THE MAXIMAL VARIANCE OF A UNILATERALLY TRUNCATED CHI DISTRIBUTION: THE NEGATIVE-DIMENSION REGIME, TWO SHARP MILLS-RATIO INEQUALITIES, AND A CERTIFIED TABLE

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Abstract. For a radially symmetric Gaussian in n dimensions restricted to the shell $R \geq a$, Petrella (arXiv:2511.11566) writes the variance of R at a fixed mean M as $M^2\, V(n,r)$ with $r = a/\sigma$, $x = r^2/2$ and $V(n,r) = \Gamma(\frac{n}{2},x)\Gamma(\frac{n+2}{2},x)/\Gamma(\frac{n+1}{2},x)^2 - 1$, an expression that makes sense for every real n once $r > 0$, and conjectures on numerical evidence that $V(n,\cdot)$ is maximal in the limit $r \to 0$ for every n in $-100 \leq n \leq 512$. We prove the conjectured maximum in the negative-dimension regime: for every integer $n \leq -3$ and every $a > 0$, $V(n,a) < 1/(n(n+2))$, the value the source derives for the limit; the bound is never attained, and the constant is sharp (certified at $n = -3$ and $n = -6$, and for every integer $n \leq -5$ through an explicit lower bound tending to $1/(n(n+2))$). The mechanism is that $W_k(a) = (k-1)a^{k-1}\int_a^\infty x^{-k}e^{-x^2/2}\,dx$ is a strictly concave, hence log-concave, function of the integer order k; in incomplete-gamma terms this is the sharp reverse Turán inequality $\Gamma(\nu-\frac{1}{2},x)\Gamma(\nu+\frac{1}{2},x)/\Gamma(\nu,x)^2 \leq \nu^2/(\nu^2-\frac{1}{4})$ at half-integer $\nu \leq -1$, whose unit-shift analogue is a theorem of Baricz and Ismail. For $n \geq 1/2$ the conjecture is a corollary of a 2023 lemma of Cao and Vempala on truncations of log-concave densities; we record the reduction, and at $n = 1$ and $n = 2$ we give machine-checked proofs over the whole half line in the form of the sharp Mills-ratio inequalities $Q(a)^2 + aQ(a) \leq \pi/2$ and $(a + Q(a))^2 \geq (\pi/4)(a^2 + 2)$. Twelve cells of the source's table are certified as rational enclosures, its $r = 0$ column is given in closed form for $n = 1,\ldots,6$, and one printed value is corrected: at $n = 6$, $r = 0$ the spread is $16/(15\sqrt{2\pi}) = 0.42554$, not 0.42553. The strip $0 < n < 1/2$ remains open. Every theorem is verified in Lean 4 with Mathlib, with no axioms beyond the three standard ones.

1. Introduction

1.1. The objects. Let $x = (x_1,\ldots,x_n)$ be a centred Gaussian vector in $\mathbb{R}^n$ with covariance $\sigma^2 I_n$ and let $R = (x_1^2 + \cdots + x_n^2)^{1/2}$, a chi variable with n degrees of freedom scaled by σ; its density is proportional to $R^{n-1}e^{-R^2/2\sigma^2}$. Restrict R to $[a,\infty)$ (the source's "inner truncation") and write $r = a/\sigma$. Petrella [1] asks for the variance of the restricted variable when its mean is held fixed at M, and derives (his "Form II", Eq. (40), p. 19 of the arXiv PDF)

(1)

$$\mathrm{Var}(R;M,r,n)_{|R\geq a} = M^2\Big(\frac{\Gamma(\frac{n}{2},\frac{r^2}{2})\,\Gamma(\frac{n+2}{2},\frac{r^2}{2})}{\Gamma(\frac{n+1}{2},\frac{r^2}{2})^2} - 1\Big), \qquad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt.$$

The bracket is scale free, so

$$V(n,r) := \frac{\Gamma(\frac{n}{2},\frac{r^2}{2})\,\Gamma(\frac{n+2}{2},\frac{r^2}{2})}{\Gamma(\frac{n+1}{2},\frac{r^2}{2})^2} - 1 = \frac{\operatorname{Var}(R \mid R \geq a)}{\mathbb{E}(R \mid R \geq a)^2} \tag{2}$$

is the squared coefficient of variation of the left-truncated chi distribution, a function of (n,r) alone. Throughout we take $\sigma = 1$, so that $r = a$.

Two features of (2) shape everything below. First, no incomplete gamma function is needed at integer n: with the Gaussian tail moments

$$I_m(a) = \int_a^\infty x^m e^{-x^2/2}\,dx, \qquad \Gamma\big(\tfrac{m+1}{2},\tfrac{a^2}{2}\big) = 2^{(1-m)/2}\, I_m(a), \tag{3}$$

the powers of two cancel and $V(n,a) = I_{n+1}(a)I_{n-1}(a)/I_n(a)^2 - 1$, which the recurrence $I_{m+2} = a^{m+1}e^{-a^2/2} + (m+1)I_m$ turns into a rational function of a and the Mills ratio $Q(a) = I_0(a)/I_1(a) = \Phi(-a)/\varphi(a)$ (Section 2). Second, (2) is meaningful for *every real* n as soon as $a > 0$, since $x^{n-1}e^{-x^2/2}$ is then a bounded positive integrable function on $[a,\infty)$; $V(n,a)$ is the squared coefficient of variation of an honest probability distribution for every real n, and only the limit $a \to 0$ fails to be one when $n \leq 0$. The source says as much (p. 18): "$\Gamma(,) > 0$ when the function's second argument is positive, so this expression is well defined and describes a positive probability distribution for all $\{n,r\} \in \mathbb{R}$". At $n = -k$ the relevant moments are the negative-order tail moments

$$J_k(a) = \int_a^\infty x^{-k} e^{-x^2/2}\,dx = 2^{-(k+1)/2}\,\Gamma\big(\tfrac{1-k}{2},\tfrac{a^2}{2}\big), \tag{4}$$

and again the powers of two cancel: $V(-k,a) = J_{k-1}(a)J_{k+1}(a)/J_k(a)^2 - 1$.

1.2. **The source and its conjecture.** The central claim of [1] is numerical, and the paper says so. In Section 7.1 (p. 20 of the arXiv PDF) it states:

> "As is illustrated in Figure 7, Panel (a), and Table 4 for the case of $M = 1$, the variance appears to decrease monotonically with increasing $|r| = a/\sigma$ for any n, given a fixed mean, $M > a$. For fixed $n \in \mathbb{R}$ and $M > a \in \mathbb{R}^+$, the maximal variance, $V_{\max,r}(n)$ always appears to occur in the limit as $|r| \to 0$. Although this is not proven here, it was numerically verified for over 1000 values of $n \in \mathbb{R}$ in a range of $-100 \leq n \leq 512$."

