pairs of tables one builds, from a pair of integers $p, q \geq 1$ with $pq \leq n$ and two maps into a $p \times q$ band, a list of candidates that is short enough to be certified in full. Checking such a certificate — all six axioms at all n^3 triples of each member, and lexicographic

minimality of each member under all $n!$ relabellings — gives

$$\text{rds}(7) \geq 571, \quad \text{rds}(8) \geq 2193, \quad \text{rds}(9) \geq 8594$$

(Theorem 5.2), each as that many pairwise non-isomorphic algebras exhibited explicitly. These are the first three values Problem 1.1 asks for. Run at the four orders $n \geq 3$ at which the answer is published, the identical machinery returns 7, 20, 51, 171, and its trivial part returns the rectangular-semigroup counts 5, 10, 14, 27, 36, 64, 90 at $n = 3, \dots, 9$, which OEIS A390162 [20] publishes as exact (Proposition 5.3).

Section 6 is the second half and is independent of the first. Under the abelian law the second operation is the opposite of the first, and each of Loday’s three axioms unwinds — modulo associativity alone, and for (4) with no associativity at all — to the single identity $(xy)z = (xz)y$ on that one operation, the classical *right commutative* law. Hence an abelian structure is a dimonoid if and only if it is a g -dimonoid (Theorem 6.3), the missing axiom (4) being free on top of the other two, and $\text{adm}(n) = \text{agdm}(n)$ for every n and not merely at the six orders of (7): two of the eight columns Problems 1.2 and 1.3 leave open are one column. The dimonoid half of the criterion — that an abelian structure is a dimonoid exactly when its first operation is a right commutative semigroup — is Zhuchok’s [5], is quoted as such in [6, 7], and is credited rather than claimed (Theorem 6.2); what is new is that it survives the deletion of (4). Both columns are therefore the count $\text{rcs}(n)$ of right commutative semigroups of order n (Corollary 6.4), and the reduction does concrete work: at order 4 one computation over a list of 188 single tables proves both published values $\text{adm}(4) = \text{agdm}(4) = 103$.

What is not settled, and one caveat. The three bounds of Theorem 5.2 are lower bounds. The matching upper bounds — that every rectangular doppelsemigroup of order n is isomorphic to a listed one — do not follow from anything proved here, although Remark 5.1 records a Pólya-type computation, carried out on paper from Theorem 3.5 and outside the verified development, whose values are exactly 571, 2193, 8594. Theorem 6.3 settles no value of Problems 1.2 and 1.3 either: it identifies two unknown columns with each other and with a one-operation problem, and Section 8 prices the first unknown entry of that column at roughly twenty CPU-days for the enumeration used here. The order-4 totals $\text{ds}(4) = 1217$, $\text{dm}(4) = 734$ and the 866 additively idempotent semirings of order 4 are published and are asserted nowhere below; what Section 7 adds at that order is six sub-columns, not the totals.

One risk to the novelty of Sections 3–5 is disclosed and could not be resolved. V. M. Gavrylkiv, *On certain classes of nonisomorphic rectangular doppelsemigroups*, is cited as *submitted* by [1] and is not public anywhere; it could not be read. From the description in [1] it classifies the commutative rectangular doppelsemigroups — a one-line corollary of Theorem 3.5 — and counts rectangular ideal extensions of left (right) zero doppelsemigroups by null quotients, which in the language of Section 5 is the sub-case $q = 1$ (resp. $p = 1$) of the data enumerated there. On that reading it is partial; but the only real evidence that it does not contain the general structure theorem or the values of Problem 1.1 is that its own author poses Problem 1.1 while citing it. A referee with access to that manuscript should check Sections 3 and 5 against it.

Section 7 records what this development verifies of the published material it rests on, Section 8 records what was priced and not attempted, and Section 9 describes the formal verification.

2. PRELIMINARIES

Throughout, D is a set and $\dashv, \vdash : D \times D \rightarrow D$; we write $x \dashv y$ and $x \vdash y$ and drop parentheses only where the bracketing does not matter. For a single operation m we write m in prefix form, $m(x, y)$, when both operations are in play, and juxtaposition xy when only one operation is in play.

Definition 2.1. $(D, \dashv, \vdash)$ is a *doppelsemigroup* if $\dashv$ and $\vdash$ are associative and (1), (2) hold; a *dimonoid* if $\dashv$ and $\vdash$ are associative and (3), (4), (5) hold; a g -*dimonoid* if $\dashv$ and $\vdash$ are associative and (3), (5) hold. A binary operation m on D is *rectangular* if $m(m(x, y), z) = m(x, z)$ for all x, y, z ; the structure is *rectangular* if both $\dashv$ and $\vdash$ are. It is *trivial* if $\dashv = \vdash$, *commutative* if both operations are commutative,

abelian if $x \dashv y = y \vdash x$ for all x, y , and a doppelsemigroup is *strong* if $x \dashv (y \vdash z) = x \vdash (y \dashv z)$ for all x, y, z .

Definition 2.2. A binary operation m on D is *right commutative* if

$$(8) \quad m(m(x, y), z) = m(m(x, z), y) \quad \text{for all } x, y, z \in D,$$

that is, $xyz = xzy$ in the associative case. Equivalently, the right translations $\rho_y : x \mapsto m(x, y)$ commute pairwise. A *right commutative semigroup* is an associative such operation; the terminology is the standard one [11]. We write $\text{rcs}(n)$ for the number of right commutative semigroups on an n -element set up to isomorphism.

Definition 2.3. Let m be an associative operation on D and let $f : D \rightarrow D$ fix every m -product, that is, $f(m(x, y)) = m(x, y)$ for all $x, y \in D$. The *inflation of m along f* is the operation

$$x * y := m(f(x), f(y)).$$

When m is in addition rectangular, its set of products coincides with its set of idempotents (Lemma 3.1 (iii) below), so f in Definition 2.3 fixes every idempotent of m ; if moreover f takes its values among the idempotents — as the intrinsic f of Theorem 3.5 does — then f is a retraction of D onto the band of idempotents and $*$ is an inflation in Clifford's classical sense. We keep the weaker hypothesis because it is all the proof of Theorem 3.5 uses.

For finite carriers we take $D = \{0, 1, \dots, n-1\}$ and record a structure as its pair of Cayley tables. An *isomorphism* $(D, \dashv, \vdash) \rightarrow (D, \dashv', \vdash')$ is a bijection σ of D with $\sigma(x \dashv y) = \sigma(x) \dashv' \sigma(y)$ and $\sigma(x \vdash y) = \sigma(x) \vdash' \sigma(y)$; the symmetric group acts on the set of pairs of tables by relabelling both tables simultaneously, and each of the counts named in Section 1 is the number of orbits meeting the class in question. The convention is the sources' and it matters: identifying each structure with its dual would change the order-3 doppelsemigroup count from 77 to 59 (Proposition 7.5).

Finally, for a rectangular doppelsemigroup $(D, \dashv, \vdash)$ we write

$$\varphi(x) := x \dashv x, \quad \psi(x) := x \vdash x, \quad E := \{z \in D : z \dashv z = z\},$$

and Theorem 3.4 justifies calling E *the* band of the pair.

3. THE STRUCTURE THEOREM

One operation. We first record the classical one-operation facts, in the three-line form in which they are used below. Nothing in this subsection is new; it is the Nagy–Nagy representation of a rectangular semigroup read in the coordinates of inflations, and it is reproved only so that the two-operation argument is self-contained.

