THE SIXTH AND EIGHTH MOMENTS OF THE TREND SCORE OF A RANDOM ℓ-ARY SEQUENCE

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ABSTRACT. Fix $\ell \geq 2$ ordered symbols and let all ℓ^n words $x_1x_2\cdots x_n$ be equally likely. The *trend score* $S = S^+ - S^-$ is the number of increasing pairs of positions minus the number of decreasing ones: Kendall's score with repetitions allowed and — unlike in the classical theory of ties — with the tie pattern random rather than held fixed. Lindqvist, in a paper that is a revision of a 1986 Helsinki technical report, derives the variance of S for every ℓ, states the fourth moment for every ℓ but gives it, in its own words, "without proof, when $\ell \geq 3$", gives the sixth moment for $\ell = 2$ alone, and ends its variance section by asking for "a more effective method … for higher moments". We answer that for two more moments. We give closed forms for $\mu_6(n,\ell)$ and $\mu_8(n,\ell)$ at every ℓ, in the Newton presentation $\mu_r(n,\ell) = \sum_V \binom{n}{V} a_V(\ell)$ with explicit rational $a_V(\ell)$, together with the binary specialisation $\mu_8(n,2) = n(n^2-1)\big(175n^9 - 1260n^8 + \cdots + 310896\big)/34560$, the next nonzero entry past the source's own binary list. That presentation is forced by the mathematics rather than chosen: $\mu_{2k}(\cdot,\ell)$ is a polynomial in n of degree at most $3k$ — and of degree exactly $3k$ in every case we can evaluate — so $3k+1$ exact values determine it, and we prove the degree bound by a cancellation over a pair of positions that nothing else in the term touches. Both closed forms are verified inside the Lean 4 kernel, from the definition of S as a sum over all ℓ^n words, at every point of a grid of 49 parameter pairs with $2 \leq \ell \leq 7$; so are the source's own published formulas. We also record that the source's binary sixth moment is not the general-ℓ one — it already fails at $\ell = 3$ — and correct an exchanged pair of labels in the source's worked example.

1. INTRODUCTION

The statistic. Let $\ell \geq 1$ and let the symbols $0 < 1 < \cdots < \ell - 1$ be ordered. A word $x = x_1x_2\cdots x_n \in \{0,\ldots,\ell-1\}^n$ is scored by comparing its positions pairwise:

$$S^+(x) = \#\{(i,j) : i < j,\ x_i < x_j\}, \qquad S^-(x) = \#\{(i,j) : i < j,\ x_i > x_j\},$$
$$S(x) = S^+(x) - S^-(x).$$

A large positive S is evidence of an upward trend; the test against trend that this statistic supports is what the statistic is for. The probability model is the one that makes the problem hard: *all ℓ^n words are equally likely.* Repetitions are allowed and the multiset of tie multiplicities is therefore a random object, not a parameter.

This is exactly the point at which the setting departs from the classical theory. There the sample space is the set of arrangements of one *fixed* multiset: Kendall [4] introduced the score for the permutations of n distinct values, Sillitto [5] gives the distribution of the same score when the rankings contain ties, and Silverstone [6] its cumulants; throughout, the tie multiplicities $p_1,\ldots,p_r$ are held constant. Averaging such formulas over the multinomial law of the tie pattern is possible in principle and unpleasant in practice, and no such average appears in the literature. Lindqvist's §1 names all three as the version his setting departs from, and says of that version that "the numbers of ties, i.e., $p_1, p_2, ..., p_r$, are regarded as fixed"; it is why the classical moment formulas do not answer the question asked here.

Write

$$\mu_r(n,\ell) \;=\; \ell^{-n} \sum_{x\in\{0,\ldots,\ell-1\}^n} S(x)^r$$

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for the r-th moment of S under the uniform law. Reversing a word negates its score (Proposition 2.2), so the law of S is symmetric, $\mu_1 = 0$, every odd moment vanishes, and μ_r is simultaneously the r-th moment and the r-th central moment.

The source and where it stops. The one paper that studies this sample space is Lindqvist [1]; its arXiv comment records that it is "essentially a revision of" the Helsinki technical report [2] of 1986, with some details from [3] of 1978. We name its displayed formulas by the mnemonic labels they carry in the LaTeX source of the e-print. In the printed v1 they are numbered: (var) is [1, (4)], (four) is [1, (5)], the definition of S is [1, (2)], and the asymptotic normality statement (normal) is [1, (6)]. The paper contains, in the notation above:

- the variance for general ℓ — its formula (var), displayed in §2 — *with* a proof, carried out in its §6 by a recursion on the tie-count vector;
- the fourth moment for general ℓ — its formula (four), displayed in the same §2 — stated, in the source's own words, "without proof, when $\ell \geq 3$", because "the corresponding calculations for (5) are, to say the least, a laborious task";
- for $\ell = 2$ only, in its §3 and obtained from the binary characteristic function [1, (10)] — a product of squared cosines, in two cases according to the parity of n — the values $\mu_2(n,2) = n(n^2-1)/12$, $\mu_4(n,2)$ and $\mu_6(n,2)$, the list closing with $\mu_7(n,2) = 0$ and an ellipsis;
- the exact distribution of S for $\ell = 2$ and $n \leq 7$, as a figurate table.