(The source's sign convention is $r = -a/\sigma$; only r^2 enters.) The limiting value is given in three regimes. For $n > 0$, Eq. (42), p. 20,

$$V_{\max,r}(n) = \frac{n\,\Gamma(\frac{n}{2})^2}{2\,\Gamma(\frac{n+1}{2})^2} - 1, \tag{5}$$

and on the same page: "Under fixed mean, for all tested values of $n \in [-2,0] \subset \mathbb{R}$ (using Eq. 40), $V_{\max,r}(n) \to \infty$ as $|r| \to 0$; and for $n \in (-\infty,-2) \subset \mathbb{R}$, $V_{\max,r}(n) \to M^2/n(n+2)$ as $|r| \to 0$ (see Tables 4 and 5)." Table 5 (p. 24) tabulates the three cases with the caption "The expressions were derived analytically and confirmed numerically", and Appendix Section 9.7 (p. 42) derives them. Table 4 (p. 19) prints σ, a and $\operatorname{Var}(R)_{|R\geq a}$ to five decimals at $M = 1$ for $n \in \{-6,-3,-2,-1,0,\frac{1}{2},1,2,3,6\}$ and $r \in \{-4,-2,-1,-\frac{1}{2},0\}$; its two negative-dimension $r = 0$ cells read 0.33333 at $n = -3$ and 0.04167 at $n = -6$, which are $1/3 = 1/((-3)(-1))$ and $1/24 = 1/((-6)(-4))$. Also on p. 18 the source warns that "some of the results in this section are inferred from numerical analysis rather than derived, and these conjectures are indicated as such". As of 2026-09-03 the paper exists only as arXiv version 1 (14 November 2025), with no journal reference and no citing work recorded by OpenAlex.

1.3. **What was already known.** The conjecture is not open for $n \geq 1/2$. Cao and Vempala [2], in a paper on unsupervised halfspace learning, prove (their Lemma 3, stated on p. 3 and proved on p. 18 in the printed pagination of the arXiv PDF):

> "**Lemma 3** (Monotonicity of Moment Ratio). *Let q be a logconcave distribution in one dimension with nonnegative support. For any $t \geq 0$, let q_t be the distribution obtained by restricting q to $[t, \infty)$. Then the moment ratio of q_t, defined as $\mathrm{var}_{q_t}(X^2)/(\mathbb{E}_{q_t} X^2)^2$, is strictly decreasing with t.*"

Take $q(x) \propto x^{2n-1} e^{-x^4/2}$ on $[0, \infty)$. Then $Y = X^2$ has density $\propto y^{n-1} e^{-y^2/2}$, a chi density with n degrees of freedom, restricting X to $[t, \infty)$ restricts Y to $[t^2, \infty)$, and the moment ratio of q_t is exactly $V(n, t^2)$. The log-density $(2n-1)\log x - x^4/2$ has second derivative $-(2n-1)/x^2 - 6x^2$, which is ≤ 0 on $(0, \infty)$ if and only if $n \geq 1/2$. Hence *for every real $n \geq 1/2$, $V(n, \cdot)$ is strictly decreasing on* $[0, \infty)$, and the source's conjecture holds there, maximum and monotonicity both. Cao and Vempala call the lemma "a new monotonicity property" (abstract and pp. 3 and 16; "a novel monotonicity property" on p. 15) and never mention chi distributions; log-concavity is used in their proof, so the negative-dimension regime is outside it.

At $n = 1$ the maximum also follows from a classical Mills-ratio bound: Boyd's upper bound $Q(x) < \pi/(\sqrt{(\pi-2)^2 x^2 + 2\pi} + 2x)$ for $x > 0$ [4] (see also [7]) is uniformly stronger than $Q(x) \leq \pi/(\sqrt{x^2 + 2\pi} + x)$, since $(\pi - 2)^2 > 1$, and the latter is equivalent to $Q^2 + xQ \leq \pi/2$, i.e. to $V(1, x) \leq V(1, 0)$ by the closed form of Section 2.

For the negative-dimension regime, the closest published statement is a Turán-type inequality of Baricz and Ismail [3] for the Tricomi function $\psi(a, c, x)$ (their Theorem 4, eq. (3.16), p. 8 of arXiv:1110.4699v2):

> "*The function $c \mapsto \psi(a, c, x)$ is strictly logarithmically convex on $\mathbb{R}$ for all $a, x > 0$ fixed, and the following sharp Turán type inequality*
>
> $$\frac{a}{c(1+a-c)}\,\psi^2(a,c,x) < \psi^2(a,c,x) - \psi(a,c-1,x)\psi(a,c+1,x) < 0 \tag{3.16}$$
>
> *is valid for all $a > 0 > c$ and $x > 0$. Furthermore, the right-hand side of (3.16) holds for all $a, x > 0$ and $c \in \mathbb{R}$. These inequalities are sharp in the sense that the constants $a(c(1+a-c))^{-1}$ and 0 are best possible.*"

(The theorem continues with a second inequality, (3.17), for another range of the parameters, not needed here.) By their integral representation (3.12), $\psi(1, 1+\nu, x) = \int_0^\infty e^{-xt}(1+t)^{\nu-1}\,dt = x^{-\nu} e^x \Gamma(\nu, x)$, so at $a = 1$, $c = 1 + \nu$ the left inequality of (3.16) is

$$\frac{\Gamma(\nu-1,x)\Gamma(\nu+1,x)}{\Gamma(\nu,x)^2} < \frac{\nu^2}{\nu^2-1} \qquad (\nu < -1,\ x > 0), \tag{6}$$

the *unit*-shift reverse Turán inequality with a sharp constant. Form II needs the *half* shift, $\Gamma(\frac{n}{2}), \Gamma(\frac{n+1}{2}), \Gamma(\frac{n+2}{2})$, which (3.16) neither states nor implies (a function can satisfy $f(\nu-1) + f(\nu+1) = 2f(\nu)$ everywhere without being concave: $\cos 2\pi\nu$).

Its right-hand side, log-convexity of $c \mapsto \psi(a,c,x)$, gives $V(n,a) \geq 0$; for positive order this is also Corollary 3 of Qi [10], which states that $\Gamma(r,x)$ is logarithmically convex in $r > 0$ for $x > 0$.

1.4. **Results.** Write $n = -(j+3)$ with $j \in \mathbb{N}$ for the integers $n \leq -3$.

(A) *The conjectured maximum at negative dimension* (Theorems 3.3 and 3.6). For every integer $n \leq -3$ and every $a > 0$, $V(n,a) < 1/(n(n+2))$. The constant is the source's own limiting value and is sharp: $V(-3, 10^{-3}) > 1/3 - 1/500$, $V(-6, 10^{-3}) > 1/24 - 10^{-5}$, and for every integer $n \leq -5$ an explicit lower bound whose limit as $a \to 0$ is $1/(n(n+2))$. The supremum is never attained. The two negative-dimension $r = 0$ cells of the source's Table 4 are therefore the suprema of their rows: certified to within $1/500$ at $n = -3$ and, through the explicit bound at $n \leq -5$, exactly at $n = -6$, the passage to the limit $a \to 0$ being the one step taken in prose.

(B) *The mechanism* (Theorem 3.1). For every $a > 0$ the sequence $W_k(a) = (k-1)a^{k-1}J_k(a)$, $k \geq 2$, is strictly concave in k, the concavity defect being the integral of a positive function. A positive concave sequence is log-concave, and $W_{k-1}W_{k+1} \leq W_k^2$ is precisely (A). We could not find the concavity in the literature; its consequence (6) at the unit shift is the Baricz–Ismail theorem, and the half-shift case is what (A) needs.