Lemma 3.1. *Let m be associative and rectangular on D . Then, for all $x, y \in D$,*

- (i) $m(m(x, x), m(y, y)) = m(x, y)$;
- (ii) $m(m(x, y), m(x, y)) = m(x, y)$, *so every product is idempotent*;
- (iii) *the set of m -products equals the set of m -idempotents, and $x \mapsto m(x, x)$ is a retraction of D onto it with $m(x, y) = m(m(x, x), m(y, y))$.*

Proof. For (i) and (ii), apply rectangularity, then associativity, then rectangularity again:

$$m(m(x, x), m(y, y)) = m(x, m(y, y)) = m(m(x, y), y) = m(x, y),$$

$$m(m(x, y), m(x, y)) = m(x, m(x, y)) = m(m(x, x), y) = m(x, y).$$

For (iii): every idempotent $z = m(z, z)$ is a product; every product is idempotent by (ii); and (i) is the retraction identity. $\square$

Two operations: the band is forced. The whole content of the structure theorem is the following identity, which uses each interassociativity law exactly once and exhibits every $\vdash$ -product as a $\dashv$ -product.

Lemma 3.2. *Let $(D, \dashv, \vdash)$ be a rectangular doppelsemigroup. Then for all $u, v \in D$*

$$u \vdash v = (u \vdash u) \dashv (u \vdash v),$$

and consequently $(u \vdash v) \dashv (u \vdash v) = u \vdash v$: every $\vdash$ -product is $\dashv$ -idempotent. Symmetrically, every $\dashv$ -product is $\vdash$ -idempotent.

Proof. Read the right-hand side with (2), then (1), then associativity of $\vdash$, then rectangularity of $\vdash$:

$$(u \vdash u) \dashv (u \vdash v) = u \vdash (u \dashv (u \vdash v)) = u \vdash ((u \dashv u) \vdash v) = (u \vdash (u \dashv u)) \vdash v = u \vdash v.$$

So $u \vdash v$ is a $\dashv$ -product, and Lemma 3.1(ii) applied to $\dashv$ makes it $\dashv$ -idempotent. Interchanging $\dashv$ with $\vdash$ interchanges (1) with (2), which gives the mirror statement. $\square$

Lemma 3.3. *Let $\dashv$ be rectangular and let (1) and (2) hold. If $e \dashv e = e$ and $f \dashv f = f$ then $e \dashv f = e \vdash f$.*

Proof. Put $u := e \vdash f$. Instance (1) at (e, e, f) reads $(e \dashv e) \vdash f = e \dashv (e \vdash f)$, i.e. $u = e \dashv u$. Instance (2) at (e, f, f) reads $(e \vdash f) \dashv f = e \vdash (f \dashv f)$, i.e. $u \dashv f = u$. Hence, using rectangularity of $\dashv$ in the first step, $e \dashv f = (e \dashv u) \dashv f = u \dashv f = u = e \vdash f$. $\square$

Note the hypotheses of Lemma 3.3: neither associativity nor rectangularity of $\vdash$ is used. This is not economy for its own sake — it is what makes the negative control of Proposition 3.7 (a) the right one, since it shows exactly which hypothesis the conclusion of Theorem 3.4 needs.

Theorem 3.4 (The band is intrinsic). *Let $(D, \dashv, \vdash)$ be a rectangular doppelsemigroup. Then*

- (i) *for every $z \in D$, $z \dashv z = z \iff z \vdash z = z$; the two operations have the same idempotents, and we write E for that set;*
- (ii) *$\dashv$ and $\vdash$ agree on E : if $e, f \in E$ then $e \dashv f = e \vdash f$;*
- (iii) *the two operations have the same image: every $\dashv$ -product is a $\vdash$ -product and conversely, and that common image is E .*

Proof. (i) If $z \vdash z = z$ then Lemma 3.2 at $u = v = z$ gives $z \dashv z = z$; the mirror half of Lemma 3.2 gives the converse. (ii) is Lemma 3.3 together with (i). (iii) Lemma 3.2 writes $u \vdash v$ as $(u \vdash v) \dashv (u \vdash v)$ and its mirror writes $u \dashv v$ as $(u \dashv v) \vdash (u \dashv v)$; that the common image is E is Lemma 3.1(iii) for $\dashv$. $\square$

Theorem 3.5 (Structure theorem). *Let $\dashv$ and $\vdash$ be binary operations on a set D . Then $(D, \dashv, \vdash)$ is a rectangular doppelsemigroup if and only if there exist an associative rectangular operation m on D and maps $f, g : D \rightarrow D$, each fixing every m -product, such that*

$$x \dashv y = m(f(x), f(y)) \quad \text{and} \quad x \vdash y = m(g(x), g(y)) \quad \text{for all } x, y \in D;$$

that is, both operations are inflations of one and the same associative rectangular operation. No relation between f and g is required, and f and g need be neither idempotent nor surjective.

Proof. ($\Rightarrow$) Take $m := \dashv$, $f := \varphi$ and $g := \psi$. Then m is associative and rectangular by hypothesis; f fixes every $\dashv$ -product by Lemma 3.1(ii) and g fixes every $\vdash$ -product by the mirror half of Lemma 3.2; the identity $x \dashv y = m(f(x), f(y))$ is Lemma 3.1(i); and for the second identity, $\psi(x), \psi(y) \in E$ by Lemma 3.2, so

$$m(g(x), g(y)) = \psi(x) \dashv \psi(y) = \psi(x) \vdash \psi(y) = x \vdash y$$

by Theorem 3.4(ii) and Lemma 3.1(i) applied to $\vdash$.

($\Leftarrow$) Suppose $x \dashv y = m(f(x), f(y))$ and $x \vdash y = m(g(x), g(y))$ with m associative and rectangular and f, g fixing every m -product. Associativity and rectangularity of m give $m(a, m(b, c)) = m(m(a, b), c) = m(a, c)$, so every bracketing of a triple collapses to the outer pair. Now each of the six required laws is two rewrites. For instance

$$(x \dashv y) \vdash z = m(g(m(f(x), f(y))), g(z)) = m(m(f(x), f(y)), g(z)) = m(f(x), g(z)),$$

using that g fixes m -products and then rectangularity, while

$$x \dashv (y \vdash z) = m(f(x), f(m(g(y), g(z)))) = m(f(x), m(g(y), g(z))) = m(f(x), g(z)),$$

which is (1); (2) is the same computation with f and g interchanged. Replacing g by f throughout the two displays gives $(x \dashv y) \dashv z = m(f(x), f(z)) = x \dashv z$ and $x \dashv (y \dashv z) = m(f(x), f(z))$, that is, rectangularity and associativity of $\dashv$; replacing f by g gives the same for $\vdash$. $\square$

Corollary 3.6 (Both retractions, explicitly). *In a rectangular doppelsemigroup, for all x, y ,*

$$x \dashv y = \varphi(x) \dashv \varphi(y), \quad x \vdash y = \psi(x) \dashv \psi(y),$$

where $\varphi(x) = x \dashv x$ and $\psi(x) = x \vdash x$ are retractions of D onto the common band E . Thus the pair is determined by the single band $(E, \dashv)$ together with the two retractions; and conversely, given a rectangular band $B \subseteq D$ and any two retractions $\rho, \rho' : D \rightarrow B$, the operations $x \dashv y := \rho(x)\rho(y)$ and $x \vdash y := \rho'(x)\rho'(y)$ form a rectangular doppelsemigroup, by Theorem 3.5 applied to $m(x, y) := \rho(x)\rho(y)$.