Its §6, in the paragraph that follows the completed proof of (var), states the open problem this paper addresses:

> "The fourth moment μ_4 in (5) is the result of a similar, although more tedious, calculation. However, a more effective method should be invented for higher moments."

We stress what is and is not missing, because it is easy to overstate. A sixth moment of S *is* in the source: the binary one. What is absent is a sixth moment at general ℓ, and the two are genuinely different objects — the binary polynomial already fails at $\ell = 3$ (Remark 4.3).

Results. Throughout, $\binom{n}{V}$ is the binomial coefficient, extended by $\binom{n}{V} = 0$ for $V > n$, and

$$A = A(\ell) = \frac{\ell - 1}{\ell}, \qquad B = B(\ell) = \frac{\ell^2 - 1}{\ell^2}.$$

All closed forms below are written in *Newton form* $\mu_r(n,\ell) = \sum_V \binom{n}{V} a_V(\ell)$. This is not a cosmetic choice. The coefficient $a_V(\ell)$ is the total contribution to μ_r of those r-tuples of index pairs whose union is a *prescribed* set of exactly V positions, so the Newton coefficients are the combinatorially meaningful quantities, and the vanishing of a_V for $V > 3r/2$ is the whole content of the closed form. Expanded into powers of n the same formulas are unreadable: at $r = 6$ the numerator over $476280\,\ell^6$ is a bivariate integer polynomial with 60 terms.

Let

$$\mathcal{G} = \big\{(n,\ell) \;:\; 0 \leq n \leq N(\ell),\ 2 \leq \ell \leq 7\big\}, \qquad \big(N(2),\dots,N(7)\big) = (12,8,7,6,5,5), \tag{1}$$

a set of 49 parameter pairs. On $\mathcal{G}$ the moments are evaluated by summing over all ℓ^n words, so a statement about $\mathcal{G}$ is a statement checked against the definition and nothing else.

Theorem 1.1 (Sixth moment; Theorem 4.1)**.** *For every $(n,\ell) \in \mathcal{G}$,*

$$\mu_6(n,\ell) = \sum_{V=2}^{9} \binom{n}{V} a_V^{(6)}(\ell)$$

with the eight explicit rational functions $a_V^{(6)}$ of (5) below.

Theorem 1.2 (Eighth moment; Theorem 5.1 and Corollary 5.2)**.** *For every $(n,\ell)\in\mathcal{G}$,*

$$\mu_8(n,\ell)=\sum_{V=2}^{12}\binom{n}{V}a_V^{(8)}(\ell)$$

with the eleven explicit rational functions $a_V^{(8)}$ of (6)–(7) *below; and for every $n<40$ that Newton sum at $\ell=2$ collapses to*

$$\sum_{V=2}^{12}\binom{n}{V}a_V^{(8)}(2)=\frac{n(n^2-1)}{34560}\big(175n^9-1260n^8+3111n^7+1380n^6-23919n^5 \\ +37740n^4+54509n^3-192084n^2-33876n+310896\big).$$

Granting Theorem 1.3 below, both are polynomials in n of degree at most 9 and at most 12; at $\ell=2$ those degrees are exactly 9 and exactly 12 (Proposition 3.5). So the grid, which contains the 13 values $n=0,1,\dots,12$ at $\ell=2$, is not merely a sample there: it exhausts the degrees of freedom. The passage from the grid to all n, and then to all ℓ, is Theorem 3.3 together with its Corollary 3.4, both proved below as ordinary mathematics and, at present, formalised only in the substantive step of the first:

Theorem 1.3 (Degree bound)**.** *For every even r and every $\ell\ge 1$, the function $n\mapsto\mu_r(n,\ell)$ is a polynomial in n of degree at most $3r/2$, and consequently*

$$\mu_r(n,\ell)=\sum_{V=0}^{3r/2}\binom{n}{V}a_V(\ell),\qquad a_V(\ell)=\sum_{m=0}^{V}(-1)^{V-m}\binom{V}{m}\mu_r(m,\ell).$$

Theorem 1.3 is what turns a finite table into a proof, and it is the answer to the source's request for "a more effective method": no better method is needed, because $\mu_r(\cdot,\ell)$ is pinned by its values at $m=0,1,\dots,3r/2$ — ten values for μ_6, thirteen for μ_8. In particular it reduces the source's (four), which the source declined to prove, to a finite computation at each ℓ: seven exact values of $\mu_4(\cdot,\ell)$ decide it.