(C) *The cases $n = 1$ and $n = 2$ over the whole half line* (Theorems 4.1 and 4.2): $Q(a)^2 + aQ(a) \leq \pi/2$ and $(a + Q(a))^2 \geq (\pi/4)(a^2+2)$ for all $a \geq 0$, with equality at $a = 0$. The mathematics is not new (Section 1.3); the proofs are machine-checked certificates with a handful of cells, and we did not find either inequality stated in this Mills-ratio form.

(D) *The table* (Section 5). Twelve cells of Table 4 are enclosed in intervals of width 10^{-7}; the $r = 0$ column is $(\pi-2)/2$, $4/\pi - 1$, $3\pi/8 - 1$, $32/(9\pi) - 1$, $45\pi/128 - 1$, $768/(225\pi) - 1$ for $n = 1, \ldots, 6$; and the printed spread 0.42553 at $n = 6$, $r = 0$ is a truncation of $16/(15\sqrt{2\pi}) = 0.4255384\ldots$, whose five-decimal rounding is 0.42554.

Items (A) and (B) are new; (C) and (D) are verification of published material, with one correction. Every statement is proved in Lean 4 with Mathlib (Section 7), and no claim in this paper rests on floating-point computation.

1.5. **What remains open.** After (A) and Section 1.3, the maximality half of the source's conjecture is settled outside a single strip: it holds for $n \geq 1/2$ (Cao–Vempala), for every integer $n \leq -3$ (here), and on $-2 \leq n \leq 0$ the source reports, and the small-x asymptotics of $\Gamma(s,x)$ at $s \leq 0$ confirm, that $V(n,r) \to +\infty$ as $r \to 0$, so the assertion that the supremum is approached as $r \to 0$ has no content there beyond monotonicity. The strip $0 < n < 1/2$ is open: no published lemma applies, the supremum (5) is finite and non-elementary, and Section 6 discusses what a proof there would need. Also open, in the sense that nothing below proves it, is the *monotonicity* of $V(n,\cdot)$ (as opposed to its maximum) for $n < 1/2$; and the statements of (A) at non-integer $n < -2$ are not formalized (Remark 3.9 gives a proof in prose).

2. Preliminaries

2.1. **Gaussian tail moments and the Mills ratio.** Let $I_m(a)$ be as in (3) for $m \in \mathbb{N}$ and $a \geq 0$, and put $G(a) = \int_0^a e^{-t^2/2}\,dt$, so that

(7)
$$I_0(a) = \tfrac{\sqrt{2\pi}}{2} - G(a), \qquad I_1(a) = e^{-a^2/2}, \qquad I_{m+2}(a) = a^{m+1}e^{-a^2/2} + (m+1)\,I_m(a),$$

the last by one integration by parts. The Mills ratio is $Q(a) = I_0(a)/I_1(a)$; then $I_0 = Qe^{-a^2/2}$ and, by (7), $I_m = p_m\,e^{-a^2/2}$ with p_m a polynomial in a and $Q = Q(a)$ with integer coefficients:

$$\begin{aligned} &p_0 = Q, && p_1 = 1, && p_2 = a + Q, \\ &p_3 = a^2 + 2, && p_4 = a^3 + 3a + 3Q, && p_5 = a^4 + 4a^2 + 8, \\ &p_6 = a^5 + 5a^3 + 15a + 15Q, && p_7 = a^6 + 6a^4 + 24a^2 + 48. \end{aligned} \tag{8}$$

Proposition 2.1. $Q(0) = \sqrt{2\pi}/2$.

Proposition 2.2 (two free Mills bounds)**.** *For every $a \geq 0$: (i) $aQ(a) \leq 1$; (ii) $Q(a)^2 + aQ(a) \geq 1$.*

Proof. (i) is $\int_a^\infty (x-a)e^{-x^2/2}\,dx \geq 0$, i.e. $I_1 \geq aI_0$. (ii) is Cauchy–Schwarz for the moments: $\int_a^\infty (x-\lambda)^2 e^{-x^2/2}\,dx \geq 0$ at $\lambda = I_1/I_0$ gives $I_0 I_2 \geq I_1^2$, and $I_2 = aI_1 + I_0$ turns this into $Q^2 + aQ \geq 1$; it says that the variance $V(1,a)$ below is nonnegative. Both are classical: (i) is the upper bound of Gordon [5] and (ii) is equivalent to Birnbaum's lower bound $Q(a) \geq 2/(\sqrt{a^2+4} + a)$ [6]. □

For $k \in \mathbb{N}$ let $\operatorname{tcv}(k,a) = I_{k+2}(a)I_k(a)/I_{k+1}(a)^2 - 1$; this is $V(k+1,a)$, the squared coefficient of variation of the chi distribution with $n = k+1$ degrees of freedom truncated to $[a,\infty)$, because $\mathbb{E}(R^p \mid R \geq a) = I_{n-1+p}/I_{n-1}$.

Proposition 2.3 (closed forms)**.** *For every $a \geq 0$ and $n = 1,\dots,6$, with the polynomials* (8)*,*

$$V(n,a) + 1 = \frac{p_{n+1}(a)\,p_{n-1}(a)}{p_n(a)^2} \qquad (n = 1,\dots,6),$$

$$\textit{i.e.} \quad V(1,a) + 1 = Q^2 + aQ, \qquad V(2,a) + 1 = \frac{a^2+2}{(a+Q)^2},$$

$$V(3,a) + 1 = \frac{(a^3+3a+3Q)(a+Q)}{(a^2+2)^2}, \qquad V(4,a) + 1 = \frac{(a^4+4a^2+8)(a^2+2)}{(a^3+3a+3Q)^2},$$

and so on.

These are the source's Form II at $n = 1,\dots,6$, re-derived from the defining integrals through (7). At $a = 0$ they give (5) exactly (Section 5). The source's conjecture at integer $n \geq 1$ is thus a family of sharp Mills-ratio inequalities, each tight only at $a = 0$: $Q^2 + aQ \leq \pi/2$ at $n = 1$, $(a+Q)^2 \geq (\pi/4)(a^2+2)$ at $n = 2$, $(a^3+3a+3Q)(a+Q) \leq (3\pi/8)(a^2+2)^2$ at $n = 3$, the direction alternating with the parity of n.

2.2. **Negative-order tail moments.** For $k \in \mathbb{N}$ and $a > 0$ let $J_k(a)$ be as in (4); the integral converges for every k since $x^{-k} \leq a^{-k}$ on $[a,\infty)$, and $J_k(a) > 0$. Differentiating $-x^{-(k+1)}e^{-x^2/2}$ gives the downward recurrence

$$(k+1)\,J_{k+2}(a) + J_k(a) = a^{-(k+1)}e^{-a^2/2} \qquad (k \in \mathbb{N},\ a > 0). \tag{9}$$

For $j \in \mathbb{N}$ put $\operatorname{negtcv}(j,a) = J_{j+2}(a)J_{j+4}(a)/J_{j+3}(a)^2 - 1$; by (4) this is $V(-(j+3),a)$, the squared coefficient of variation of the distribution with density $\propto x^{-(j+4)}e^{-x^2/2}$ on $[a,\infty)$, i.e. Form II at dimension $n = -(j+3)$. Finally, for $k \geq 2$ let

$$W_k(a) = (k-1)\, a^{k-1}\, J_k(a) = e^{-a^2/2} - a^{k-1} J_{k-2}(a), \tag{10}$$

the second expression being (9) multiplied by a^{k-1}. In the formal development W_k is $\mathrm{Wq}(j,a)$ with $k = j+2$. In incomplete-gamma terms, $W_k(a) = -\nu\, x^{-\nu}\, \Gamma(\nu, x)$ with $\nu = (1-k)/2 < 0$ and $x = a^2/2$, and (10) is the elementary representation $-\nu x^{-\nu}\Gamma(\nu,x) = e^{-x} - x\int_1^\infty t^\nu e^{-xt}\, dt$. The normalisation is chosen so that $W_k(a) \to 1$ as $a \to 0$ for every $k \geq 2$.