Proof. The two identities are Lemma 3.1(i) and the last display in the proof of Theorem 3.5; that φ and ψ land in E and fix it pointwise is Lemma 3.1(ii) and Lemma 3.2 with Theorem 3.4(i). $\square$

Proposition 3.7 (Both hypotheses are load-bearing). (a) “Both operations rectangular” cannot be weakened to “ $\dashv$ rectangular”. On $\{0, 1\}$ put $x \dashv y = 0$ and $x \vdash y = \min(x, y)$. This is a doppelsemigroup, $\dashv$ is rectangular, $\vdash$ is not, and the two idempotent sets differ: $1 \vdash 1 = 1$ while $1 \dashv 1 \neq 1$. So Theorem 3.4(i) fails without the second rectangularity hypothesis and is not a consequence of the doppelsemigroup axioms alone.

(b) “One and the same m ” in Theorem 3.5 is not decoration. On $\{0, 1\}$ the left-zero operation $x \dashv y = x$ and the right-zero operation $x \vdash y = y$ are both rectangular bands, yet the pair is not a doppelsemigroup: two inflations of two different bands need not be interassociative.

Proof. Each assertion is the evaluation of finitely many instances of the axioms on a two-element carrier, that is, of four-entry tables; both are checked by kernel evaluation. $\square$

4. CONSEQUENCES

Interassociates. A doppelsemigroup is a pair of mutually interassociative semigroups, so Theorem 3.5 answers a question in the vocabulary of [9]: which rectangular semigroups are interassociative with a given one?

Theorem 4.1. *Let $\dashv$ be an associative rectangular operation on D and let $\vdash$ be any operation on D . Then $(D, \dashv, \vdash)$ is a rectangular doppelsemigroup if and only if there is a map $g : D \rightarrow D$ fixing every $\dashv$ -product with $x \vdash y = g(x) \dashv g(y)$ for all x, y . The rectangular interassociates of a rectangular semigroup are therefore exactly the inflations of that semigroup along the maps fixing its products, one for each such map.*

Proof. Necessity: take $g = \psi$; that ψ fixes every $\dashv$ -product is the mirror half of Lemma 3.2, and $x \vdash y = \psi(x) \dashv \psi(y)$ is Corollary 3.6. Sufficiency is Theorem 3.5 with $m = \dashv$ and $f(x) = x \dashv x$. $\square$

Corollary 4.2. *If $(D, \dashv)$ is a rectangular band — rectangular and idempotent — then its only rectangular interassociate is itself.*

Proof. By Theorem 3.4(i) every element is $\vdash$ -idempotent, so Corollary 3.6 gives $x \vdash y = \psi(x) \dashv \psi(y) = x \dashv y$. $\square$

Corollary 4.2 is not new: it is the rectangular-partner case of a statement [2] records, citing [9, Lemma 5.5] — “a semigroup coincides with each of its interassociates if and only if it is a rectangular band” — and applies correctly to the three-element left-zero and right-zero bands. (The volume [9] appeared in is not held here; the wording and the lemma number are quoted from [2], and nothing below rests on them.) It is recorded here because it shows where that paper’s enumeration lost the two structures: no tool was available for the interassociates of a rectangular semigroup that is *not* a band, and the inflation lemma that [2] does use — ascribed there to [9] — is for inflations of inverse

Clifford semigroups, a class meeting the rectangular bands only in the trivial one. Theorem 4.1 is exactly the missing case.

Two dichotomies. The published dichotomy comes out of the structure theorem without the Nagy–Nagy representation.

Theorem 4.3 ([1, Proposition 8]). *A rectangular doppelsemigroup is strong if and only if it is trivial.*

Proof. Trivial pairs are strong because the strong law is then an instance of associativity. Conversely, Corollary 3.6 and Lemma 3.1(i) reduce $x \dashv (y \vdash z)$ to $\varphi(x) \dashv \psi(z)$ and $x \vdash (y \dashv z)$ to $\psi(x) \dashv \varphi(z)$, so the strong law at (x, x, x) says that the two idempotents $\varphi(x)$ and $\psi(x)$ commute in $(E, \dashv)$. In a rectangular semigroup two commuting idempotents are equal: rectangularity gives $e = e \dashv e = (e \dashv f) \dashv e = (f \dashv e) \dashv e = f \dashv e$ and, in the same way, $f = f \dashv f = (f \dashv e) \dashv f = (e \dashv f) \dashv f = e \dashv f$, so $e = f \dashv e = e \dashv f = f$. Hence $\varphi = \psi$, and Corollary 3.6 gives $\dashv = \vdash$. $\square$

The second dichotomy has the same shape but different hypotheses, and neither statement follows from the other by a substitution. Call the pair

- (A) for all $y, y' \in D$: $(\forall u \in D, u \vdash y = u \vdash y') \implies (\forall u \in D, u \dashv y = u \dashv y')$,
- (B) for all $x, x' \in D$: $(\forall u \in D, x \dashv u = x' \dashv u) \implies (\forall u \in D, x \vdash u = x' \vdash u)$;

(A) says that $\vdash$ -right-equivalent elements are $\dashv$ -right-equivalent, and (B) is its opposite-handed mirror.

Theorem 4.4. *A rectangular doppelsemigroup satisfies (A) and (B) together if and only if it is trivial.*

Proof. A trivial pair satisfies both trivially. Conversely, two identities come for free from associativity and (6):

$$u \vdash \psi(w) = u \vdash (w \vdash w) = (u \vdash w) \vdash w = u \vdash w, \quad \varphi(w) \dashv u = (w \dashv w) \dashv u = w \dashv u.$$

The first says that w and $\psi(w)$ are $\vdash$ -right-equivalent, so (A) gives $x \dashv y = x \dashv \psi(y)$ for all x, y ; the second says that w and $\varphi(w)$ are $\dashv$ -left-equivalent, so (B) gives $x \vdash y = \varphi(x) \vdash y$ for all x, y . Now

$$x \dashv y = x \dashv \psi(y) = \varphi(x) \dashv \varphi(\psi(y)) = \varphi(x) \dashv \psi(y) = \varphi(x) \vdash \psi(y) = \varphi(x) \vdash y = x \vdash y,$$

using in turn (A); Corollary 3.6; that $\psi(y) \in E$, so φ fixes it; Theorem 3.4(ii), both arguments now lying in E ; the first free identity again; and (B). $\square$

Proposition 4.5 (Neither half suffices). *For every $n \geq 3$ there is a non-trivial rectangular doppelsemigroup of order n satisfying (A) but not (B), and one satisfying (B) but not (A).*

Proof. The witnesses are W_n and its dual W_n^{op} , defined before Theorem 4.6 below. In W_n both operations ignore their second argument, so the conclusion of (A) holds outright: $u \dashv y = \varphi_n(u)$ does not depend on y . And (B) fails at $x = 0, x' = 2$, whose hypothesis holds because $\varphi_n(0) = \varphi_n(2) = 0$ while its conclusion fails because $\psi_n(0) = 0 \neq 1 = \psi_n(2)$. For W_n^{op} the two sides are interchanged, and the witnessing pair is 1, 2. All four statements are proved uniformly in n , with no case analysis on the carrier. $\square$

The threshold, and where the erratum comes from. Let $n \geq 3$ and define two maps $\{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ by

$$\varphi_n(x) = \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise,} \end{cases} \quad \psi_n(x) = \begin{cases} 0, & x = 0 \\ 1, & \text{otherwise,} \end{cases}$$

two retractions onto the two-element left-zero band $\{0, 1\}$ that differ at the point 2, and let

$$W_n := (\{0, \dots, n-1\}, x \dashv y = \varphi_n(x), x \vdash y = \psi_n(x)),$$

the pair of inflations of that band along them; W_n^{op} is the same with both operations reading their second argument instead.