Alongside these we record the supporting material: the source's own published formulas re-derived from the definition inside the kernel (§6), the exactness of the degrees 3, 6, 9, 12, negative controls that refute one-coefficient perturbations of both new formulas, and one correction — the source's §1 worked example carries an exchanged pair of labels (Proposition 6.3). The correction is inconsequential, and we say why: the law of S is symmetric, so every even moment, and the source states only even ones, is the same under either sign convention.

Method, in one paragraph. Write $S=\sum_e Y_e$ over the $\binom{n}{2}$ index pairs e, with $Y_{(i,j)}(x)$ the sign of x_j-x_i. Expanding S^r gives a sum over r-tuples g of index pairs. Grouping the tuples by the set of positions their pairs cover and inverting a binomial sum (§3) identifies $\binom{n}{V}$-coefficients with the contribution of tuples covering a prescribed V-set. If some pair of the tuple meets no other pair, transposing the two values at that pair's positions is a sign-reversing involution of the word space, and the contribution is zero (Lemma 3.1). A counting argument shows that above $V=3r/2$ every tuple has such a free pair. The same expansion bounds the degree of $\ell^V a_V$ in ℓ by V, so finitely many ℓ settle the ℓ-dependence too. What remains is a finite exact computation of $\mu_r(m,\ell)$ for $m\le 3r/2$ at finitely many ℓ, which we did in exact rational arithmetic and then re-checked, at $2\le\ell\le 7$, inside the Lean kernel.

2. Preliminaries

Definition 2.1. For a,b in an ordered set put $\operatorname{sgn}(a,b)=1$ if $a<b$, -1 if $b<a$, and 0 if $a=b$. For $n\ge 0$ let $\mathcal{P}_n=\{(i,j):1\le i<j\le n\}$ and, for a word $x\in\{0,\dots,\ell-1\}^n$,

$$S(x)=\sum_{(i,j)\in\mathcal{P}_n}\operatorname{sgn}(x_i,x_j),\qquad \mu_r(n,\ell)=\frac{1}{\ell^n}\sum_{x\in\{0,\dots,\ell-1\}^n}S(x)^r\in\mathbb{Q}.$$

Definition 2.1 is [1, (2)]; the identification is Proposition 6.3(1).

Proposition 2.2 (Reversal)**.** *Let ρ reverse the positions, $(\rho x)_i = x_{n+1-i}$. Then $S(\rho x) = -S(x)$ for every n, every ℓ and every word x. Consequently the law of S is symmetric about 0 and, for every odd r, every n and every ℓ,*

$$\sum_x S(x)^r = 0, \qquad \textit{hence} \qquad \mu_r(n,\ell) = 0.$$

Proof. The map $(i,j) \mapsto (n+1-j,\, n+1-i)$ is an involution of $\mathcal{P}_n$ and it exchanges the two arguments of sgn, which is antisymmetric; summing gives $S(\rho x) = -S(x)$. Since ρ is an involution of the word space, $\sum_x S(x)^r = \sum_x S(\rho x)^r = (-1)^r \sum_x S(x)^r$, which for odd r forces the sum to vanish. □

The source asserts $\mu_1 = \mu_3 = \mu_5 = \cdots = 0$ "by symmetry" and does not argue it; Proposition 2.2 is the argument, uniform in n and ℓ.

We shall use the source's two general-ℓ formulas as printed. With A and B as above, its variance formula is

$$\mu_2(n,\ell) = A \cdot \frac{n(n-1)}{2} + B \cdot \frac{n(n-1)(n-2)}{9}, \tag{2}$$

and its fourth-moment formula, the one it states without proof for $\ell \geq 3$, is

$$\begin{aligned} \mu_4(n,\ell) = {} & B^2 \cdot \frac{100n^4 + 328n^3 - 127n^2 - 997n - 372}{2700}\, n(n-1) \\ & + \frac{\ell^2 - 1}{\ell^4} \cdot \frac{252n^3 + 507n^2 - 3623n + 3652}{900}\, n(n-1) \\ & - \frac{\ell^2 - 1}{\ell^3} \cdot \frac{2n^3 + 3n^2 - 5n - 15}{6}\, n(n-1) + \frac{\ell - 1}{\ell^3} \cdot \frac{n^2 + 11n - 25}{2}\, n(n-1). \end{aligned} \tag{3}$$

Both displays are transcribed from the v1 e-print, where they stand in its §2 as [1, (4)] and [1, (5)]; the four summands of (3) are reproduced in the source's own order and with its own signs. Their mathematical content is verified from Definition 2.1 in Theorem 6.1.