3. The negative-dimension regime

Theorem 3.1 (strict concavity in the order)**.** *For every integer $k \geq 2$ and every $a > 0$,*

$$2W_{k+1}(a) - W_k(a) - W_{k+2}(a) = \int_a^\infty a^{k-1}\, x^{-k}\, (x-a)^2\, e^{-x^2/2}\, dx \; > \; 0.$$

In particular $W_k(a) + W_{k+2}(a) < 2W_{k+1}(a)$: the sequence $k \mapsto W_k(a)$ is strictly concave.

Proof. Subtract three instances of (10): $2W_{k+1} - W_k - W_{k+2} = a^{k-1}J_{k-2} - 2a^k J_{k-1} + a^{k+1}J_k$ (the three $e^{-a^2/2}$ cancel), and $a^{k-1}x^{-(k-2)} - 2a^k x^{-(k-1)} + a^{k+1}x^{-k} = a^{k-1}x^{-k}(x-a)^2$ pointwise. The integrand is nonnegative on (a,∞), which gives the weak inequality; it is strictly positive there, and integrating over $[a+1,\infty)$ alone already gives a positive lower bound, which gives the strict one. The formal statements (`Wq_concave`, `Wq_concave_strict`, Appendix A) are the two inequalities, the identity entering as a bound against the integral. □

Lemma 3.2 (log-concavity)**.** *For every $k \geq 2$ and $a > 0$, $W_k(a)W_{k+2}(a) < W_{k+1}(a)^2$.*

Proof. $W_k > 0$ by (10) and $J_k > 0$, so by AM–GM and Theorem 3.1, $W_k W_{k+2} \leq \left(\frac{W_k + W_{k+2}}{2}\right)^2 < W_{k+1}^2$. □

Unwinding (10), $W_{k-1}W_{k+1} < W_k^2$ is $k(k-2)\, J_{k-1}J_{k+1} < (k-1)^2\, J_k^2$, which by (4) is $V(-k,a) < (k-1)^2/(k(k-2)) - 1 = 1/(k(k-2))$. With $n = -k$ this is $1/(n(n+2))$.

Theorem 3.3 (the conjectured maximum at negative dimension)**.** *For every integer $n \leq -3$ and every truncation point $a > 0$,*

$$V(n,a) \; < \; \frac{1}{n(n+2)}.$$

The right-hand side is the value the source derives for $\lim_{r\to 0} V(n,r)$ when $n < -2$ (its Table 5 and Section 9.7). The formal development proves the non-strict inequality (`negtcv_le`) and the strict one (`negtcv_lt`) separately.

Proof. Lemma 3.2 at $k = -n \geq 3$, as just computed. □

Corollary 3.4 (the two printed cells are suprema)**.** *$V(-3,a) \leq 1/3$ and $V(-6,a) \leq 1/24$ for every $a > 0$. These are the $r = 0$ cells 0.33333 and 0.04167 of the source's Table 4 (p. 19).*

Proposition 3.5 (nonnegativity)**.** *$V(n,a) \geq 0$ for every integer $n \leq -3$ and $a > 0$.*

Proof. Cauchy–Schwarz: $\int_a^\infty (1-\lambda x^{-1})^2 x^{-(j+2)} e^{-x^2/2}\,dx \geq 0$ at $\lambda = J_{j+3}/J_{j+4}$ gives $J_{j+3}^2 \leq J_{j+2}J_{j+4}$. This is the log-convexity of $\Gamma(\cdot, x)$ in its order, the right-hand side of (3.16) in [3]; it is included so that Theorem 3.3 pins a nonempty interval. ☐

Theorem 3.6 (sharpness)**.** (i) $V(-3, \frac{1}{1000}) > \frac{1}{3} - \frac{1}{500}$.

(ii) $V(-6, \frac{1}{1000}) > \frac{1}{24} - \frac{1}{100000}$.

(iii) *For every integer $i \geq 0$, with $n = -(i+5)$, and every $0 < a \leq 1$,*

$$V(n,a) \;\geq\; \frac{(i+4)^2}{(i+3)(i+5)}\Big(1 - \frac{a^2}{i+1}\Big)\Big(1 - \frac{a^2}{i+3}\Big) - 1,$$

and the right-hand side tends to $\frac{(i+4)^2}{(i+3)(i+5)} - 1 = \frac{1}{(i+3)(i+5)} = \frac{1}{n(n+2)}$ as $a \to 0$.

Proof. All three are lower bounds for W_{k-1} and W_{k+1} against an upper bound for W_k, fed into $V(-k,a)+1 = \frac{(k-1)^2}{k(k-2)} \cdot \frac{W_{k-1}W_{k+1}}{W_k^2}$. The upper bound is $W_k \leq e^{-a^2/2}$ (the second expression in (10)). For $k \geq 4$ the lower bound is $W_k \geq e^{-a^2/2}(1 - a^2/(k-3))$, from $W_k = e^{-a^2/2} - a^{k-1}J_{k-2}$ and $J_{k-2} \leq a^{-(k-3)}e^{-a^2/2}/(k-3)$, the latter being (9) with its positive term J_{k-4} dropped; this gives (iii) and, at $k = 6$, (ii). At $k = 2$ the recurrence is unavailable and one uses instead $W_2 = e^{-a^2/2} - aJ_0 \geq e^{-a^2/2} - a\sqrt{2\pi}/2$, with the certified bound $\sqrt{2\pi}/2 \leq 1.2533141374$; this gives (i). The remaining inequalities are between explicit rationals ($e^{-t} \geq 1-t$ and $e^{-t} \leq 1$ at $t = 5 \cdot 10^{-7}$). At $a = 10^{-3}$ the bounds are $W_2 \geq 0.99874$, $W_3 \leq 1$, $W_4 \geq 0.999998$, whence $V(-3, 10^{-3}) \geq 663/2000$, and $W_5, W_7 \geq 0.999998$, $W_6 \leq 1$, whence $V(-6, 10^{-3}) \geq 2083/50000$; the true values are $0.3316808\ldots$ and $0.0416665\ldots$ (30-digit arithmetic, outside the formal development). ☐

Corollary 3.7. *For $n = -3$, $n = -6$, and every integer $n \leq -5$, $\sup_{a>0} V(n,a) = 1/(n(n+2))$, approached as $a \to 0$ and never attained.*

Proof. Theorem 3.3 gives the upper bound and non-attainment. For $n \leq -5$ (which includes $n = -6$) the lower bound of Theorem 3.6(iii) tends to $1/(n(n+2))$ as $a \to 0$; only the inequalities are formalized, the limit is continuity in a. At $n = -3$ the certified part (i) pins the supremum only within $(1/3 - 1/500, 1/3]$; that it equals $1/3$ follows from $W_k(a) \to 1$ as $a \to 0$ (Section 2.2), a limit that is not formalized. The case $n = -4$ is not covered by (iii), no sharpness certificate was produced for it, and it is left out of the statement. ☐

Proposition 3.8 (controls)**.** *(i) It is false that $V(-3,a) \leq 1/3 - 1/500$ for all $a > 0$: a smaller constant fails. (ii) For every $a > 0$ it is false that $1/3 \leq V(-3,a)$: the bound is not attained.*

These are formalized alongside the theorems so that Corollary 3.4 and Theorem 3.6(i) cannot both be satisfied by a misplaced quantifier.

