TIGHT SINGLE-CHANGE COVERING DESIGNS AT SMALL ORDERS: EXACT COUNTS AND UNCONDITIONAL NON-EXISTENCE

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Abstract. A *tight single-change covering design* $\mathrm{tsccd}(v,k)$ is an ordered sequence of k-subsets of $\{1,\dots,v\}$ in which each block after the first is obtained from its predecessor by removing one point and inserting another, every pair of points occurs together in some block, and the point inserted at each step has never before shared a block with any of its new block-mates. Preece asked at the Nineteenth British Combinatorial Conference for such a design with block size greater than five, and Bean's $\mathrm{tsccd}(26,6)$ answers that question. We settle several small orders exactly. There are exactly 32 standardised $\mathrm{tsccd}(7,3)$; we have found no source that states this number, the standard survey printing six representatives and no total. There are exactly 1440 tight single-change covering designs on six labelled points with block size three, and there is no $\mathrm{tsccd}(6,4)$ and no $\mathrm{tsccd}(7,4)$ at all — with no canonical form assumed anywhere. At five further orders, $(9,4)$, $(10,4)$, $(11,6)$, $(12,5)$ and $(10,6)$, no standardised design exists. What makes these statements about designs rather than about a program is a completeness theorem for the enumeration procedure, proved with no computation, together with a proof of the counting identity $\binom{v}{2}=\binom{k}{2}+(b-1)(k-1)$ for all parameters, which forces the number of blocks and yields congruence obstructions ruling out a positive proportion of orders with no search at all. We also record a verification, inside the proof kernel, of Bean's design and of every statistic published about it. All statements are machine-checked in Lean 4.

1. Introduction

1.1. **The objects.** Let $S=\{1,\dots,v\}$ and let $k<v$. A *single-change covering design* is a sequence $\mathcal{D}=(B_1,\dots,B_b)$ of k-subsets of S such that every pair of points of S lies in at least one block, and each block after the first is obtained from its predecessor by deleting one point and inserting one point. Write e_j for the point inserted to form B_j, so that $B_j\setminus B_{j-1}=\{e_j\}$ for $2\le j\le b$. The design is *tight* if for every $j\ge 2$ and every $f\in B_j\setminus\{e_j\}$ the pair $\{e_j,f\}$ occurs in none of $B_1,\dots,B_{j-1}$: each newly inserted point is new to all of its block-mates. A tight single-change covering design on v points with block size k is a $\mathrm{tsccd}(v,k)$.

Single-change covering designs were introduced by Wallis, Yucas and Zhang [14]; the question behind them is older. Nelder [7] asked in 1969 how to form the triangular array of pairwise statistics of a data matrix when only k of its v rows can be held in store at a time, and the block sequences that answer that question were taken up by Gower and Preece [4]. Section 1 of the survey of Preece, Constable, Zhang, Yucas, Wallis, McSorley and Phillips [12] — which remains the reference for the subject — describes a tight single-change covering design as a special type among the block sequences considered by Gower and Preece in response to the problem posed by Nelder.

Tightness has an immediate numerical consequence, and it is the organising fact of the subject. Under tightness each covered pair occupies exactly one run of consecutive blocks, so the first block creates $\binom{k}{2}$ pairs and each later block creates exactly $k-1$; since every pair must be created, the number of blocks is determined by v and k:

$$\binom{v}{2} \;=\; \binom{k}{2}+(b-1)(k-1), \qquad \text{so} \qquad b \;=\; b(v,k) \;:=\; \frac{\binom{v}{2}-\binom{k}{2}}{k-1}+1. \tag{1}$$

This is criterion (3.1) of [12] and equation (2) of [1]. In particular $k-1$ must divide $\binom{v}{2}-\binom{k}{2}$, which already excludes most orders.

1.2. **The source problem.** Existence is settled for the two smallest block sizes. Chafee and Stevens [3, Theorem 4] record the two statements, crediting them to [14] and to [15, 12]: an economical single-change covering design with block size 3 exists for every $v \geq 6$, and is tight if and only if $v \equiv 2, 3 \pmod 4$; one with block size 4 exists for every $v \geq 12$, and is tight if and only if $v \equiv 0, 1 \pmod 3$. For $k = 5$ the smallest example, a $\mathrm{tsccd}(20, 5)$, was found by Phillips [9] in 2001. Beyond that the subject stalled for a quarter of a century: the abstract of [12] still read *"No tsccd with $k > 4$ has yet been found."*, and at the Nineteenth British Combinatorial Conference, held at Bangor in 2003,[1] D. A. Preece posed as Problem 1 of the conference's problem list [2]:

Find tight single-change covering designs with block size greater than 5.

In July 2026 Bean [1] exhibited a $\mathrm{tsccd}(26, 6)$, found by a constraint solver after fixing a configuration of six *anchors* and three *persistent pairs*, and thereby answered the problem. For $k = 6$ the criteria of [12] and [15] leave $v \in \{21, 25, 26, 30, 31, 35, 36, \dots\}$ admissible; $v = 26$ is now settled positively and the others are open, $v = 21$ having resisted the search reported in [1, §4.1] under an extra structural restriction.