Its two binary formulas, valid at $\ell = 2$ only, are

$$\begin{aligned} \mu_4(n,2) &= \frac{n(n^2-1)(5n^3 - 6n^2 - 5n + 14)}{240}, \\ \mu_6(n,2) &= \frac{n(n^2-1)\big(35n^6 - 126n^5 + 74n^4 + 420n^3 - 829n^2 - 294n + 1488\big)}{4032}. \end{aligned} \tag{4}$$

3. The degree bound

The whole of §4 and §5 rests on one cancellation.

Lemma 3.1 (Free-pair cancellation)**.** *Let $n, \ell \geq 0$, let a, b be positions and let F be any integer-valued function on $\{0, \ldots, \ell-1\}^n$ that is unchanged by transposing the entries at a and b. Then*

$$\sum_x \operatorname{sgn}(x_a, x_b)\, F(x) = 0.$$

Consequently, if $g = (g_1, \ldots, g_r)$ is an r-tuple of index pairs and some g_t is disjoint from every other g_s, then

$$\sum_x \prod_{s=1}^{r} \operatorname{sgn}\big(x_{(g_s)_1}, x_{(g_s)_2}\big) = 0.$$

Proof. Precomposition with the transposition of the positions a and b is an involution of the word space; it fixes F by hypothesis and reverses the sign of $\operatorname{sgn}(x_a, x_b)$. So the sum equals its own negative. For the second statement factor the product as $\operatorname{sgn}(x_{(g_t)_1}, x_{(g_t)_2})$ times the product over

$s \neq t$; the latter depends only on entries at positions outside g_t, hence is invariant under the transposition, and the first statement applies. □

Remark 3.2. Neither hypothesis is vacuous. Dropping the invariance hypothesis of the first statement makes it false — $\sum_{x\in\{0,1\}^2} \operatorname{sgn}(x_1,x_2)^2 = 2 \neq 0$ — and dropping disjointness in the second makes it false as well, as the tuple $g = ((1,2),(1,2))$ over the alphabet of size 3 already shows.

Theorem 3.3 (Degree bound)**.** *Let r be even and $\ell \geq 1$. Then $n \mapsto \mu_r(n,\ell)$ is a polynomial in n of degree at most $3r/2$, and*

$$\mu_r(n,\ell) = \sum_{V=0}^{3r/2} \binom{n}{V} a_V(\ell), \qquad a_V(\ell) = \sum_{m=0}^{V} (-1)^{V-m} \binom{V}{m} \mu_r(m,\ell)$$

for every $n \geq 0$.

Proof. Step 0 (Newton). For any $f : \mathbb{N} \to \mathbb{Q}$ one has $f(n) = \sum_{V=0}^{n} \binom{n}{V} \Delta^V f(0)$ with $\Delta^V f(0) = \sum_m (-1)^{V-m}\binom{V}{m} f(m)$; this is binomial inversion and says nothing about S. Since $\binom{n}{V} = 0$ for $V > n$, it suffices to prove $a_V := \Delta^V \mu_r(0) = 0$ for $V > 3r/2$.

Step 1 (restriction). Fix V and work with words of length V. For a set $U \subseteq \{1,\dots,V\}$ of positions put $\mathrm{sc}_U(x) = \sum \operatorname{sgn}(x_i,x_j)$, the sum over the pairs $(i,j) \in \mathcal{P}_V$ with $i,j \in U$. If $|U| = m$, the increasing bijection $U \to \{1,\dots,m\}$ carries $\mathrm{sc}_U(x)$ to $S(x|_U)$, and every word of length m has exactly ℓ^{V-m} preimages, so $\ell^{-V}\sum_x \mathrm{sc}_U(x)^r = \mu_r(m,\ell)$.

Step 2 (inclusion–exclusion). Grouping the subsets U by size therefore gives

$$a_V = \ell^{-V} \sum_{U\subseteq\{1,\dots,V\}} (-1)^{V-|U|} \sum_x \mathrm{sc}_U(x)^r.$$

Expand $\mathrm{sc}_U(x)^r$ as a sum over r-tuples g of pairs contained in U and exchange the order of summation. For a fixed g with support $\operatorname{supp} g = \bigcup_t g_t$, the inner sum $\sum_{U\supseteq \operatorname{supp} g}(-1)^{V-|U|}$ is 0 unless $\operatorname{supp} g = \{1,\dots,V\}$, in which case it is 1. Hence

$$a_V = \ell^{-V} \sum_{g:\ \operatorname{supp} g = \{1,\dots,V\}} \ \sum_x \prod_t \operatorname{sgn}\bigl(x_{(g_t)_1}, x_{(g_t)_2}\bigr).$$

Step 3 (cancellation). Call t *free* in g if the pair g_t meets no other g_s. By Lemma 3.1 every g with a free index contributes 0.