Remark 3.9 (real order and every step)*.* The identity (10) holds verbatim at real order: for every $x > 0$, $-\nu x^{-\nu}\Gamma(\nu, x) = e^{-x} - x\int_1^\infty t^\nu e^{-xt}\,dt$, whose integrand is convex in ν for each $t \geq 1$, so $\nu \mapsto -\nu x^{-\nu}\Gamma(\nu, x)$ is positive and concave on $\nu < 0$, hence log-concave, which gives the sharp reverse Turán inequality

$$\frac{\Gamma(\nu - h, x)\,\Gamma(\nu + h, x)}{\Gamma(\nu, x)^2} \;\leq\; \frac{\nu^2}{\nu^2 - h^2} \qquad (\nu < -h < 0,\; x > 0),$$

with the constant attained in the limit $x \to 0^+$, where $\Gamma(s, x) \sim x^s/(-s)$ for $s < 0$. At $h = 1$ this is (6), the Baricz–Ismail theorem (which is proved there by a different route, through recurrence relations for ψ and its derivative, and states only the unit shift); at $h = 1/2$, $\nu = (n+1)/2$ it is $V(n, a) \leq 1/(n(n+2))$ for every real $n < -2$. Only the integer-order, half-integer-ν case is formalized (Theorem 3.1); the real-order statement is a remark, its identities checked numerically (25 digits at a few points) but not machine-verified. Read the novelty claim accordingly: the unit-shift constant is published and declared best possible in [3]; what is new is the concavity, hence every other step, including the one Form II needs.

Remark 3.10 (what the argument does not give). For $\nu > 0$, i.e. $n > -1$, the quantity $-\nu x^{-\nu}\Gamma(\nu, x)$ is negative and the AM–GM step reverses; the argument gives nothing there, and is exactly complementary to the log-concavity hypothesis $n \geq 1/2$ of [2]. Nor does it give monotonicity of $V(n, \cdot)$ in a at $n < -2$: the concavity identity bounds the endpoint, not the derivative.

4. One and two degrees of freedom over the whole half line

By Proposition 2.3, the source's conjecture at $n = 1$ says $Q(a)^2 + aQ(a) \leq Q(0)^2 = \pi/2$, and at $n = 2$ it says $(a + Q(a))^2 \geq (\pi/4)(a^2 + 2)$. Both follow from Cao–Vempala's Lemma 3 (Section 1.3), and the first also from Boyd's bound, so neither is new mathematics; what is recorded here is that each admits a short certificate checked by the proof kernel, with no interval-arithmetic library and no floating point.

Theorem 4.1 ($n = 1$)**.** *For every $a \geq 0$, $Q(a)^2 + aQ(a) \leq \pi/2$, i.e. $V(1, a) \leq V(1, 0) = (\pi - 2)/2$.*

Proof sketch. Write $\kappa = \sqrt{2\pi}/2$, $g = e^{-a^2/2}$, $G = G(a)$, so that $Q = (\kappa - G)/g$ and the claim is $\Psi(a) := \kappa^2(g^2 - 1) + \kappa(2G - ag) + G(ag - G) \geq 0$. Three regions.

- $0 \leq a \leq 1/2$: the Taylor sandwiches $g^2 \geq 1 - a^2$, $a - a^3/6 \leq G \leq a$, $1 - a^2/2 \leq g \leq 1$ give $\Psi \geq a\,[\kappa - \kappa^2 a - \kappa a^2/3 - a^3/2]$, and the bracket is at least 0.3 on $[0, 1/2]$.
- $1/2 \leq a \leq 11/8$: seven cells $[p, q]$ of width $1/8$. On each, G increasing and g decreasing give $Q(a) \leq (\kappa_{\mathrm{hi}} - G_{\mathrm{lo}}(p))/g_{\mathrm{lo}}(q)$, where G_{lo} is a six-term and g_{lo} a twelve-term alternating Taylor lower bound evaluated at rational points; the resulting rational bound B satisfies $B^2 + qB < \pi/2$ on every cell, with worst margin $1.4716 < 1.5708$.
- $a \geq 11/8$: $Q \leq 1/a$ (Proposition 2.2(i)) gives $Q^2 + aQ \leq 1/a^2 + 1 \leq 1 + 64/121 < \pi/2$.

The seven cells are the only computation, and each is one rational inequality. Only π to seven digits and $\sqrt{2\pi}/2$ to ten decimals enter. □

Theorem 4.2 ($n = 2$)**.** *For every $a \geq 0$, $\pi(a^2+2) \leq 4(a+Q(a))^2$, and consequently $V(2, a) \leq 4/\pi - 1 = V(2, 0)$.*

Proof sketch. The inequality is now a *lower* bound on Q, so the certificate is the mirror image. On $[0, 1/2]$, $g^2 \le 1 - a^2 + a^4/2$ and $ag + \kappa - G \ge \kappa - a^3/2$ reduce the claim to $2\kappa a \le \kappa^2 + a^4(1-\kappa^2)/2$, with slack ≥ 0.29. On $[1/2, 7/8]$, twelve cells of width $1/32$, on each of which $Q(a) \ge (\kappa_{\mathrm{lo}} - G_{\mathrm{hi}}(q))/g_{\mathrm{hi}}(p)$ with seven- and thirteen-term Taylor upper bounds; the worst cell has $4(p+Q_{\mathrm{lo}})^2 = 7.2398$ against $\pi(q^2+2) \le 7.1698$. Beyond $7/8$ nothing but Proposition 2.2: from $Q(Q+a) \ge 1$ and $aQ \le 1$ one gets $Q(a^2+1) \ge a$, and then $4(a+Q)^2 \ge 4a^2 + 4aQ + 4 \ge \pi(a^2+2)$ reduces to a quartic in a that is positive from $a \ge 7/8$ on. The second statement is the first divided by $4(a+Q)^2 > 0$. □

5. The source's table, certified

Twelve of the finite cells of Table 4 of [1] are re-derived from the defining integrals: a two-sided enclosure of the Mills ratio at the three truncation points, $Q(\frac{1}{2}) \in [0.876364455, 0.876364459]$, $Q(1) \in [0.655679541, 0.655679545]$, $Q(2) \in [0.421369221, 0.421369234]$ (alternating Taylor bounds for G and $e^{-a^2/2}$ with between ten and twenty-seven terms), propagated through Proposition 2.3. All 45 finite cells of that table were first reproduced from (2) in 40-digit arithmetic, outside the formal development.