Theorem 4.6 (The threshold is exactly three). (i) *A rectangular doppelsemigroup with $\dashv \neq \vdash$ has at least three elements; every rectangular doppelsemigroup on two elements is trivial.*
(ii) *For every $n \geq 3$, W_n and W_n^{op} are non-trivial rectangular doppelsemigroups of order n .*
(iii) *W_3 and W_3^{op} are, on the nose, the two structures [1, §6] exhibits as the ones [2] overlooked.*

Proof. (i) Suppose $\dashv \neq \vdash$. If every element were $\dashv$ -idempotent then Theorem 3.4(ii) would give $x \dashv y = x \vdash y$ for all x, y ; so some x has $\varphi(x) \neq x$, and $x \notin E$. If E had a single element then φ and ψ would both be constant with that value, and Corollary 3.6 would again force $\dashv = \vdash$; so E contains two distinct elements e, f . Then x, e, f are three distinct elements of D . The second assertion is the contrapositive at $n = 2$. (ii) The four identities $\varphi_n \varphi_n = \varphi_n$, $\psi_n \psi_n = \psi_n$, $\psi_n \varphi_n = \varphi_n$ and $\varphi_n \psi_n = \psi_n$ hold because both maps take values in $\{0, 1\}$ and both fix 0 and 1; the first two give associativity and (6) for $\dashv$ and for $\vdash$, the third gives (1) and the fourth (2), each side of each law being $\varphi_n(x)$ or $\psi_n(x)$. Equivalently, both operations are inflations of the associative rectangular operation $m(x, y) = \varphi_n(x)$, whose products are exactly $\{0, 1\}$, along φ_n and ψ_n , so Theorem 3.5 applies. Non-triviality is the value at $x = 2$, where $\varphi_n(2) = 0 \neq 1 = \psi_n(2)$. The proof is uniform in n and involves no search. (iii) The two identifications are equalities of pairs of 3×3 tables and are checked by kernel evaluation. $\square$

So the published correction sits at order 3 because that is the smallest order at which the class it concerns is non-empty: a non-trivial member needs at least two idempotents to make the band and at least one non-idempotent to make the inflation proper. What [1] exhibits at order 3 is the first member of a family that exists at every order, and Theorem 3.5 says what the whole family is.

5. COUNTING: THE FIRST VALUES OF rds

The shape of the count. Corollary 3.6 suggests the following description of the class, which we use *only to generate candidates*; it is not proved here, and no statement of this section depends on it. A rectangular doppelsemigroup of order n is a rectangular band E — up to isomorphism a $p \times q$ rectangular band for a pair of integers $p, q \geq 1$ with $pq \leq n$ — together with the pair $(\varphi(x), \psi(x)) \in E \times E$ attached to each of the $n - pq$ elements outside E ; isomorphisms act on that data through the isomorphisms of the band, that is, through $S_p \times S_q$ acting diagonally, the outside elements being permuted freely. So the isomorphism classes should be indexed by a pair (p, q) together with a multiset of size $n - pq$ of elements of $([p] \times [q])^2$ modulo $S_p \times S_q$. This is what makes the class *constructible* rather than merely testable: instead of filtering the n^{2n^2} pairs of tables one builds one algebra per datum. At $n = 7$ that is 28 855 algebras, at $n = 8$ it is 359 435, and at $n = 9$ it is 5 214 314, against 7^{98} pairs of tables at $n = 7$ alone.

Remark 5.1 (Outside the verified development). Burnside’s lemma applied to the description above gives a closed form, $\text{rds}(n) = \sum_{pq \leq n} [t^{n-pq}] Z_{p,q}(t)$ with $Z_{p,q}$ the cycle index of $S_p \times S_q$ acting diagonally on $([p] \times [q])^2$. Evaluated in exact rational arithmetic it returns, for $n = 1, \dots, 12$,

$$\text{rds}(n) = 1, 3, 7, 20, 51, 171, 571, 2193, 8594, 38249, 189931, 1031692,$$

$$\text{rs}(n) = 1, 3, 5, 10, 14, 27, 36, 64, 90, 150, 213, 358.$$

Both agree with the published table at every order it reaches. *This computation is on paper and in an ordinary program; it is not part of the formal development and nothing below depends on it.* It is recorded because it says what the answers to Problem 1.1 should be, and because the three values proved below as lower bounds are exactly its values — so the gap between what is proved and what is expected is precisely the completeness half.

Certificates, and what is checked. Fix n . A *certificate* is a finite list of pairs of $n \times n$ tables, written as a string of $2n^2$ digits per member. Four properties of a certificate are decided by direct computation:

- *range*: every digit is $< n$ and every block has length $2n^2$;
- *soundness*: each block satisfies all six laws — associativity and (6) for each operation and (1), (2) — at every one of the n^3 triples;

- *canonicity*: for each block and each of the $n!$ relabellings of $\{0, \dots, n-1\}$, the block's digit string is lexicographically $\leq$ the relabelled string;
- *distinctness*: the blocks are pairwise different strings.

Canonicity plus distinctness is what turns a list into a lower bound: if two blocks were isomorphic then each would be $\leq$ the other in the lexicographic order, hence equal, hence not distinct. For this argument the relabelling scan must be over the *whole* symmetric group, so the permutation enumerator used by the checker is itself proved complete, and the string the checker compares is proved to be the digit string of the relabelled structure and not of something else.

Theorem 5.2 (Lower bounds at the first three open orders). *There are at least 571 pairwise non-isomorphic rectangular doppelsemigroups of order 7, at least 2193 of order 8, and at least 8594 of order 9:*

$$\text{rds}(7) \geq 571, \quad \text{rds}(8) \geq 2193, \quad \text{rds}(9) \geq 8594.$$

Proof sketch. For each $n \in \{7, 8, 9\}$ a certificate is produced outside the proof by building one algebra per datum of the description above, canonicalising each under S_n , and discarding duplicates; this is where Theorem 3.5 is used, and it is used only to *choose* what to certify. The certificate is then checked exhaustively for the four properties listed above. For $n = 7$ that is 571 blocks, each of the six laws tested at all 343 triples of every block, and $571 \cdot 5040$ relabelling comparisons; for $n = 8$, 2193 blocks and $2193 \cdot 40320$ comparisons; for $n = 9$, 8594 blocks and $8594 \cdot 362880$ comparisons. Each check succeeded, and the block counts were verified to be 571, 2193 and 8594 exactly. Soundness gives that every block is a rectangular doppelsemigroup; canonicity and distinctness give that no two blocks are isomorphic, by the argument above. The generation step is not trusted: a wrongly generated block simply fails soundness, and a duplicate fails distinctness. What the bounds *do* rest on is the exhaustiveness of the relabelling scan and of the axiom scan, both of which are checked, and on compiled evaluation of those scans — see Section 9. $\square$

Proposition 5.3 (The machinery against every value it can be checked against). (i) *The identical procedure at the four orders $n \geq 3$ at which the answer is published returns 7, 20, 51 and 171 pairwise non-isomorphic rectangular doppelsemigroups at $n = 3, 4, 5, 6$, matching [1]'s table.*

(ii) *Its trivial part — the blocks with $\dashv = \vdash$, that is, the rectangular semigroups — has 5, 10, 14, 27 members at $n = 3, 4, 5, 6$ and 36, 64, 90 members at $n = 7, 8, 9$; so $\text{rs}(7) \geq 36$, $\text{rs}(8) \geq 64$ and $\text{rs}(9) \geq 90$. OEIS A390162 [20] publishes those seven numbers as the exact counts of rectangular semigroups, so the trivial part of each certificate is complete at every order, including the three at which the total is unpublished.*

(iii) *At order 3 the certificate route agrees with an independent route: filtering the complete transversal of all 77 order-3 doppelsemigroups of Section 7, which never uses Theorem 3.5, gives the same number 7.*

(iv) *Each of the four certificate checks can fail, and is exhibited failing on an explicit two-element algebra: a rectangular doppelsemigroup that is not in canonical form; an algebra satisfying (1), (2) and associativity but not (6); a list containing the left-zero band twice; and a block with a digit out of range. The count at order 7 is moreover refuted at 570 and at 572.*

Proof sketch. (i)–(ii) are the same four exhaustive checks run on the certificates for $n = 3, \dots, 9$, together with a count of the blocks whose two tables are equal. (iii) compares the length of the order-3 certificate with the length of the rectangular part of the order-3 transversal of Theorem 7.1, which is obtained from a proved-complete enumeration of the associative tables of order 3; the two computations share no code. (iv) is four evaluations on two-element data. $\square$

Item (ii) is the strongest external control available for Theorem 5.2. The trivial blocks are not a corner of the construction: they are the data with $\varphi = \psi$, spread across every pair (p, q) with $pq \leq n$, so a construction that dropped a class would be liable to drop one there. That the trivial column

is exactly right at three orders where the total is unpublished is evidence — not a proof — that the total is right as well.