Step 4 (counting). Let g have $\operatorname{supp} g = \{1,\dots,V\}$ and no free index, and let $d(v) = \#\{t : v \in g_t\}$. Each pair has two distinct elements, so $\sum_v d(v) = 2r$, and $d(v) \geq 1$ on the support. Put $D = \{v : d(v) \geq 2\}$. "No free index" means every g_t shares a position with some g_s, $s \neq t$; that shared position lies in g_t and has degree ≥ 2, so every pair of the tuple meets D. Counting incidences between D and the r pairs in two ways,

$$r \ \leq \ \sum_{v\in D} d(v) \ = \ 2r - \sum_{v\notin D} d(v) \ = \ 2r - (V - |D|), \qquad 2|D| \ \leq \ \sum_{v\in D} d(v) \ = \ 2r - V + |D|,$$

whence $V \leq r + |D|$ and $|D| \leq 2r - V$, so $V \leq r + (2r - V)$, that is $V \leq 3r/2$. Therefore every g with support of size $V > \lfloor 3r/2 \rfloor$ has a free index and $a_V = 0$. □

The same expansion controls the dependence on ℓ, and that is what will let finitely many values of ℓ settle a formula stated for all of them.

Corollary 3.4 (Degree in ℓ)**.** *Fix r and V and let $a_V(\ell)$ be as in Theorem 3.3. Then $\ell \mapsto \ell^V a_V(\ell)$ is the restriction to the positive integers of a polynomial of degree at most V, integer-valued there. Consequently, if a rational function $\tilde{a}_V$ with $\ell^V \tilde{a}_V(\ell)$ polynomial of degree at most V satisfies $\tilde{a}_V(\ell) = a_V(\ell)$ at $V+1$ distinct positive integers ℓ, then $\tilde{a}_V = a_V$ at every $\ell \geq 1$.*

Proof. Step 2 of the previous proof gives $\ell^V a_V(\ell) = \sum_g \sum_x \prod_t \operatorname{sgn}(x_{(g_t)_1}, x_{(g_t)_2})$, the outer sum over the finitely many r-tuples g with $\operatorname{supp} g = \{1, \dots, V\}$ and the inner one over $x \in \{0, \dots, \ell-1\}^V$. Each product depends on x only through the order pattern of its V entries. A word of length V that uses exactly k distinct symbols out of the ℓ available is a choice of those k symbols together with a word in $\{0, \dots, k-1\}^V$ using all k of them, and the product is unchanged when the k chosen symbols are replaced by $0, \dots, k-1$ in order. Grouping by k, which runs over $0, \dots, V$, therefore gives $\ell^V a_V(\ell) = \sum_{k=0}^{V} \binom{\ell}{k} c_k$ with integers c_k independent of ℓ. That is an integer-valued polynomial in ℓ of degree at most V. The last sentence is the uniqueness of polynomial interpolation applied to $\ell^V(a_V - \tilde{a}_V)$. □

The bound in n is attained where we can check it. Taking $k = r/2$ vertex-disjoint paths with two edges each uses $2k = r$ pairs and $3k = 3r/2$ positions and leaves no free pair, so Lemma 3.1 does not annihilate the sum defining $a_{3r/2}$; that $a_{3r/2}$ is in fact nonzero is a separate matter, settled by evaluation at $r = 2, 4, 6, 8$ in the following proposition, and predicted in general by Observation 5.5.

Proposition 3.5 (Exact degrees)**.** *Write $\Delta^V \mu_r(0, \ell)$ for the V-th forward difference at 0 of $n \mapsto \mu_r(n, \ell)$, computed from Definition 2.1. Then*

$$\Delta^3 \mu_2(0,2) \neq 0, \qquad \Delta^V \mu_2(0,\ell) = 0 \ \text{ on } \ \{4,5,6\} \times \{2\} \cup \{(4,3),(5,3),(4,4),(4,5)\};$$
$$\Delta^6 \mu_4(0,2) \neq 0, \qquad \Delta^V \mu_4(0,\ell) = 0 \ \text{ on } \ \{7,8,9\} \times \{2\} \cup \{(7,3),(8,3),(7,4),(7,5)\};$$
$$\Delta^9 \mu_6(0,\ell) \neq 0 \ \text{ for } \ell = 2,3, \qquad \Delta^V \mu_6(0,\ell) = 0 \ \text{ on } \ \{10,11,12\} \times \{2\} \cup \{(10,3)\},$$

the pairs (V, ℓ) ranging over the sets displayed. Likewise $\Delta^{12} \mu_8(0,2) \neq 0$ while $\Delta^{13} \mu_8(0,2) = \Delta^{14} \mu_8(0,2) = 0$ (Proposition 5.3). So the degrees $3, 6, 9, 12$ of $\mu_2, \mu_4, \mu_6, \mu_8$ in n are exact, not upper estimates, wherever the moments can be evaluated.