1.3. **What this paper settles.** Our subject is the small orders, where a different question can be asked than existence: *how many designs are there?* Because relabelling the points carries designs to designs, counts are taken in the *standardised* form of [12, §1] (Definition 2.3 below): first block $\{1, \dots, k\}$, later points introduced in increasing order, first-block points removed in decreasing order. The literature contains one such count — [12, §6] proves that its designs (1.1) and (2.2) are the only two standardised $\mathrm{tsccd}(6, 3)$ — and, for the next order, six representatives with no total.

- **Exactly** 32 **standardised** $\mathrm{tsccd}(7, 3)$ (Theorem 5.3). We have found no source that states this number; §5.1 records exactly what was searched and what was not. The six designs of [12, Table 4] are among the 32 — a comparison made against the enumerated list outside the proof kernel, and the one assertion in this paper that is not machine-checked.
- **Exactly** 1440 **tight single-change covering designs on six labelled points with block size three** (Theorem 6.2), with no canonical form assumed at any point. Since $1440 = 2 \cdot 6!$, this is an independent confirmation, at one order, of the reduction to standardised form which the counts otherwise take from the literature.
- **No** $\mathrm{tsccd}(6, 4)$ **and no** $\mathrm{tsccd}(7, 4)$ **at all** (Theorem 6.2), again with nothing assumed; and no standardised design at $(9, 4)$, $(10, 4)$, $(11, 6)$, $(12, 5)$, $(10, 6)$ (Theorem 5.1). All seven orders pass the divisibility test in (1) and fail Zhang's published bound $v > 3k - 2$ [15], so these confirm a known exclusion; what is new is that the confirmation is unconditional rather than relative to an unverified program.
- **The enumeration is complete** (Theorem 4.2). Every standardised $\mathrm{tsccd}(v, k)$ is produced by the breadth-first procedure of §4. This is what turns the computations into statements about designs, and it is proved with no computation of any kind.
- **The counting identity holds for all parameters** (Proposition 3.1), and with it non-existence with no search: a $\mathrm{tsccd}(v, 3)$ forces $v \equiv 2, 3 \pmod 4$, a $\mathrm{tsccd}(v, 4)$ forces $v \equiv 0, 1 \pmod 3$, a $\mathrm{tsccd}(v, 5)$ forces $v \equiv 4, 5 \pmod 8$, and a $\mathrm{tsccd}(v, 6)$ forces $v \equiv 0, 1 \pmod 5$ (Corollary 3.2). These are the necessity halves of the published spectra, proved here rather than quoted, and each rules out a positive proportion of all orders at once.
- **Bean's design, verified** (§7). The 63 blocks of [1, Appendix A], all four defining conditions, and every statistic the paper states about its design are checked inside the proof kernel.

Nothing here touches the frontier of [1]: no claim is made about $v = 21$, 25, 30 or 31 for $k = 6$. Section 8 says what stands in the way.

Every numbered statement below has been machine-checked; §9 says what that means, and §4 says which claims rest on exhaustive computation and how large each search was.

[1] The problem list [2] dates the conference to 2001, the year of the Eighteenth; the British Combinatorial Committee's record places the Nineteenth at Bangor, 30 June–4 July 2003.

2. Preliminaries

Throughout, $S = \{1, \dots, v\}$, blocks are k-subsets of S written as increasing sequences, and a *design* is a finite sequence $\mathcal{D} = (B_1, \dots, B_b)$ of blocks. We do not fix b in advance: the number of blocks is an output of the definition, by Proposition 3.1.

Definition 2.1. $\mathcal{D} = (B_1, \dots, B_b)$ is a tsccd(v, k) if $k < v$, every B_j is a k-subset of S, and

(i) (covering) every pair $\{x, y\} \subseteq S$ satisfies $\{x, y\} \subseteq B_j$ for some j;
(ii) (single change) $|B_j \setminus B_{j-1}| = |B_{j-1} \setminus B_j| = 1$ for $2 \le j \le b$;
(iii) (tightness) writing $B_j \setminus B_{j-1} = \{e_j\}$, for $2 \le j \le b$ and $f \in B_j \setminus \{e_j\}$ there is no $i < j$ with $\{e_j, f\} \subseteq B_i$.

This is the definition of [14, 12] and the one quoted in [1, §1]. Note that (iii) refers to (ii) for the meaning of e_j; in the formal development the three conditions are separate Boolean predicates and (iii) is false at any index where the single change fails.

Definition 2.2. Let $\mathcal{D}$ be a design. The point e is *transferred* $t(e)$ times, where $t(e)$ counts the occurrence of e in B_1 together with every $j \ge 2$ with $e_j = e$; the total is $T = \sum_{e \in S} t(e)$, and $t_i = \#\{e : t(e) = i\}$. An *anchor* is a point with $t(e) = 1$. The *run* of a pair is the number of blocks containing both its points, and a pair is *persistent* if its run is at least $v - k$ [11, 12]; there are at most $\lfloor k/2 \rfloor$ of them, which is inequality (4.6) of [12]. Write $\alpha(e)$ for the least j with $e \in B_j$ (and $\alpha(e) = 0$ if there is none), and $\rho(e)$ for the least $j \ge 2$ with $e \in B_{j-1} \setminus B_j$ (and $\rho(e) = b + 1$ if e is never removed, so that a point surviving to the end counts as removed last).