6. ABELIAN DIMONOIDS ARE ABELIAN g -DIMONOIDS

This section is independent of Sections 3–5: it concerns Loday’s axioms rather than interassociativity, and its theorems use no computation of any kind — only the controls at the end are evaluations, on two-element tables. The freeness phenomenon is the same one. There, two interassociativity laws turned out to be free on top of a shared band; here, the axiom (4) turns out to be free on top of (3) and (5), as soon as the structure is abelian.

Under the abelian law there is only one operation. Let $(D, \dashv, \vdash)$ be abelian, so that $x \dashv y = y \vdash x$ for all x, y , equivalently $x \vdash y = y \dashv x$: the second operation is the opposite of the first and carries no information of its own. Write $xy := x \dashv y$ for the remaining operation. Since the opposite of an associative operation is associative, $\vdash$ is associative as soon as $\dashv$ is, and the associativity axiom on $\vdash$ may be dropped throughout.

Lemma 6.1 (The three axioms are one). *Let $\cdot$ be a binary operation on a set D and let $x \vdash y := y \cdot x$ be its opposite. Then, for the pair $(\cdot, \vdash)$:*

- (i) (4) holds if and only if $\cdot$ is right commutative, and neither direction uses associativity;
- (ii) if $\cdot$ is associative, then (3) holds if and only if $\cdot$ is right commutative;
- (iii) if $\cdot$ is associative, then (5) holds if and only if $\cdot$ is right commutative.

In particular, for an associative $\cdot$ any one of the three axioms implies the other two.

Proof. Substituting $x \vdash y = y \cdot x$ into the three axioms turns them into the three identities

$$(3): (xy)z = x(zy), \quad (4): (yx)z = (yz)x, \quad (5): z(xy) = (zy)x.$$

The middle one is (8) after renaming the bound variables, which is (i). For (ii), $(xy)z = x(zy) = (xz)y$ by (3) and then associativity backwards, and conversely $(xy)z = (xz)y = x(zy)$ by right commutativity and then associativity. For (iii), $(xy)z = x(yz) = (xz)y$ by associativity and then (5) at (y, z, x) , and conversely $z(xy) = (zx)y = (zy)x$ by associativity and then right commutativity. Each of the six implications is a single rewrite. $\square$

Theorem 6.2 (Zhuchok’s criterion). *An abelian structure $(D, \dashv, \vdash)$ is a dimonoid if and only if $(D, \dashv)$ is a right commutative semigroup.*

Proof. If it is a dimonoid then $\dashv$ is associative by definition and right commutative by Lemma 6.1(i), which needs (4) only. Conversely, if $\dashv$ is an associative right commutative operation then $\vdash$ is associative because it is its opposite, and Lemma 6.1 returns all three axioms. $\square$

Theorem 6.2 is **not new** and is not claimed. It is A. V. Zhuchok’s Lemma 3 of [5, p. 300], which reads there, verbatim: “Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ be a dual semigroup to $(D, \dashv)$. Then an algebra $(D, \dashv, \vdash)$ is a dimonoid if and only if $(D, \dashv)$ is a right commutative semigroup.” “Dual” is defined immediately above it, by $x \vdash y = y \dashv x$, which is the abelian law. The lemma is restated in that form in [6, §2] and, with “dimonoid” replaced by “abelian dimonoid”, in [7, §4.2]; it is reproved here only so that the g -dimonoid version can be compared with it line by line. Zhuchok’s own proof of the forward direction goes through (3) rather than (4), and his paper — which predates the introduction of the g -dimonoids [8] by a year — does not mention them.

Theorem 6.3 (The free theorem). *An abelian structure $(D, \dashv, \vdash)$ is a g -dimonoid if and only if $(D, \dashv)$ is a right commutative semigroup. Consequently:*

- (i) an abelian structure is a dimonoid if and only if it is a g -dimonoid — on abelian structures Loday’s (4) is a consequence of (3) and (5);
- (ii) for every n the abelian dimonoids of order n and the abelian g -dimonoids of order n are the same algebras, and therefore

$$\text{adm}(n) = \text{agdm}(n) \quad \text{for every } n.$$

Proof. Forward: $\dashv$ is associative by definition, and (5) gives right commutativity by Lemma 6.1(iii) — note that (4), which is not available, is not needed. Backward: as in Theorem 6.2, Lemma 6.1 returns (3) and (5) (and (4) as well, which the g -dimonoid axioms do not ask for). Part (i) is this equivalence combined with Theorem 6.2. For (ii), the two defining conditions on a structure of order n — “abelian dimonoid” and “abelian g -dimonoid” — are by (i) the same condition, so the two sets of isomorphism classes they meet are one set and their cardinalities agree. No finiteness argument and no computation enters at any point. $\square$

The economy of the argument is worth recording: (4) is right commutativity *verbatim* — two bound variables change places and nothing else happens — while (3) and (5) each need exactly one application of associativity in each direction. That is the whole proof; there is no case analysis and no appeal to the finiteness of D .

Corollary 6.4 (The one-operation reduction). *For every n ,*

$$\text{adm}(n) = \text{agdm}(n) = \text{rcs}(n),$$

the number of right commutative semigroups of order n up to isomorphism.

Proof. The map $T \mapsto (T, T^{\text{op}})$ sending a single table to the abelian structure it generates is injective, commutes with relabelling, and has as its image exactly the abelian structures of that order. Two abelian structures are isomorphic if and only if their first operations are isomorphic, so the map descends to an injection of isomorphism classes; by Theorems 6.2 and 6.3 the image of the classes of right commutative semigroups is exactly the set of classes of abelian dimonoids, equivalently of abelian g -dimonoids. Counting the two sides gives the two equalities. $\square$

For dimonoids, Corollary 6.4 is **known** in prose: [6, §2] draws from Zhuchok’s lemma the sentence “it follows that the task of describing abelian dimonoids reduces to the task of describing right commutative semigroups”, and the isomorphism half — that abelian structures are isomorphic exactly when their $\dashv$ -semigroups are — is [4]’s proposition on abelian g -dimonoids. What is new is the g -dimonoid column of the statement, which is Theorem 6.3’s.

Remark 6.5 (The abelian column read as a one-operation sequence). Corollary 6.4 turns the two abelian rows of (7) into a statement about a single operation:

$$\text{rcs}(n) = 1, 4, 17, 103, 791, 10870 \quad (n = 1, \dots, 6).$$

The entry at $n = 1$ is trivial and those at $n = 2, 3, 4$ are verified here, through Theorem 7.1 and Theorem 7.3; those at $n = 5, 6$ combine Corollary 6.4 with the published values of adm and are no better established than the tables they come from. We have not found this sequence in the literature as a count of right commutative semigroups, and OEIS returns no entry for it. Independently of the formal development, an ordinary program that enumerates associative right commutative tables and counts them up to isomorphism returns the same six integers, the last of them (10870 classes from 6683929 labelled tables) in about 31 minutes; that computation is outside the verification and nothing above rests on it.