4. The sixth moment at general ℓ

$$\begin{aligned}
\ell\, a_2^{(6)}(\ell) &= \ell - 1, \\
3\ell^2 a_3^{(6)}(\ell) &= 722\ell^2 - 1800\ell + 1078, \\
\ell^3 a_4^{(6)}(\ell) &= 3970\ell^3 - 10140\ell^2 + 7250\ell - 1080, \\
\ell^4 a_5^{(6)}(\ell) &= 19920\ell^4 - 36920\ell^3 - 9480\ell^2 + 47000\ell - 20520, \\
3\ell^5 a_6^{(6)}(\ell) &= 137290\ell^5 - 171630\ell^4 - 234890\ell^3 + 300150\ell^2 + 98680\ell - 129600, \\
7\ell^6 a_7^{(6)}(\ell) &= 377404\ell^6 - 286944\ell^5 - 897680\ell^4 + 558600\ell^3 \\
&\qquad + 682276\ell^2 - 271656\ell - 162000, \\
\ell^6 a_8^{(6)}(\ell) &= 31808\ell^6 - 11200\ell^5 - 87808\ell^4 + 22400\ell^3 + 80192\ell^2 - 11200\ell - 24192, \\
3\ell^6 a_9^{(6)}(\ell) &= 22400\,(\ell^2 - 1)^3,
\end{aligned} \tag{5}$$

and $a_V^{(6)} = 0$ for $V \leq 1$ and for $V \geq 10$. The last line says $a_9^{(6)} = \frac{22400}{3} B^3$; see Observation 5.5.

Theorem 4.1. *Let $a_V^{(6)}$ be as in (5). Then for every pair (n, ℓ) of the grid $\mathcal{G}$ of (1) — that is, for $\ell = 2$ and $n \leq 12$, for $\ell = 3$ and $n \leq 8$, for $\ell = 4$ and $n \leq 7$, for $\ell = 5$ and $n \leq 6$, and for $\ell = 6$ and $\ell = 7$ with $n \leq 5$ —*

$$\mu_6(n, \ell) = \sum_{V=2}^{9} \binom{n}{V} a_V^{(6)}(\ell).$$

Granting Theorem 3.3 and Corollary 3.4, together with the off-kernel evaluations recorded in the proof, the same identity holds for every $n \geq 0$ and every $\ell \geq 1$.

Proof sketch. Two independent claims are being made and they rest on different things.

The identity *on* $\mathcal{G}$ rests on exhaustive computation, and on nothing else. For each of the 49 pairs, $\mu_6(n,\ell)$ is computed from Definition 2.1 by summing $S(x)^6$ over all ℓ^n words — $2^{12} = 4096$ words at the largest binary point, $3^8 = 6561$, $4^7 = 16384$, $5^6 = 15625$, $6^5 = 7776$ and $7^5 = 16807$ at the largest points of the other five columns — and compared, in exact rational arithmetic, with the right-hand side. All 49 comparisons are equalities. No induction, no generating function and no algebraic identity enters; the finite search is the enumeration of those $\sum_{(n,\ell)\in\mathcal{G}} \ell^n$ words.

The passage *from* $\mathcal{G}$ *to all* n rests on Theorem 3.3, which is proved above as ordinary mathematics: both sides are polynomials in n of degree at most 9, and the grid contains the 13 values $n = 0, 1, \ldots, 12$ at $\ell = 2$, which is three more than the ten a degree-9 polynomial needs. For $\ell \geq 3$ the grid alone does not contain 10 values of n and the coefficients were instead determined, off the kernel, as follows: exact values $\mu_6(m,\ell)$ for $m \leq 9$ and for consecutive ℓ starting at $\ell = 1$ were produced by a count-vector dynamic program in exact rational arithmetic, the $a_V^{(6)}(\ell)$ were obtained from them by the forward-difference formula of Theorem 3.3, and each numerator was then fitted as a polynomial in ℓ from one more value of ℓ than its degree and *verified at six further values of* ℓ *off the fitting grid*. The finished formula was checked against 134 exact values of μ_6 with $\ell \leq 10$ ($n \leq 15$ for $\ell \leq 4$, $n \leq 12$ for $\ell \leq 6$, $n \leq 10$ for $\ell \leq 10$) and at $\ell = 11, 12, 13, 17, 25$ for $n \leq 10$, with no mismatch; the values with $n \geq 10$ are held out from the determination and are what tests the degree bound.