Definition 2.3 ([12, §1], [1, §1])**.** A design is *standardised* if $B_1 = \{1, \dots, k\}$, the points outside B_1 first appear in increasing order, $\alpha(k+1) < \alpha(k+2) < \cdots < \alpha(v)$, and the points of B_1 are first removed in decreasing order, $\rho(k) < \rho(k-1) < \cdots < \rho(1)$. It is *front-loaded* [1, §4.1] if moreover $\alpha(e) = e - k + 1$ for every $e > k$, i.e. if every new point appears in the earliest block standardisation permits.

Every design can be brought to standardised form by relabelling the points; this is stated in [1, §1] and used without comment throughout [12], and standardised designs are the canonical form of the subject. We use that reduction where we say so and never prove it; Remark 5.2 isolates exactly where it is load-bearing, and Theorem 6.2 dispenses with it at three orders.

Two published necessary conditions are used for context. Criterion (3.3) of [12], from Theorem 3.3 of [15], is $v > 3k - 2$ for $k > 3$; criterion (3.4), also from [15], is $v(v-1) \ge (6v - 7k)(k-1)$, which at $k = 6$ reads $(v - 10)(v - 21) \ge 0$.

3. The counting identity, and non-existence with no search

Identity (1) is stated in both sources as a consequence of tightness. Because everything in §4 enumerates sequences of a fixed length, we need it as a theorem about all designs rather than as an accounting rule.

Proposition 3.1. *Let $\mathcal{D} = (B_1, \dots, B_b)$ be a* tsccd(v, k) *with $b \ge 1$. Then*

$$\binom{v}{2} = \binom{k}{2} + (b-1)(k-1), \qquad \textit{equivalently} \qquad v^2 + k = k^2 + v + 2(b-1)(k-1).$$

If moreover $k \ge 2$ then $\mathcal{D}$ has exactly $b(v, k)$ blocks, with $b(v, k)$ as in (1)*.*

Proof sketch. Let $P(\mathcal{D})$ be the set of pairs $\{x, y\}$ sharing a block of $\mathcal{D}$. A block of size n carries $\binom{n}{2}$ pairs, in the division-free form $2|P| + n = n^2$. For the inductive step, let $e = e_j$ be the point inserted to form B_j. Every pair of B_j avoiding e lies inside B_{j-1}, by single change, hence was already covered; and by tightness none of the $k - 1$ pairs $\{e, f\}$ with $f \in B_j \setminus \{e\}$ was covered before. So passing from $(B_1, \dots, B_{j-1})$ to $(B_1, \dots, B_j)$ adds exactly $k-1$ pairs, and induction along the design gives $2|P(B_1, \dots, B_m)| + k = k^2 + 2(m-1)(k-1)$. Covering identifies $P(\mathcal{D})$ with all $\binom{v}{2}$ pairs of S, and the identity follows; solving for b needs $k \ge 2$. □

The formal proof is a finite-set cardinality induction quantified over all v, k and $\mathcal{D}$: no search, no finite data, and condition (ii) is never used — covering and tightness suffice. As controls, the identity forces $b = 63$ at $(26, 6)$, holds at the designs of §7, fails at a two-block sequence that is not a design, and the hypothesis $k \geq 2$ is not decoration: the empty design satisfies Definition 2.1 vacuously at $(v, k) = (1, 0)$ and has $0 \neq b(1, 0) = 1$ blocks.

Corollary 3.2. *Let $k \geq 2$. If $k - 1$ does not divide $\binom{v}{2} - \binom{k}{2}$ then there is no* $\mathrm{tsccd}(v, k)$. *In particular, for every v:*

$$\mathrm{tsccd}(v,3) \Rightarrow v \equiv 2,3 \pmod 4, \qquad \mathrm{tsccd}(v,4) \Rightarrow v \equiv 0,1 \pmod 3,$$
$$\mathrm{tsccd}(v,5) \Rightarrow v \equiv 4,5 \pmod 8, \qquad \mathrm{tsccd}(v,6) \Rightarrow v \equiv 0,1 \pmod 5.$$

These are the necessity halves of the classical spectra for $k = 3, 4$ [14, 15, 12], the list that [12, §3] prints as the orders surviving its criterion (3.1) for $k = 5$, and the divisibility condition of [1, §1] for $k = 6$. Stated as congruences they hold at every order rather than on a filtered list: for instance there is no $\mathrm{tsccd}(8, 3)$, no $\mathrm{tsccd}(9, 3)$, no $\mathrm{tsccd}(14, 4)$ and no $\mathrm{tsccd}(22, 6)$, none of which any search below touches. On the ranges where [12] and [1] print lists, the congruences reproduce those lists exactly. The condition is necessary and never sufficient: $(10, 6)$ passes it, and Theorem 5.1 shows that order is empty.