The hypotheses are load-bearing.

Proposition 6.6 (g -dimonoids are a strictly larger variety). *Every dimonoid is a g -dimonoid, and the inclusion is strict already at order 2: on $\{0, 1\}$ the pair $x \dashv y = 0, x \vdash y = 1$ satisfies (3) and (5) and both associativity laws, and fails (4) at $(0, 0, 0)$. The class of g -dimonoids is invariant under relabelling of the carrier, so filtering a transversal by it counts isomorphism classes.*

Proof. The inclusion is the deletion of an axiom. The two-element assertion is the evaluation of the axioms on four-entry tables. Invariance is the observation that each axiom is an identity, and identities are preserved by isomorphism. $\square$

- Proposition 6.7** (Controls for the free theorem). (a) *The abelian hypothesis is load-bearing: the order-2 structure of Proposition 6.6 is a g -dimonoid that is not a dimonoid, and it is not abelian, so it is consistent with Theorem 6.3 and shows that the theorem fails without its hypothesis. It is moreover the only such structure up to isomorphism, since $\text{dm}(2) = 8$ and $\text{gdm}(2) = 9$ (Theorems 7.1 and 7.2) and every dimonoid is a g -dimonoid: the published gap $\text{gdm}(2) - \text{dm}(2) = 1$ is this one algebra.*
- (b) *Right commutativity is a proper restriction inside associativity, in both directions: on $\{0, 1\}$ the left-zero operation $xy = x$ is associative and right commutative, while the right-zero operation $xy = y$ is associative and not right commutative.*
- (c) *The class is not empty and not trivial: Loday’s $x \dashv y = x$, $x \vdash y = y$ is an abelian dimonoid — hence an abelian g -dimonoid — on every carrier, and its two operations differ as soon as the carrier has two elements.*

Proof. (b) and the two-element parts of (a) and (c) are evaluations on four-entry tables; the general statement in (c) is five instances of $x = x$. The uniqueness assertion in (a) is the arithmetic $9 - 8 = 1$ together with the inclusion of Proposition 6.6 and the invariance of both classes under relabelling. $\square$

7. THE PUBLISHED MATERIAL THIS RESTS ON

The results above use, at order 3, an independent classification of all doppelsemigroups of that order; it is recorded here because it is what makes Proposition 5.3(iii) a cross-check and because the erratum of [1] is where this family of questions starts. Nothing in this section is claimed as new: the counts are published, and the development re-derives them from the axioms rather than transcribing a catalogue.

Two neighbouring classes of two-operation algebra are classified alongside, because they sit on the same enumeration and because their published counts are a further check on it. Dimonoids and g -dimonoids were defined in Section 1. An *additively idempotent semiring* is a set with a commutative idempotent semigroup operation $+$, a semigroup operation $\cdot$, and both distributive laws, with no identity, zero or absorption required. Doppelsemigroups and dimonoids share only the second of the dimonoid laws, (4), which is (2), and they are incomparable varieties (Proposition 7.5(iii)).

Theorem 7.1 ([1], [3], [13], [15]; verified). *Up to isomorphism there are exactly 8 doppelsemigroups, 8 dimonoids and 6 additively idempotent semirings of order 2, and exactly 77 doppelsemigroups, 52 dimonoids and 61 additively idempotent semirings of order 3, together with the sources’ subsidiary counts: of the 77, exactly 41 are commutative, 12 abelian, 65 strong, 7 rectangular and 24 trivial; of the 8 dimonoids of order 2, exactly 3 are commutative, 4 abelian, 6 rectangular and 5 trivial; and of the 52 of order 3, exactly 14, 17, 18 and 24.*

Proof sketch. Each count is four statements, not one: a list is produced by a row-by-row enumeration of the associative tables whose soundness and completeness are both proved, so that the list is *exactly* the set of associative tables of that order; pairs from it are filtered by the axioms and by a canonical-form predicate; and the three transversal theorems — everything on the list satisfies the axioms, every algebra is isomorphic to something on the list, no two members are isomorphic — are proved from the injectivity of the encoding used for the canonical form. Only the resulting lengths are computed; the structural halves involve no computation. The subsidiary counts are counts of isomorphism classes because each named property is invariant under relabelling and filtering a transversal by an invariant property is again a transversal. $\square$

The order-3 doppelsemigroup counts and their subsidiary columns are [1], and so are the order-2 doppelsemigroup counts; the dimonoid counts are [3]; the count 61 of additively idempotent semirings of order 3 is [14], quoted in [13, §1]; and the count 6 at order 2 is [15, §1] — “up to isomorphism, there are exactly 6 ai-semirings of order 2” — which attributes it to [16]. The corresponding order-4 *totals* — 1217 doppelsemigroups [1], 734 dimonoids [3] and 866 additively idempotent semirings [15] — are published, and they are *not* part of the development described here; nothing in this paper asserts them. What is reached at that order is the six sub-columns of Theorem 7.3.

Theorem 7.2 ([4]; verified). *Up to isomorphism there are exactly 9 g -dimonoids of order 2 and 78 of order 3, with the source's subsidiary columns:*

n	2	3
$\text{gdm}(n)$	9	78
$\text{cgdm}(n)$	4	22
$\text{agdm}(n)$	4	17
$\text{rgdm}(n)$	7	27

Moreover the trivial members — those with $\dashv = \vdash$, which are exactly the semigroups — number 5 and 24, and the trivial rectangular ones 3 and 5.

Proof sketch. The same four statements as in Theorem 7.1, for one more axiom set: the pairs of order-2 (resp. order-3) associative tables whose first table is canonical are filtered by (3), (5) and the canonical-form predicate, and soundness, completeness, pairwise non-isomorphism and distinctness are proved for the resulting list; only the lengths and the lengths of the filtered sublists are computed. Sub-class completeness is proved separately for each named property, so the subsidiary numbers are class counts and not merely list lengths. The last sentence is a control the computation knows nothing about: a trivial g -dimonoid is a semigroup, so those two entries must be the numbers 5 and 24 of semigroups of orders 2 and 3 (OEIS A027851 [20], and the source's own table), and the trivial rectangular ones must be $\text{rs}(2) = 3$ and $\text{rs}(3) = 5$ (OEIS A390162 [20]). Both come out right. Each headline is also refuted one below and one above. $\square$

The abelian entries 4 and 17 of Theorem 7.2 are the abelian entries 4 and 17 of Theorem 7.1, while the totals 9 and 78 differ from the dimonoid totals 8 and 52: the agreement of the abelian columns is not an artefact of the two varieties coinciding, since they do not. This is Theorem 6.3 seen at two orders.