The passage *to all* ℓ is Corollary 3.4, and this is where those checks are spent. At each of the fifteen values $\ell \in \{1, 2, \ldots, 13, 17, 25\}$ the check above covers $n = 0, 1, \ldots, 9$, so at each of them binomial inversion turns the agreement of the two sides into $a_V^{(6)}(\ell) = \Delta^V \mu_6(0,\ell)$ for every $V \leq 9$. Both $\ell^V a_V^{(6)}(\ell)$, by inspection of (5), and $\ell^V \Delta^V \mu_6(0,\ell)$, by Corollary 3.4, are polynomials in ℓ of degree at most $V \leq 9$; fifteen agreements are more than the ten such a polynomial admits, so the two agree at every $\ell \geq 1$. □

Proposition 4.2 (Consistency with the source's printed shapes)**.** *Write* $\mu_2^{\mathrm{N}}(n,\ell) = \sum_{V\leq 3} \binom{n}{V} a_V^{(2)}(\ell)$ *and* $\mu_4^{\mathrm{N}}(n,\ell) = \sum_{V\leq 6} \binom{n}{V} a_V^{(4)}(\ell)$ *with*

$$a_2^{(2)} = A, \quad a_3^{(2)} = \tfrac{2}{3}B; \qquad a_2^{(4)} = A, \quad a_3^{(4)} = \frac{74\ell^2 - 144\ell + 70}{3\ell^2}, \quad a_4^{(4)} = \frac{74\ell^3 - 84\ell^2 - 62\ell + 72}{\ell^3},$$

$$a_5^{(4)} = \frac{384\ell^4 - 200\ell^3 - 600\ell^2 + 200\ell + 216}{5\ell^4}, \qquad a_6^{(4)} = \tfrac{80}{3}B^2.$$

Then $\mu_2^{\mathrm{N}}(n,\ell)$ *equals the right-hand side of* (2) *and* $\mu_4^{\mathrm{N}}(n,\ell)$ *equals the right-hand side of* (3)*, for every* $n < 40$ *and every* $\ell \in \{2, 3, 4, 5, 6, 7, 11, 25\}$*; and* $\sum_{V\leq 9} \binom{n}{V} a_V^{(6)}(2)$ *equals the right-hand side of the second formula of* (4)*, for every* $n < 40$.

Proposition 4.2 is a pointwise identity between two printed expressions, not a new result; its role is to show that the Newton presentation is the same object the source writes in expanded form, so that (2) and (3) can be read off a table of a_V, and that at $\ell = 2$ the new formula collapses onto the source's published binary μ_6.

Remark 4.3 (The binary formula is not the general one). The collapse at $\ell = 2$ does not make Theorem 4.1 a restatement of (4). At $\ell = 3$ and $n = 4$ the true value is $\mu_6(4,3) = 45412/27$, while the source's binary polynomial evaluates to $1405/2$. A general-ℓ sixth moment is therefore a different object from the published one.

Proposition 4.4 (Negative controls)**.** *Each of the following four perturbations of Theorem 4.1 is refuted by a single exact value of* μ_6*:*

(1) *replacing* $a_9^{(6)}$ *by* $a_9^{(6)} + \frac{1}{3}$ *— refuted at* $(n,\ell) = (9,2)$*;*

(2) *replacing the coefficient* 722 *of* $a_3^{(6)}$ *by* 723 *— refuted at* $(n,\ell) = (3,3)$*;*

(3) *truncating the sum at $V \le 8$, i.e. asserting degree ≤ 8 in n — refuted at $(n,\ell) = (9,2)$;*
(4) *adjoining a tenth term $\binom{n}{10}A$, i.e. asserting degree 10 — refuted at $(n,\ell) = (10,2)$.*

Moreover $\mu_2(3,2) \neq 0$, so Proposition 2.2 is not the vacuous assertion that all moments vanish, and S is not identically zero.

Items (3) and (4) bracket the answer from both sides: one degree too small and one degree too large both fail, at the first value of n where they can.

5. The eighth moment

The same machine runs once more. For $r = 8$ the bound of Theorem 3.3 is 12, so thirteen exact values of n determine the polynomial; the expensive corner of the determination is a count-vector dynamic program at $\ell = n = 12$, some $2.7 \cdot 10^6$ states, which took 647 seconds and 1.2 gigabytes for the whole sweep $\ell = 1, \dots, 12$ in exact rational arithmetic.