4. Exhaustive enumeration, and why it is complete

4.1. **The procedure.** Fix (v, k) and let $b = b(v, k)$. Define, for a block B, the list of all single changes available at B: for each $r \in B$ and each $e \in S \setminus B$, the block $(B \setminus \{r\}) \cup \{e\}$; there are $k(v - k)$ of them. Let Π be the following prefix-closed rendering of standardisation, which can be tested on a partial design:

> $B_1 = \{1, \ldots, k\}$; the points introduced so far, listed in order of introduction, form a prefix of $(k + 1, k + 2, \ldots, v)$; and the points of B_1 removed so far, listed in order of first removal, form a prefix of $(k, k - 1, \ldots, 1)$.

The procedure builds layers: $L_0 = \{(\{1, \ldots, k\})\}$, and L_{m+1} consists of all extensions of a member of L_m by one available single change that satisfy Π and pass a tightness test at the newly added block. The output is

$$\Sigma(v, k) \;=\; \{\, \mathcal{D} \in L_{b-1} \;:\; \mathcal{D} \text{ is a } \mathrm{tsccd}(v, k) \,\},$$

the last step filtering by Definition 2.1 itself. Consequently *everything the procedure returns is a design*; the content of §4.2 is the converse.

Theorem 4.1 (The enumeration)**.** *The following nine instances of $\Sigma(v, k)$ have been computed in full.*

(a) $\Sigma(6, 3)$ *has two members, namely the designs (2.2) and (1.1) of* [12]*, in that order.*
(b) $\Sigma(7, 3)$ *has* 32 *members, one of which is the* $\mathrm{tsccd}(7, 3)$ *printed in the statement of BCC Problem 1* [2]*.*
(c) $\Sigma(v, k) = \emptyset$ *for* $(v, k) \in \{(6, 4), (7, 4), (9, 4), (10, 4), (11, 6), (12, 5), (10, 6)\}$.

Proof sketch. Each item is a finite computation: the layers are built to depth $b - 1$ and the survivors filtered. These are the only exhaustive computations in the paper, and they are delegated to compiled evaluation rather than to symbolic reduction; §9 says how. Their sizes, measured as the number of partial designs the search visits, are in Table 1; the same nine cells were independently recomputed by a backtracking program in C sharing no code with the verified enumerator, with identical verdicts. $\square$

TABLE 1. The nine enumerated orders. Each b is the value forced by (1): at $(12, 5)$, for instance, $b = (66 - 10)/4 + 1 = 15$. "Nodes" counts partial designs visited, as measured by an independent implementation in C with its optional pruning heuristic switched off, so that the figure is the size of the bare search tree; the last column is the same search with the first block allowed to be an arbitrary k-subset and no standardisation imposed, where it was run.

(v, k)	b	standardised designs	nodes	nodes, unrestricted
$(6, 3)$	7	2	36	23,960
$(7, 3)$	10	32	1,171	—
$(6, 4)$	4	0	4	615
$(7, 4)$	6	0	10	22,295
$(9, 4)$	11	0	1,590	—
$(10, 4)$	14	0	89,081	—
$(11, 6)$	9	0	77	—
$(12, 5)$	15	0	137,907	—
$(10, 6)$	7	0	24	7,716,450

4.2. **Completeness.** Nothing so far rules out a standardised design that the procedure fails to generate. That gap is what the next theorem closes.

Theorem 4.2 (Completeness)**.** *Let $k \geq 2$. Every* $\mathrm{tsccd}(v, k)$ *satisfying* Π *belongs to* $\Sigma(v, k)$*. Hence* $\Sigma(v, k)$ *is exactly the set of* $\mathrm{tsccd}(v, k)$ *satisfying* Π.

Proof sketch. The load-bearing lemma is that *a strictly increasing sequence is determined by its set of entries*. With it, the $k(v - k)$ blocks listed at B really are *all* the single changes available: if B' is a k-subset of S with $|B' \setminus B| = |B \setminus B'| = 1$, say $B \setminus B' = \{r\}$ and $B' \setminus B = \{e\}$, then B' and $(B \setminus \{r\}) \cup \{e\}$ are increasing sequences with the same entries, hence equal. The rest is bookkeeping along a design $\mathcal{D}$: the cheap one-block tightness test applied at the head of a prefix is exactly clause (iii) of Definition 2.1 at that index, because the inserted point, the last block and the earlier prefix agree between $\mathcal{D}$ and its prefix; and Π is prefix-closed, since the introduction list and the removal list of a prefix are prefixes of those of $\mathcal{D}$. Induction on m then gives $(B_1, \ldots, B_{m+1}) \in L_m$ for every $m < b$, and Proposition 3.1 supplies $b = b(v, k)$, the layer at which the procedure stops — so the number of blocks is not an assumption either. No computation of any kind enters this proof. □

Two controls show the hypotheses are load-bearing. The bound $k \geq 2$ cannot be dropped: at $(v, k) = (1, 0)$ the empty design satisfies Definition 2.1 and Π, and is not in $\Sigma(1, 0)$, because $b(1, 0) = 1$ is the wrong length there. And Π cannot be dropped: the design (1.1) of [12] relabelled by the transposition $1 \leftrightarrow 2$ is a $\mathrm{tsccd}(6, 3)$, fails Π, and is not in $\Sigma(6, 3)$.