Theorem 7.3 ([3], [4]; verified). *At order 4,*

$$\text{adm}(4) = \text{agdm}(4) = 103, \quad \text{cdm}(4) = 101, \quad \text{cgdm}(4) = 189, \quad \text{rdm}(4) = 62, \quad \text{rgdm}(4) = 128.$$

Proof sketch. All six values are published; what is added is a derivation from the axioms on the proved-complete order-4 enumeration. Two devices make the computation affordable, and both are consequences of results above. First, by Corollary 6.4 an abelian structure is (T, T^{op}) for a single table T , so the abelian column is a scan of *one* table at a time: the pool is the 188 canonical associative tables of order 4 (out of 3492 labelled ones), not the $188 \times 3492 = 6.6 \times 10^5$ pairs the full order-4 count would need; and by Theorem 6.3 the resulting list is a transversal for the abelian dimonoids *and* for the abelian g -dimonoids at once — one computation, two published values, with $\text{adm}(4) = \text{agdm}(4)$ proved rather than observed twice. Second, both operations of a commutative structure are commutative and both operations of a rectangular one are rectangular, so those two columns are pair scans over pre-filtered pools of 1140 tables (58 canonical) and 84 tables (10 canonical) respectively; completeness against a pool that does not contain every associative table is a separate lemma, proved here. Soundness, completeness and pairwise non-isomorphism are proved for each of the five lists; only the lengths are computed, and all of them in a single evaluation, together with the two anchors 3492 (associative binary operations on a labelled 4-set, OEIS A023814 [20]) and 188 (semigroups of order 4 up to isomorphism, OEIS A027851 [20]). The values 103, 101 and 128 are each refuted one below and one above. $\square$

Proposition 7.4 (The erratum, re-derived). *Every non-trivial rectangular doppelsemigroup of order 3 is isomorphic to one of the two structures of [1, §6]; those two are non-isomorphic rectangular non-trivial doppelsemigroups; and of the 7 rectangular order-3 doppelsemigroups exactly these 2 are non-trivial. Hence $77 = 75 + 2$ with the two missing classes identified, and the published value 75 is refuted.*

Proof sketch. The two Cayley tables of [1, §6] are transcribed and all their asserted properties — being doppelsemigroups, rectangular, non-trivial, non-isomorphic to each other, and one being isomorphic to

the opposite of the other by the transposition the source gives — are checked directly. Exhaustiveness, which the source does not claim, follows by filtering the order-3 transversal of Theorem 7.1 to the non-trivial rectangular members and checking that every one of them is isomorphic to one of the two. Theorem 4.6 explains the same fact conceptually and at every order. $\square$

- Proposition 7.5** (Controls on the classification). (i) *Each count above is refuted on both sides; in particular the published value 75 is refuted.*
- (ii) *Neither interassociativity law follows from the other together with associativity: with $\dashv$ the left-zero and $\vdash$ the right-zero operation on three elements, (2) holds and (1) fails, and interchanging the two operations interchanges the two laws, so one witness settles both directions.*
 - (iii) *Doppelsemigroups and dimonoids are incomparable varieties, with witnesses in both directions at order 2; this is why there are more doppelsemigroups than dimonoids at order 3 although the dimonoid axiom list is longer.*
 - (iv) *The axioms are neither vacuous nor universal on the enumerated pool.*
 - (v) *The counting convention matters: exactly 41 of the 77 order-3 doppelsemigroups are isomorphic to their own dual — precisely the commutative ones — so identifying duals would give $41 + (77 - 41)/2 = 59$ classes, not 77.*

Proof sketch. (i) and (v) are arithmetic on the lengths of Theorem 7.1 together with one further traversal of the order-3 transversal; (ii), (iii) and (iv) are evaluations on explicit two- and three-element tables. $\square$

8. WHAT WAS PRICED AND NOT ATTEMPTED

Several of the cells left open by Problems 1.1, 1.2 and 1.3 were attacked, measured and abandoned, and the measurements are more useful than the attempts. All timings below are single-core, on a shared 20-core machine.

The abelian column at order 7. Corollary 6.4 converts $\text{adm}(7) = \text{agdm}(7)$ into a one-operation problem, counting right commutative semigroups of order 7, and that looked cheap enough to attempt: an initial estimate put it at about one CPU-hour. That estimate is withdrawn. The enumeration used here — a depth-first search over the n^2 cells of a table with incremental pruning, followed by a lexicographic canonical form at the leaves — takes 0.02 s at $n = 4$, 1.97 s at $n = 5$ and 1862.5 s at $n = 6$, the labelled counts growing $1985 \rightarrow 76\,821 \rightarrow 6\,683\,929$. The step from $n = 5$ to $n = 6$ is a factor of 945; one more step of that shape puts $n = 7$ at roughly *twenty CPU-days* on one core — some five hundred times the estimate it replaces, and far past the budget adopted here for a single statement. So the reduction makes the abelian column a one-operation problem without making order 7 reachable, and the honest reading is that a different algorithm is needed rather than a larger budget: the cost is not the isomorphism rejection, which runs only at the leaves, but the search tree itself, whose partial tables survive far longer than they should. The two named remedies are to fill columns rather than cells and prune on the identity that right commutativity is the pairwise commuting of right translations (Definition 2.2), and to count by Burnside’s lemma rather than by canonical form, which removes the leaf test but not the dominant identity term.

The rectangular columns at order 7. $\text{rdm}(7)$ and $\text{rgdm}(7)$ look cheaper — the rectangular pool is tiny and the whole order-6 computation takes 62.2 s — and they are not: the run was stopped after 56 minutes still inside the single-operation enumeration. The fix is the one Section 5 already uses for rds: build the rectangular tables from the structure theorem instead of searching for them.

The totals. $\text{dm}(6)$ and $\text{gdm}(6)$ are out of reach by this method for anyone: the pool of associative tables of order 6 has 1706118 members and the pair scan 2.9×10^{14} pairs. At order 4 the full pair scan is 1.2×10^7 pairs, a second of ordinary computation; the verified route cuts that to the $188 \times 3492 = 6.6 \times 10^5$ pairs whose first table is canonical, and has still twice failed to finish (about 50 and 57 minutes, both stopped), which is why Theorem 7.3 states six sub-columns and not the order-4 totals. For rds, order 10 — 38 249 blocks against $10! = 3\,628\,800$ relabellings — is about 6.9 CPU-hours of the computation of Theorem 5.2, past the same budget.

9. VERIFICATION

Every lemma, proposition, theorem and corollary stated above has been formally verified in Lean 4 (toolchain v4.32.0) against *Mathlib* [21] at the matching tag; the repository is private, repository reference to be added at submission. The development contains no `sorry` and no hand-declared axiom; the only non-standard axioms occurring anywhere are those `native_decide` generates for the compiled evaluations named below. The structure theory of Sections 3 and 4 is proved on an arbitrary carrier with no computation at all: Lemmas 3.2 and 3.3, the backward half of Theorem 3.5, Theorem 3.4, Theorem 4.3 and Theorem 4.4 depend on no axioms whatever in their carrier-free form, and their finite-carrier corollaries add only `propext`, `Classical.choice` and `Quot.sound`. The same holds, more sharply, of Section 6: the six implications of Lemma 6.1 are proved for an arbitrary type carrying an arbitrary binary operation, each is a single rewrite, and Theorem 6.2, Theorem 6.3 and the equivalence in part (i) of the latter — which are recorded at each finite order, their proofs being those carrier-free lemmas — *depend on no axioms at all*; the counting equality $\text{adm}(n) = \text{agdm}(n)$ of Theorem 6.3(ii) and the two equalities of Corollary 6.4, which quantify over isomorphism classes and so speak about quotients, use `propext`, `Classical.choice` and `Quot.sound` and no evaluation of any kind. The finite content splits into two kinds. Kernel evaluations, which the proof checker performs itself: the two-element controls of Proposition 3.7, Proposition 6.6 and Proposition 6.7, the axiom-independence, variety-separation and non-vacuity controls of Proposition 7.5, and the identification of W_3 , W_3^{op} with the erratum’s two structures in Theorem 4.6(iii). And compiled evaluations, which are delegated to compiled code and recorded as such in the axiom list of the affected statements: the certificate checks behind Theorem 5.2 and Proposition 5.3, and the lengths in Theorem 7.1, Theorem 7.2, Theorem 7.3, Proposition 7.4 and Proposition 7.5(i),(v). For the certificates the delegation is confined to the four decision procedures; the bridge that turns each of their Boolean verdicts into the corresponding mathematical statement — including the completeness of the permutation enumerator and the identity between the string the checker relabels and the encoding of the relabelled algebra — uses no evaluation of any kind. The certificate checks are driven by a small *Mathlib*-free core compiled into a shared library, without which the order-9 scan is not affordable: with the library the checks for orders 3 through 8 take 23.7 seconds and those for order 9 take 467 seconds, while the same order-7 checks interpreted rather than compiled take 27.6 times longer, and the interpreted order-8 run was stopped after twenty minutes without finishing, which makes the corresponding figure a lower bound of about 70 times rather than a measurement. In Theorem 7.3 the structural half — soundness, completeness and pairwise non-isomorphism of all five lists, and the pool-relative completeness lemma — carries no compiled evaluation at all, and all seven numbers are stated in a *single* one, because each mention of the order-4 enumeration costs a fresh run of it (4 minutes 18 seconds interpreted, against 25 minutes 21 seconds for all seven numbers together). The searches that *found* the certificates, the independent implementations that reproduced every published count before any of this was formalised, the right commutative enumeration of Remark 6.5, the timings of Section 8 and the Pólya computation of Remark 5.1 lie outside the verification; they are how the data was obtained and no statement above rests on them.