Proposition 5.1 (twelve table cells)**.** *For* $n \in \{1, 2, 3, 6\}$ *and* $a \in \{\frac{1}{2}, 1, 2\}$*,* $V(n, a)$ *lies strictly between the two seven-decimal bounds of Table 1. In every case the source's printed five-decimal value is the correct rounding.*

n	$\lvert r\rvert = a$	certified enclosure of $V(n,a)$	printed in Table 4
1	1/2	$0.2061968 < V < 0.2061969$	0.20620
2	1/2	$0.1877242 < V < 0.1877243$	0.18772
3	1/2	$0.1565793 < V < 0.1565794$	0.15658
6	1/2	$0.0862826 < V < 0.0862827$	0.08628
1	1	$0.0855952 < V < 0.0855953$	0.08560
2	1	$0.0943813 < V < 0.0943814$	0.09438
3	1	$0.0977226 < V < 0.0977227$	0.09772
6	1	$0.0801326 < V < 0.0801327$	0.08013
1	2	$0.0202904 < V < 0.0202905$	0.02029
2	2	$0.0233618 < V < 0.0233619$	0.02336
3	2	$0.0266677 < V < 0.0266678$	0.02667
6	2	$0.0363631 < V < 0.0363632$	0.03636

Table 1. Certified enclosures of twelve cells of Table 4 of [1] (the $\mathrm{Var}(R)_{|R\ge a}$ column at $M = 1$), with the printed values.

Proposition 5.2 (the $r = 0$ column)**.**

$$V(1,0) = \frac{\pi-2}{2}, \qquad V(2,0) = \frac{4}{\pi} - 1, \qquad V(3,0) = \frac{3\pi}{8} - 1,$$
$$V(4,0) = \frac{32}{9\pi} - 1, \qquad V(5,0) = \frac{45\pi}{128} - 1, \qquad V(6,0) = \frac{768}{225\pi} - 1.$$

Proof. Proposition 2.3 at $a = 0$ with $Q(0)^2 = \pi/2$ (Proposition 2.1). □

These are instances of (5); numerically 0.57080, 0.27324, 0.17810, 0.13177, 0.10447, 0.08650, and the four that appear in Table 4 ($n = 1, 2, 3, 6$) are printed correctly there. The rational parts are, by parity, partial Wallis products (OEIS A069955 and relatives), where the truncated-chi reading is not recorded.

The spread column of Table 4 is, by the source's Eq. (39), $\sigma(M, r, n) = (M/\sqrt{2})\,\Gamma(\frac{n}{2}, \frac{r^2}{2})/\Gamma(\frac{n+1}{2}, \frac{r^2}{2})$, which at $M = 1$ and integer $n = k+1$ is $I_k(r)/I_{k+1}(r)$ by (3).

Proposition 5.3. *At $n = 6$, $r = 0$ the spread is $\sigma = 16/(15\sqrt{2\pi})$.*

Proof. $I_5(0) = 8$ and $I_6(0) = 15\sqrt{2\pi}/2$ from (7). □

Theorem 5.4 (correction)**.** $0.4255384 < 16/(15\sqrt{2\pi}) < 0.4255385$. *Consequently the value $\sigma = 0.42553$ printed in Table 4 of* [1] *for $n = 6$, $r = 0$ is a truncation, not a rounding; the correctly rounded five-decimal value is* 0.42554.

Proof. From the certified enclosure

$$2.5066282746310005024 \leq \sqrt{2\pi} \leq 2.5066282746310005025.$$

□

Remark 5.5. The same happens once more in that table, at $n = 1/2$, $r = 0$, where the printed variance 1.18843 is a truncation of $V(\frac{1}{2}, 0) = \frac{\Gamma(1/4)^2}{4\Gamma(3/4)^2} - 1 = 1.1884396\ldots$, whose rounding is 1.18844. This value involves Γ at quarter integers and is not certified here (30-digit arithmetic, outside the formal development). The other 43 finite cells are correctly rounded. No erratum exists (the arXiv record has only version 1).

Proposition 5.6 (controls)**.** *(i) It is false that $V(1, a) \leq \pi/2 - 1 - 1/1000$ for all $a \geq 0$: the maximum $(\pi - 2)/2$ is attained at $a = 0$, so no smaller bound holds. (ii) $V(1, 1) < V(1, 0)$: Theorem 4.1 is not the statement that $V(1, \cdot)$ is constant. (iii) $V(3, 1) \neq 0.09773$: a too-large value for one printed cell is refuted. (iv) $V(1, 0) > 0$: at $a = 0$ there is no truncation and the value is the untruncated coefficient of variation, not zero.*

6. The open strip $0 < n < 1/2$

After Sections 1.3 and 3 the maximality assertion of [1] is proved for every real $n \geq 1/2$ and every integer $n \leq -3$, and has no content on $[-2, 0]$. What is left is the strip $0 < n < 1/2$ (and, formally, the non-integer $n < -2$ of Remark 3.9). We claim nothing in the strip. Two observations fix its price.

First, no published lemma applies: the Cao–Vempala density $x^{2n-1}e^{-x^4/2}$ is not log-concave for $n < 1/2$, and the concavity of Section 3 needs $\nu < 0$. The supremum (5) is finite there and non-elementary (at $n = 1/2$ it is $V(\frac{1}{2}, 0)$ above), so a formal statement of the maximum needs a certified enclosure of Γ at real arguments, and the object $V(n, a)$ itself needs the upper incomplete gamma function at real order, or equivalently the tail moment $\int_a^\infty x^{n-1}e^{-x^2/2}dx$ with real exponent. Mathlib has Γ (`Real.Gamma`, with `Real.Gamma_add_one` and log-convexity) but no incomplete gamma function at any order; the tail moment with real exponent is available as an integral. The computational cost of any such certificate is negligible; the cost is in the enclosure infrastructure.

Second, there is a candidate argument for every real n at once, recorded because it identifies the mechanism, and *not* verified. With $L(t) = \log\Gamma(t, x)$, $p = n/2$ and $x = a^2/2$, $\log(V(n, a) + 1) = L(p) - 2L(p + \frac{1}{2}) + L(p + 1)$ is a second difference of step 1/2, hence equals $\int_0^{1/2}\int_0^{1/2} L''(p + u + v)\, du\, dv$; differentiating under the integral, $L''(t) = \mathrm{Var}(\log T)$ for T with density $\propto \tau^{t-1}e^{-\tau}$ on $[x, \infty)$, and $\log T$ has the log-concave log-density $ty - e^y$ for every real t. If the variance of a log-concave variable conditioned on $[c, \infty)$ is non-increasing in c, then $L''(t)$ is non-increasing in x, hence so is the double integral, hence $V(n, \cdot)$ is non-increasing on $(0, \infty)$ for every real n. The conditioning statement is reliability theory, in the survey of Gupta [8]: Theorem 4.2 there ("DMRL(IMRL) property implies the DVRL(IVRL) property") and its Corollary 4.1 ("F is IFR implies F is DVRL"), a log-concave density (the class PF_2 of that paper) being IFR; Gupta records that the monotone behaviour of the residual variance for log-concave and log-convex densities was studied by Karlin [9], whose chapter we have not seen. The two identities were checked numerically (8–10 digits at several (t, x), including negative t); nothing here is formalized, and we do not claim the result.