The literature's standardisation (Definition 2.3) is stated through first appearances and first removals, not as a prefix condition, so one more step is needed to make Theorem 4.2 apply to it.

Proposition 4.3. *Let $k \geq 2$ and let $\mathcal{D}$ be a standardised* $\mathrm{tsccd}(v, k)$*. Then $\mathcal{D}$ satisfies* Π*. Consequently every standardised* $\mathrm{tsccd}(v, k)$ *belongs to* $\Sigma(v, k)$.

Proof sketch. Two invariants along the design. First, for every m the points introduced within the first m blocks are, in order, $k + 1, k + 2, \ldots, k + t$ for some t: if block m introduces a new point e then $e > k$, and e is not one of $k + 1, \ldots, k + t$; were $e > k + t + 1$, the chain $\alpha(k + 1) < \cdots < \alpha(v)$ would place $k + t + 1$ strictly earlier than e, hence inside the first m blocks — covering guarantees $\alpha(k + t + 1) \geq 1$ — contradicting the invariant. So $e = k + t + 1$. Second, symmetrically, the first-block points removed within the first m blocks are, in order, $k, k - 1, \ldots, k - s + 1$, using the chain $\rho(k) < \cdots < \rho(1)$. At $m = b$ the two invariants are the two prefix conditions of Π. Again no computation enters. □

As positive controls the bridge is *applied* rather than recomputed: Π is deduced for Bean's 63-block $\mathrm{tsccd}(26, 6)$ from the fact that it is standardised, and likewise for the two $\mathrm{tsccd}(6, 3)$ of [12] and for the $\mathrm{tsccd}(7, 3)$ of [2]; independently, Π is checked directly at four designs.

5. Empty cells and exact counts

Theorem 5.1. *There is no standardised tight single-change covering design at any of the seven orders*

$$(6, 4), \quad (7, 4), \quad (9, 4), \quad (10, 4), \quad (11, 6), \quad (12, 5), \quad (10, 6).$$

Proof. Combine Theorem 4.1(c), Theorem 4.2 and Proposition 4.3: a standardised design at such an order would lie in an empty $\Sigma(v, k)$. $\square$

Remark 5.2. Theorem 5.1 is a statement about designs, not about a program, but it is a statement about standardised designs. To read it as "no design at all at these orders" one needs the reduction of [1, §1] that every design can be standardised by relabelling, which we use and do not prove. That the caveat is real, and not an artefact of the argument, is visible at $(6, 3)$: the relabelled copy of (1.1) mentioned above is a genuine design outside $\Sigma(6, 3)$. Two of the seven orders need no such reduction at all — see Theorem 6.2. Each of the seven orders passes the divisibility test of (1) and fails Zhang's $v > 3k - 2$ [15], so all seven are excluded by published criteria; the point here is the removal of the hypotheses, not the verdict. As a control against reading the emptiness as an artefact of the enumeration, $\Sigma(6, 3) \neq \emptyset$.

Theorem 5.3. (a) *A design is a standardised* $\mathrm{tsccd}(6, 3)$ *if and only if it is the design (1.1) or the design (2.2) of* [12]*.*

(b) *There are exactly* 32 *standardised* $\mathrm{tsccd}(7, 3)$*: there is a list of* 32 *pairwise distinct designs such that a design is a standardised* $\mathrm{tsccd}(7, 3)$ *if and only if it is on the list.*

Proof. In both directions. Left to right is Theorem 4.2 with Proposition 4.3. Right to left is the filter in the definition of Σ, which guarantees that every member is a design, together with the finite check that every member of $\Sigma(6, 3)$ and of $\Sigma(7, 3)$ is standardised, and, for (b), that the 32 are pairwise distinct. The identification of $\Sigma(6, 3)$ with $\{(1.1), (2.2)\}$ and the count 32 are Theorem 4.1(a),(b). $\square$

Part (a) is the uniqueness statement of [12, §6] — *"(1.1) and (2.2) are the only standardised* $\mathrm{tsccd}(6, 3)$*'s"* — recovered as an equivalence over all designs. Part (b) is, as far as we have been able to determine, new.

5.1. On the count of standardised $\mathrm{tsccd}(7, 3)$.

The survey [12, Table 4] prints six standardised $\mathrm{tsccd}(7, 3)$, one for each of the six realisable combinations of a solution of its equations (4.2) with a solution of its equations (4.5), and states no total. Its six are *row-irregular* in its own terminology, which is no restriction, since the same paper shows that no row-regular $\mathrm{tsccd}(7, 3)$ exists; all six are among the 32. Searching for a published total we found none. What was searched: the On-Line Encyclopedia of Integer Sequences [8], where the queries `single-change covering` and `tight single change covering` both return nothing at all — against a control query that does return records, so the subject is simply not indexed; *zbMATH*, whose search on the phrase returns eleven records, namely [14, 11, 15, 12, 13, 10, 9, 3, 1] together with one paper on single-change *circular* covering designs and one on *double*-change designs, none of them a count; a full-text corpus of arXiv mathematics, in which the phrase "single-change covering" occurs in four files, being [1], [3], and two bibliography listings; and web search, which returns [1], [12], [3], a 2021 thesis of Chafee and work of McSorley on circular designs. What was *not* searched, so that the negative is not a clean one: MathSciNet, Google Scholar, and the back catalogues of *Utilitas Mathematica* and of *Congressus Numerantium*.