REFERENCES

- [1] V. M. Gavrylkiv, *Note on the number of doppelsemigroups of small order*, Res. Math. **34** (2026), no. 1, 16–33; doi:10.15421/242602; arXiv:2606.28908. Section, table, proposition and problem numbers cited above are those of the journal version; the arXiv version (v1, the only one) numbers its statements by section, where Problem 1 is Problem 7.1 and Proposition 8 is Proposition 4.2. The two versions agree on the section numbers, on Table 7 and on every sentence quoted here.
- [2] V. M. Gavrylkiv and D. V. Rendziak, *Interassociativity and three-element doppelsemigroups*, Algebra Discrete Math. **28** (2019), no. 2, 224–247; arXiv:2004.02505.
- [3] V. M. Gavrylkiv, *On dimonoids of small order*, arXiv:2607.17921. Cited from the e-print, which stands at v1 and carries no journal reference; the forthcoming-journal DOI that [4] lists for it neither resolved nor was registered with Crossref when checked on 30 August 2026, so no journal reference is given here.

- [4] V. M. Gavrylkiv, *On the number of g -dimonoids of small order*, arXiv:2607.29641. Cited from the e-print, likewise at v1 and likewise without a journal reference; [3] lists it as forthcoming in Mat. Stud. **66** (2026), an issue that had not appeared when checked on 30 August 2026.
- [5] A. V. Zhuchok, *Free (ℓ, rr) -dibands*, Algebra Discrete Math. **15** (2013), no. 2, 295–304. Consulted in the journal’s open-access PDF; Lemma 3, quoted verbatim above, is on p. 300.
- [6] V. M. Gavrylkiv, *On some classes of noncommutative dimonoids*, Ukrainian Math. J. **77** (2025), no. 12, 1563–1577; doi:10.1007/s11253-025-02541-w; arXiv:2504.15727. The section cited above is the e-print’s, and is §2 in both v1 and v3.
- [7] V. M. Gavrylkiv, *Classifications of dimonoids with at most three elements*, J. Math. Sci. (N.Y.) **294** (2025), no. 5, 697–710; doi:10.1007/s10958-025-08126-z; arXiv:2510.24092. The subsection cited above is the e-print’s, whose only version is v1.
- [8] Yu. Movsisyan, S. Davidov and M. Safaryan, *Construction of free g -dimonoids*, Algebra Discrete Math. **18** (2014), no. 1, 138–148. Not consulted here; it is the reference [4] gives for the introduction of the notion.
- [9] S. J. Boyd, M. Gould and A. Nelson, *Interassociativity of semigroups*, in: Proceedings of the Tennessee Topology Conference (Nashville, TN, 1996), World Scientific, River Edge, NJ, 1997, 33–51. The volume was not consulted here; the page range is the one given independently by [2], by [1] and by unrelated citing papers, and Lemma 5.5 is used only as quoted through [2].
- [10] A. Nagy and O. Nagy, *A construction of semigroups whose elements are middle units*, Int. J. Algebra **14** (2020), no. 3, 163–169; doi:10.12988/ija.2020.91248. This is the reference [1] cites for the representation of rectangular semigroups; it was not consulted here and nothing above rests on it.
- [11] A. Nagy, *Special Classes of Semigroups*, Advances in Mathematics (Dordrecht) **1**, Kluwer Academic Publishers, Dordrecht, 2001, viii+269 pp.; ISBN 0-7923-6890-8. Cited for the standard terminology “right commutative”, which is the title of its chapter on pp. 137–173; that chapter itself was not consulted here.
- [12] J.-L. Loday, *Dialgebras*, in: Dialgebras and related operads, Lecture Notes in Math. **1763**, Springer, 2001, 7–66.
- [13] M. Yue and M. Ren, *A nonfinitely based additively idempotent semiring of order four*, arXiv:2605.15493.
- [14] X. Z. Zhao, M. M. Ren, S. Crvenković, Y. Shao and P. Dapić, *The variety generated by an ai-semiring of order three*, Ural Math. J. **6** (2020), no. 2, 117–132; doi:10.15826/umj.2020.2.012.
- [15] M. M. Ren, J. Y. Liu, L. L. Zeng and M. L. Chen, *The finite basis problem for additively idempotent semirings of order four, I*, Semigroup Forum **110** (2025), no. 2, 422–457; doi:10.1007/s00233-025-10520-7; arXiv:2407.15342 (the sentence quoted above is §1 of v3).
- [16] Y. Shao and M. M. Ren, *On the varieties generated by ai-semirings of order two*, Semigroup Forum **91** (2015), no. 1, 171–184; doi:10.1007/s00233-014-9667-z. Not consulted here; it is the source [15] gives for the count 6.
- [17] A. V. Zhuchok, *Free products of doppelsemigroups*, Algebra Univers. **77** (2017), no. 3, 361–374. Cited by [1] as the paper that initiated the study of doppelsemigroups; not consulted here.
- [18] A. V. Zhuchok, Yul. V. Zhuchok and J. Koppitz, *Free rectangular doppelsemigroups*, J. Algebra Appl. **19** (2020), no. 11, 2050205; doi:10.1142/S0219498820502059.
- [19] V. M. Gavrylkiv, *On rectangular ideal extensions of left zero semigroups by null semigroups*, Semigroup Forum **113** (2026), no. 1, 73–89; doi:10.1007/s00233-026-10632-8. Not consulted here; cited only as neighbouring one-operation work.
- [20] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, <https://oeis.org>, all three entries consulted 30 August 2026. Entry A390162, “Number of nonisomorphic semigroups with n elements satisfying the equation $xyz = xz$ ” (A. Howroyd, 14 November 2025), data 1, 1, 3, 5, 10, 14, 27, 36, 64, 90, 150, 213, 358, ... from $n = 0$; entry A027851, “Number of nonisomorphic semigroups of order n ” (C. G. Bower, 13 December 1997), data 1, 1, 5, 24, 188, 1915, ... from $n = 0$; entry A023814, “Number of associative binary operations on an n -set; number of labeled semigroups” (L. Ramshaw), data 1, 1, 8, 113, 3492, 183732, 17061118, ... from $n = 0$. The three titles are quoted, not paraphrased.
- [21] The mathlib Community, *The Lean mathematical library*, in: Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; doi:10.1145/3372885.3373824.