7. Verification

Every theorem and proposition of Sections 2–5, and Corollary 3.4, is stated and proved in Lean 4 (version 4.32.0) against Mathlib (commit `81a5d257`, tag `v4.32.0`) [11, 12]; the exact statements are reproduced verbatim in Appendix A, and the repository reference is to be added at submission. (Lemma 3.2 is the declaration `Wq_logConcave_strict`; Corollary 3.7 and the remarks are prose.) The development is six modules totalling 2270 lines: the tail moments, recurrence, Mills ratio and closed forms (`Core`); the $n = 1$ certificate (`Max`); the $n = 2$ certificate (`Max2`); the table (`Values`); the negative-order moments, the concavity and Theorem 3.3 (`NegDim`); and strictness, nonnegativity, sharpness and the controls (`NegDimSharp`). They check in about four and a half minutes of one core in total.

The objects are the integrals. I_m, J_k, Q and the variances are defined as Lebesgue integrals over (a, ∞) of the literal integrands. Integrability of I_m is Mathlib's lemma for $x^s e^{-bx^2}$, that of J_k is domination by $a^{-k}e^{-x^2/2}$; the recurrences (7) and (9) are the fundamental theorem of calculus on (a, ∞) (Mathlib's `integral_Ioi_of_hasDerivAt_of_tendsto`) applied to $-x^{m+1}e^{-x^2/2}$ and $-x^{-(k+1)}e^{-x^2/2}$; and $I_0(0) = \sqrt{2\pi}/2$ is Mathlib's Gaussian integral. No incomplete gamma function appears, since Mathlib has none; the identification (3)–(4) with Form II is a change of variables outside the development, checked numerically to 28 digits at every table cell used.

Numbers. Every numerical inequality in the development is an inequality between rational literals, discharged by `norm_num` (with `nlinarith` and `linarith` for the algebra), never by compiled evaluation: there is no `native_decide`, no `sorry` and no added `axiom` anywhere. The transcendental inputs are π through Mathlib's twenty-digit bounds, $\sqrt{2\pi}$ through a twenty-digit enclosure, and $e^{-a^2/2}$ and $G(a)$ at rational a through the alternating Taylor sandwiches $\mathrm{expPoly}_n(s) \le e^s \le \mathrm{expPoly}_{n+1}(s)$ ($s \le 0$, n even) and the termwise-integrated analogue for G; each certificate cell of Theorems 4.1, 4.2 and each bracket of Section 5 is one such rational inequality. Section 3 uses no numerics except in Theorem 3.6 ($e^{-t} \ge 1 - t$, $e^{-t} \le 1$, and the $\sqrt{2\pi}$ enclosure).

Axioms. `#print axioms` was run on 2026-09-03 on each of the 48 declarations of Appendix A; every one depends on exactly `propext`, `Classical.choice`, `Quot.sound`, the three standard axioms of Mathlib, and on nothing else.

Published material re-derived. Form II at $n = 1, \ldots, 6$ (Proposition 2.3), (5) at $n = 1, \ldots, 6$ (Proposition 5.2), the spread at $(6, 0)$ and twelve printed cells of Table 4 are re-derived inside the development; all 45 finite cells of that table were reproduced in 40-digit arithmetic outside it before any statement was made. One printed value is corrected (Theorem 5.4), one more is flagged (the remark after it), and the rest agree with their printed roundings.

Appendix A. The Lean statements

The statements are quoted verbatim from the development; proof terms are omitted. The namespace `LeanProblemSpec.TruncatedChiVariance` is open throughout; `gaussHalf`, `gInt`, `expPoly` and `sq2pi_lb` come from the shared enclosure module `Cert.GaussianEnclosure`; `Ioi a` is (a, ∞).

Definitions.

```
noncomputable def gk (x : ℝ) : ℝ := Real.exp (-x ^ 2 / 2)
noncomputable def Imom (m : ℕ) (a : ℝ) : ℝ := ∫ x in Ioi a, x ^ m * gk x
noncomputable def gaussHalf (u : ℝ) : ℝ := ∫ y in (0:ℝ)..u, Real.exp (-y ^ 2 / 2)
noncomputable def mills (a : ℝ) : ℝ := Imom 0 a / Imom 1 a
noncomputable def kap : ℝ := Real.sqrt (2 * Real.pi) / 2
noncomputable def tcv (k : ℕ) (a : ℝ) : ℝ :=
  Imom (k + 2) a * Imom k a / (Imom (k + 1) a) ^ 2 - 1
noncomputable def spread (k : ℕ) (a : ℝ) : ℝ := Imom k a / Imom (k + 1) a
noncomputable def Jmom (k : ℕ) (a : ℝ) : ℝ := ∫ x in Ioi a, (x⁻¹) ^ k * gk x
noncomputable def Wq (j : ℕ) (a : ℝ) : ℝ := ((j : ℝ) + 1) * a ^ (j + 1) * Jmom (j + 2) a
noncomputable def negtcv (j : ℕ) (a : ℝ) : ℝ :=
  Jmom (j + 2) a * Jmom (j + 4) a / Jmom (j + 3) a ^ 2 - 1
```

Here `tcv k a` is $V(k+1, a)$, `spread k a` is the spread at $n = k+1$ and $M = 1$, `Wq j a` is $W_{j+2}(a)$, and `negtcv j a` is $V(-(j+3), a)$.

Section 2 (`Core.lean`).

```
theorem mills_zero : mills 0 = Real.sqrt (2 * Real.pi) / 2
theorem mills_mul_le_one {a : ℝ} (ha : 0 ≤ a) : a * mills a ≤ 1
theorem one_le_mills_quad {a : ℝ} (ha : 0 ≤ a) : 1 ≤ mills a ^ 2 + a * mills a
theorem tcv_one_eq {a : ℝ} (ha : 0 ≤ a) :
    tcv 0 a = mills a ^ 2 + a * mills a - 1
theorem tcv_two_eq {a : ℝ} (ha : 0 ≤ a) :
    tcv 1 a = (a ^ 2 + 2) / (a + mills a) ^ 2 - 1
theorem tcv_three_eq {a : ℝ} (ha : 0 ≤ a) :
    tcv 2 a = (a ^ 3 + 3 * a + 3 * mills a) * (a + mills a) / (a ^ 2 + 2) ^ 2 - 1
theorem tcv_four_eq {a : ℝ} (ha : 0 ≤ a) :
    tcv 3 a = (a ^ 4 + 4 * a ^ 2 + 8) * (a ^ 2 + 2)
      / (a ^ 3 + 3 * a + 3 * mills a) ^ 2 - 1
theorem tcv_five_eq {a : ℝ} (ha : 0 ≤ a) :
    tcv 4 a = (a ^ 5 + 5 * a ^ 3 + 15 * a + 15 * mills a) * (a ^ 3 + 3 * a + 3 * mills a)
      / (a ^ 4 + 4 * a ^ 2 + 8) ^ 2 - 1
theorem tcv_six_eq {a : ℝ} (ha : 0 ≤ a) :
    tcv 5 a = (a ^ 6 + 6 * a ^ 4 + 24 * a ^ 2 + 48) * (a ^ 4 + 4 * a ^ 2 + 8)
      / (a ^ 5 + 5 * a ^ 3 + 15 * a + 15 * mills a) ^ 2 - 1
```