One item deserves its own line: the note [13], whose title suggests it is exactly where an enumeration anecdote would be recorded. We could not obtain it — volume 13 of that Bulletin is not digitised — so it remains the one place we cannot rule out. Two descriptions of its contents agree, however, and neither mentions an enumeration: its *zbMATH* review (Zbl 0823.05023) reports that it narrates the author's involvement with these designs and the origin of [12], and [1, §1] calls it a narrative account.

6. Dropping the canonical form

At orders where the search space is small enough to start from *every* k-subset, the reduction to standardised form can be dispensed with. Let $\Sigma^*(v,k)$ be the output of the same procedure with L_0 the list of all k-subsets of S, with Π dropped and only the tightness test retained, filtered at the end by Definition 2.1 as before. The completeness proof goes through verbatim once one knows that every k-subset appears in L_0.

Lemma 6.1. *Let $k \geq 2$. Every* $\mathrm{tsccd}(v,k)$ *belongs to* $\Sigma^*(v,k)$.

Proof sketch. As in Theorem 4.2, with one addition: the first block is now arbitrary, so the initial layer must be shown to contain every k-subset of S. That is the companion of the lemma used there — a strictly increasing sequence all of whose entries lie in a strictly increasing sequence is a subsequence of it — applied to the enumeration of subsequences by length. There is no standardisation hypothesis to propagate, so the induction is shorter, and again no computation enters. □

Theorem 6.2. *There is no* $\mathrm{tsccd}(6,4)$ *and there is no* $\mathrm{tsccd}(7,4)$*. There are exactly* 1440 $\mathrm{tsccd}(6,3)$*: there is a list of* 1440 *pairwise distinct designs such that a design is a* $\mathrm{tsccd}(6,3)$ *if and only if it is on the list.*

Proof. $\Sigma^*(6,4)$ and $\Sigma^*(7,4)$ are empty and $\Sigma^*(6,3)$ has 1440 pairwise distinct members; these are exhaustive computations over 615, 22,295 and 23,960 nodes respectively (Table 1). Lemma 6.1 turns them into the stated assertions about designs. □

No canonical form, no relabelling reduction and no standardisation hypothesis occurs anywhere in Theorem 6.2. Both non-existence statements again confirm Zhang's $v > 3k-2$, now with nothing assumed at all. The count 1440 deserves a comment: $1440 = 2\cdot 6!$, and 2 is the number of standardised $\mathrm{tsccd}(6,3)$ by Theorem 5.3(a). So at this order the two standardised designs have full and disjoint relabelling orbits, which is what the reduction to standardised form predicts — an independent confirmation, at one order, of the one thing the counts of §5 take from the literature. As controls, $\Sigma^*(6,3)$ contains both (1.1) and its relabelling, while $\Sigma(6,3)$ contains only the former, so the two enumerations genuinely differ.

The larger orders stay with the standardised statements. Removing the canonical form multiplies the search: at $(10,6)$ the unrestricted search visits 7,716,450 nodes against 24 with standardisation (Table 1), which is past what the breadth-first layer construction can hold in the evaluator used here.

7. The design of block size six, and the small published designs

The results of this section are not new: they are the claims of [1] and [12], verified. We record them because they are the objects the rest of the paper is calibrated against, and because the verification is of a different character from the searches above.

Theorem 7.1 ([1, Theorem 1])**.** *The* 63 *blocks listed in* [1, Appendix A] *form a* $\mathrm{tsccd}(26,6)$*. In particular there exists a tight single-change covering design with block size greater than five, which is BCC Problem 1* [2].

Proof. The four conditions of Definition 2.1 — 63 blocks of six points of $\{1,\ldots,26\}$, a single change at each of the 62 steps, tightness, and the covering of all $\binom{26}{2} = 325$ pairs — are each decided by direct evaluation on the explicit design, and the four are then combined. The design was transcribed from [1, Appendix A] and cross-checked, block by block, against the dot array of [1, Table 1], which is an independently produced printing of the same object in the same paper; all 63 blocks agree. □

Proposition 7.2. *Bean's design has $b = 63$ blocks, which is the value $b(26,6)$ forced by (1), with pair accounting $15+62\cdot 5 = 325$; total transfers $T = 68 = b+k-1$; transfer spectrum $(t_1,t_2,t_3,t_4) = (6,5,8,7)$, summing to 26, with no point transferred five or more times; it is standardised and front-loaded, with introduction delay zero; its anchors are exactly $\{1,7,14,15,25,26\}$, each occurring in $v-1-(k-2) = 21$ blocks; its persistent pairs are exactly $(1,7)$, $(14,15)$ and $(25,26)$, so there are $\lfloor k/2 \rfloor = 3$ of them, the maximum permitted by inequality (4.6) of [12]; and every one of its 26 points obeys the identity "a point transferred m times occurs in $v-1-(k-2)m$ blocks".*

Each of these is a claim of [1, §1,§4.2] about its own design, checked here. The three persistent pairs are the feature the source reports as absent from every unsuccessful search, and they are impossible at $k = 5$, where $\lfloor k/2 \rfloor = 2$.

Remark 7.3 (The cost of not trusting the evaluator). Theorem 7.1 and Proposition 7.2 are decided by the proof kernel itself, by symbolic reduction, rather than by compiled evaluation: their proofs use no additional axiom beyond propositional extensionality. The file containing them then costs between 35 and 65 times as long to check as an otherwise identical development that delegates the same checks to compiled evaluation. The two were timed on one and the same machine, whose specification we do not record, and the spread is the difference between an idle and a loaded box; only the ratio is claimed. That factor buys the kernel's own inspection of the one object this subject exists to exhibit, and it is worth isolating precisely because the exhaustive searches of §4 cannot be paid for the same way.

Three designs are small enough to print. Writing a block as the increasing string of its points, the design displayed inside the statement of BCC Problem 1 [2] and the designs (1.1) and (2.2) of [12, §6] are

$$\begin{aligned}
\text{BCC Problem 1:}\quad & 123\ \ 124\ \ 145\ \ 146\ \ 147\ \ 347\ \ 367\ \ 357\ \ 257\ \ 256,\\
(1.1):\quad & 123\ \ 124\ \ 145\ \ 146\ \ 346\ \ 356\ \ 256,\\
(2.2):\quad & 123\ \ 124\ \ 145\ \ 156\ \ 256\ \ 356\ \ 346.
\end{aligned}$$

The first is a tsccd$(7,3)$, the other two are the two designs of Theorem 5.3(a). BCC Problem 1 prints its design as a 3×10 array, one column per block; the blocks above are its columns, each written in increasing order.

Proposition 7.4. *The* tsccd$(7,3)$ *displayed in the statement of BCC Problem 1* [2] *is a standardised* tsccd$(7,3)$ *with* $b(7,3) = 10$ *blocks. The designs (1.1) and (2.2) of* [12] *are distinct standardised* tsccd$(6,3)$*, and both have* $t_1 = t_2 = 3$*, as* [12, §4] *requires of every* tsccd$(6,3)$.

Proposition 7.5. *There are standardised designs at* $(10,3)$*,* $(11,3)$ *and* $(12,4)$*; explicit ones, too large to print here, are recorded in the formal development. In particular a* tsccd$(v,3)$ *exists for* $v = 6,7,10,11$*, the first four values of the list printed in* [12, §6]*, and a* tsccd$(12,4)$ *exists,* $v = 12$ *being the least order admitting block size four* [12, §3]*,* [10].

These three designs were found by backtracking here rather than copied, but every one of their orders lies inside a published spectrum [3, Theorem 4], so they are new representatives of known cells and nothing more. The $(12,4)$ search visited 112,362,861 nodes and the $(10,3)$ search 1,083,625; we do not state a count of the designs at those orders, and the exhaustive count at $(10,3)$ was not completed.

Proposition 7.6. *On the ranges printed in* [12, §3]*, the divisibility criterion of* (1) *yields exactly that paper's four lists of admissible orders for* $k = 3,4,5,6$*. Adding Zhang's two criteria yields* $v \in \{12,13,15,16,18,19\}$ *for* $k = 4$ *below* 20*,* $\{20,21,28,29\}$ *for* $k = 5$ *below* 30*, and* $\{21,25,26,30,31,35,36\}$ *for* $k = 6$ *below* 40 *— the list printed in* [1, §1]*. Moreover* $b(21,6) = 40$*,* $b(25,6) = 58$ *and* $b(26,6) = 63$.

Remark 7.7 (A compressed derivation in the source). [1, §1] derives its admissible orders for $k = 6$ from two conditions: the divisibility $5 \mid \binom{v}{2} - 15$, and Zhang's $v(v-1) \geq (6v - 7k)(k-1)$, which at $k = 6$ is $(v - 10)(v - 21) \geq 0$. Those two do not exclude $v = 10$: the divisibility holds, and the inequality holds at $v = 10$ with equality, that being its other root. Filtering by exactly the two stated conditions leaves $\{10, 21, 25, 26, 30, 31, 35, 36\}$ below 40. The printed list is nevertheless correct; what is missing from the derivation is Zhang's other criterion $v > 3k-2$ [15], stated as (3.3) in [12, §3] and not restated in [1], and adding it gives the printed list exactly. Independently of any criterion, Theorem 5.1 shows that $(10, 6)$ is empty. This is a remark about a compressed derivation, not about a wrong statement.

7.1. **Controls.** A definition with three clauses invites the suspicion that one clause implies another, or that a claim survives a perturbation it should not.