Section 3 (`NegDim.lean`, `NegDimSharp.lean`).

```
lemma Wq_concave (j : ℕ) {a : ℝ} (ha : 0 < a) :
    Wq j a + Wq (j + 2) a ≤ 2 * Wq (j + 1) a
theorem Wq_concave_strict (j : ℕ) {a : ℝ} (ha : 0 < a) :
    Wq j a + Wq (j + 2) a < 2 * Wq (j + 1) a
theorem negtcv_le (j : ℕ) {a : ℝ} (ha : 0 < a) :
    negtcv j a ≤ 1 / (((j : ℝ) + 1) * ((j : ℝ) + 3))
theorem negtcv_lt (j : ℕ) {a : ℝ} (ha : 0 < a) :
    negtcv j a < 1 / (((j : ℝ) + 1) * ((j : ℝ) + 3))
theorem negtcv_three {a : ℝ} (ha : 0 < a) : negtcv 0 a ≤ 1 / 3
theorem negtcv_six {a : ℝ} (ha : 0 < a) : negtcv 3 a ≤ 1 / 24
theorem negtcv_nonneg (j : ℕ) {a : ℝ} (ha : 0 < a) : 0 ≤ negtcv j a
```

```
theorem negtcv_three_sharp : (1:ℝ)/3 - 1/500 < negtcv 0 (1/1000)
theorem negtcv_six_sharp : (1:ℝ)/24 - 1/100000 < negtcv 3 (1/1000)
theorem negtcv_ge_general (i : ℕ) {a : ℝ} (ha : 0 < a) (ha1 : a ≤ 1) :
    ((i : ℝ) + 4) ^ 2 / (((i : ℝ) + 3) * ((i : ℝ) + 5))
        * ((1 - a ^ 2 / ((i : ℝ) + 1)) * (1 - a ^ 2 / ((i : ℝ) + 3))) - 1
      ≤ negtcv (i + 2) a
theorem negtcv_three_no_smaller : ¬ (∀ a : ℝ, 0 < a → negtcv 0 a ≤ 1/3 - 1/500)
theorem negtcv_three_not_attained {a : ℝ} (ha : 0 < a) : ¬ ((1:ℝ)/3 ≤ negtcv 0 a)
```

The recurrence (9) and the concavity defect of Theorem 3.1, on which the two concavity statements rest:

```
lemma Jmom_rec (k : ℕ) {a : ℝ} (ha : 0 < a) :
    ((k : ℝ) + 1) * Jmom (k + 2) a + Jmom k a = (a⁻¹) ^ (k + 1) * gk a
lemma defect_integral_pos (j : ℕ) {a : ℝ} (ha : 0 < a) :
    0 < ∫ x in Ioi a, (a ^ (j + 1) * ((x⁻¹) ^ j * gk x)
          - 2 * a ^ (j + 2) * ((x⁻¹) ^ (j + 1) * gk x)
          + a ^ (j + 3) * ((x⁻¹) ^ (j + 2) * gk x))
```

Section 4 (`Max.lean`, `Max2.lean`).

```
theorem mills_quad_le {a : ℝ} (ha : 0 ≤ a) : mills a ^ 2 + a * mills a ≤ Real.pi / 2
theorem mills_quad_two {a : ℝ} (ha : 0 ≤ a) :
    Real.pi * (a ^ 2 + 2) ≤ 4 * (a + mills a) ^ 2
theorem tcv_two_le {a : ℝ} (ha : 0 ≤ a) : tcv 1 a ≤ 4 / Real.pi - 1
```

Section 5 (`Values.lean`).

```
theorem tcv_cell_n1_half :
    ((128873/625000) : ℝ) < tcv 0 ((1/2)) ∧ tcv 0 ((1/2)) < ((2061969/10000000) : ℝ)
theorem tcv_cell_n2_half :
    ((938621/5000000) : ℝ) < tcv 1 ((1/2)) ∧ tcv 1 ((1/2)) < ((1877243/10000000) : ℝ)
theorem tcv_cell_n3_half :
    ((1565793/10000000) : ℝ) < tcv 2 ((1/2)) ∧ tcv 2 ((1/2)) < ((782897/5000000) : ℝ)
theorem tcv_cell_n6_half :
    ((431413/5000000) : ℝ) < tcv 5 ((1/2)) ∧ tcv 5 ((1/2)) < ((862827/10000000) : ℝ)
theorem tcv_cell_n1_one :
    ((53497/625000) : ℝ) < tcv 0 (1) ∧ tcv 0 (1) < ((855953/10000000) : ℝ)
theorem tcv_cell_n2_one :
    ((943813/10000000) : ℝ) < tcv 1 (1) ∧ tcv 1 (1) < ((471907/5000000) : ℝ)
theorem tcv_cell_n3_one :
    ((488613/5000000) : ℝ) < tcv 2 (1) ∧ tcv 2 (1) < ((977227/10000000) : ℝ)
theorem tcv_cell_n6_one :
    ((400663/5000000) : ℝ) < tcv 5 (1) ∧ tcv 5 (1) < ((801327/10000000) : ℝ)
theorem tcv_cell_n1_two :
    ((25363/1250000) : ℝ) < tcv 0 ((2/1)) ∧ tcv 0 ((2/1)) < ((40581/2000000) : ℝ)
theorem tcv_cell_n2_two :
    ((116809/5000000) : ℝ) < tcv 1 ((2/1)) ∧ tcv 1 ((2/1)) < ((233619/10000000) : ℝ)
theorem tcv_cell_n3_two :
    ((266677/10000000) : ℝ) < tcv 2 ((2/1)) ∧ tcv 2 ((2/1)) < ((133339/5000000) : ℝ)
theorem tcv_cell_n6_two :
    ((363631/10000000) : ℝ) < tcv 5 ((2/1)) ∧ tcv 5 ((2/1)) < ((22727/625000) : ℝ)
theorem tcv_max_one : tcv 0 0 = Real.pi / 2 - 1
theorem tcv_max_two : tcv 1 0 = 4 / Real.pi - 1
theorem tcv_max_three : tcv 2 0 = 3 * Real.pi / 8 - 1
theorem tcv_max_four : tcv 3 0 = 32 / (9 * Real.pi) - 1
theorem tcv_max_five : tcv 4 0 = 45 * Real.pi / 128 - 1
theorem tcv_max_six : tcv 5 0 = 768 / (225 * Real.pi) - 1
theorem spread_six_zero : spread 5 0 = 16 / (15 * Real.sqrt (2 * Real.pi))
theorem spread_six_zero_bracket :
    (0.4255384 : ℝ) < spread 5 0 ∧ spread 5 0 < (0.4255385 : ℝ)
theorem max_not_smaller : ¬ (∀ a : ℝ, 0 ≤ a → tcv 0 a ≤ Real.pi / 2 - 1 - 1 / 1000)
theorem max_strict_at_one : tcv 0 1 < tcv 0 0
theorem cell_n3_one_ne : tcv 2 1 ≠ (0.09773 : ℝ)
theorem tcv_zero_pos : 0 < tcv 0 0
```

Axioms, as printed by `#print axioms` on 2026-09-03: each of the 48 declarations above depends on `[propext, Classical.choice, Quot.sound]`.

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