Proposition 7.8. *Each clause of Definition 2.1 is refuted by an object satisfying the other two. The first* 62 *blocks of Bean's design satisfy single change and tightness but do not cover; extending it by one further single change gives a* 64*-block sequence that covers and is single change but is not tight — consistently with* (1)*, since* 64 *tight blocks would create* $15 + 63 \cdot 5 = 330 > 325$ *distinct pairs; and replacing one block of it by another six-subset gives a sequence in which two points change at once. The parameters are not free either: Bean's design is not a* $\mathrm{tsccd}(25, 6)$*, a* $\mathrm{tsccd}(26, 5)$ *or a* $\mathrm{tsccd}(26, 7)$*. Standardisation is a genuine restriction: (1.1) of* [12] *relabelled by* $1 \leftrightarrow 2$ *is a* $\mathrm{tsccd}(6, 3)$ *that is not standardised. Front-loading is a genuine restriction as well: the* $\mathrm{tsccd}(10, 3)$ *of Proposition 7.5 is not front-loaded and has positive introduction delay.*

The last control is why no statement is made here about $v = 21$ at $k = 6$: the negative search reported in [1, §4.1] at that order is carried out under the front-loaded restriction, and front-loading genuinely restricts the space, so that search is not a non-existence proof and we do not treat it as one.

8. What remains

Existence at $k = 6$. Of the admissible orders $21, 25, 26, 30, 31, \ldots$ only $v = 26$ is settled. The natural next target is $v = 25$ with three persistent pairs, the configuration that succeeded at $v = 26$; [1, §4.2] reports that eighteen of the twenty-one *two*-persistent-pair configurations tried at $v = 25$ were refuted, several of them only after two to eighteen hours of computation each, and leaves that order open. This is a constraint-solving problem, not a backtracking one: the source reports 4615 seconds of solver time on a single four-core task at $v = 26$, *after* an anchor configuration had been fixed. Nor does the recursive construction of [12, §9] apply immediately at $k = 6$, as [1, §5] suggests it might: besides the design one already has, it needs a second ingredient, a standardised $\mathrm{tsccd}(n(k-1), k)$ carrying an outer expansion set of locations — so an order divisible by $k-1 = 5$ — and no design at $v = 25$, 30 or 35 is known. ([1, §5] points to the construction of [10] as well; that one we have not examined.) The combination theorem stated explicitly as [3, Theorem 3], and credited there to [12], meets the same obstruction: its second ingredient must be a tight design carrying an outer expansion set, and an expansion set partitions the point set into $v/(k-1)$ unchanged subsets, so that order too is divisible by $k - 1$.

More counts. Beyond $(7, 3)$ the exact counts are open. The exhaustive count at $(10, 3)$ did not finish here; $(13, 5)$ and $(15, 6)$ are beyond the breadth-first layer construction in the evaluator used, and would need a depth-first traversal with a stronger prefix test. Since the completeness proof of §4.2 depends on the pruning predicate only through the properties that it holds of every prefix of a design, that it is testable at the last block, and that it is what the procedure applies, strengthening the predicate does not disturb the proof; proving that a sharper counting bound has the first of those properties is the remaining work.

The relabelling reduction. That every design can be standardised by relabelling is the one published claim this paper uses without proving it (Remark 5.2). It is confirmed here at $(6, 3)$ by

Theorem 6.2, and it is what would upgrade Theorem 5.1 from "no standardised design" to "no design" at the remaining five of its seven orders.

9. Verification

Every numbered statement of this paper has been formally verified in Lean 4 against *Mathlib* [5, 6], in a development free of `sorry` and of added axioms (repository reference to be added at submission). The reasoning layer is checked by the kernel alone: Proposition 3.1, Corollary 3.2, Theorem 4.2, Proposition 4.3 and Lemma 6.1 depend on no axioms beyond propositional extensionality, choice and quotient soundness, and Theorem 7.1, Proposition 7.2, Proposition 7.4, Proposition 7.5, Proposition 7.6 and Proposition 7.8 depend on propositional extensionality alone — in particular the $\mathrm{tsccd}(26, 6)$ itself, all 63 blocks and all 325 pairs of it, was inspected by the kernel by symbolic reduction rather than by compiled evaluation (Remark 7.3). What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the exhaustive searches and the finite checks on the lists they return: the nine computations of Theorem 4.1, the three of Theorem 6.2, and the checks that every member of $\Sigma(6, 3)$ and of $\Sigma(7, 3)$ is standardised and that the returned lists repeat no design. Consequently Theorems 4.1, 5.1, 5.3 and 6.2 carry, in addition to the axioms above, one evaluation axiom per finite computation they use, and no other statement in the paper carries any. Each of the twelve searches was also run by an independent backtracking program written in C, sharing no code with the verified enumerator, with the same verdicts and the node counts of Table 1; and at $(6, 3)$, $(6, 4)$, $(7, 4)$ and $(10, 6)$ the C program was additionally run with no standardisation assumed and over every one of the $\binom{v}{k}$ possible first blocks, again with the same verdicts. The size of what is trusted is therefore small and explicitly delimited: a dozen Boolean evaluations over finite lists, each independently reproduced. One assertion made in this paper has not been seen by the kernel at all: that the six designs of [12, Table 4] are among the 32 of Theorem 5.3(b). That comparison was made against the enumerated list outside Lean, and nothing else here rests on it.

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