$$
\begin{aligned}
\ell\, a_2^{(8)}(\ell) &= \ell - 1,\\
3\ell^2 a_3^{(8)}(\ell) &= 6554\ell^2 - 18144\ell + 11590,\\
\ell^3 a_4^{(8)}(\ell) &= 147714\ell^3 - 510804\ell^2 + 573258\ell - 210168,\\
5\ell^4 a_5^{(8)}(\ell) &= 11465664\ell^4 - 35032200\ell^3 + 27237000\ell^2 + 5800200\ell - 9470664,\\
\ell^5 a_6^{(8)}(\ell) &= 15097684\ell^5 - 36285900\ell^4 - 1783460\ell^3\\
&\qquad + 60694620\ell^2 - 44577344\ell + 6854400,\\
\ell^6 a_7^{(8)}(\ell) &= 52973144\ell^6 - 96276096\ell^5 - 91897120\ell^4 + 233635920\ell^3\\
&\qquad - 6768664\ell^2 - 151391184\ell + 59724000,
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\ell^7 a_8^{(8)}(\ell) &= 109333336\ell^7 - 145093088\ell^6 - 295573040\ell^5 + 403281760\ell^4\\
&\qquad + 253634584\ell^3 - 359815232\ell^2 - 67374720\ell + 101606400,\\
3\ell^8 a_9^{(8)}(\ell) &= 412340224\ell^8 - 377992512\ell^7 - 1354858176\ell^6 + 1110162816\ell^5\\
&\qquad + 1621675776\ell^4 - 1087668288\ell^3 - 827333824\ell^2 + 355497984\ell + 148176000,\\
5\ell^8 a_{10}^{(8)}(\ell) &= 518528384\ell^8 - 294470400\ell^7 - 1888195200\ell^6 + 882470400\ell^5\\
&\qquad + 2562306432\ell^4 - 881529600\ell^3 - 1534140800\ell^2 + 293529600\ell + 341501184,\\
3\ell^8 a_{11}^{(8)}(\ell) &= 129704960\ell^8 - 34496000\ell^7 - 500881920\ell^6 + 103488000\ell^5\\
&\qquad + 724416000\ell^4 - 103488000\ell^3 - 465006080\ell^2 + 34496000\ell + 111767040,\\
9\ell^8 a_{12}^{(8)}(\ell) &= 68992000\,(\ell^2 - 1)^4,
\end{aligned}
\tag{7}
$$

and $a_V^{(8)} = 0$ for $V \le 1$ and for $V \ge 13$.

Theorem 5.1. *Let $a_V^{(8)}$ be as in (6)–(7). Then for every pair (n,ℓ) of the grid $\mathcal{G}$ of (1) — $\ell = 2$ with $n \le 12$, $\ell = 3$ with $n \le 8$, $\ell = 4$ with $n \le 7$, $\ell = 5$ with $n \le 6$, and $\ell = 6, 7$ with $n \le 5$ —*

$$\mu_8(n,\ell) = \sum_{V=2}^{12} \binom{n}{V} a_V^{(8)}(\ell),$$

and, granting Theorem 3.3 and Corollary 3.4 together with the off-kernel evaluations recorded in the proof, for every $n \ge 0$ and every $\ell \ge 1$.

Proof sketch. As for Theorem 4.1. On $\mathcal{G}$ the statement is 49 exact comparisons, each obtained by summing $S(x)^8$ over all ℓ^n words; that exhaustive enumeration is the entire content, and it is what the machine check reproduces. Off the grid, the $a_V^{(8)}(\ell)$ were determined by the forward-difference formula of Theorem 3.3 from exact values with $n \le 12$ obtained by the count-vector dynamic program, each numerator being fitted as a polynomial in ℓ and verified at values of ℓ outside the fitting set. The finished formula agrees with 176 exact values at $\ell = 1, \dots, 12$ ($n \le 16$ for $\ell \le 4$, $n \le 14$ for $\ell \le 6$, $n \le 12$ above), of which 20 have $n \ge 13$ and are held out from the determination, and with the values at $\ell = 13, 17, 25$ for $n \le 12$; there is no mismatch. The passage to all ℓ is again Corollary 3.4: at each of the fifteen values $\ell \in \{1, 2, \dots, 13, 17, 25\}$ those checks cover $n = 0, 1, \dots, 12$ and so give $a_V^{(8)}(\ell) = \Delta^V \mu_8(0, \ell)$ for every $V \le 12$, and fifteen agreements exceed the thirteen that a polynomial of degree at most 12 in ℓ admits. □

Corollary 5.2 (The binary eighth moment)**.** *For every $n < 40$,*

$$\sum_{V=2}^{12} \binom{n}{V} a_V^{(8)}(2) = \frac{n(n^2-1)P(n)}{34560},$$

where

$$P(n) = 175n^9 - 1260n^8 + 3111n^7 + 1380n^6 - 23919n^5 + 37740n^4 + 54509n^3 - 192084n^2 - 33876n + 310896.$$

Together with Theorem 5.1 this evaluates $\mu_8(n, 2)$ for $n \le 12$, and, granting Theorem 3.3, for every n.

Corollary 5.2 is the next nonzero entry of the source's own §3 list, which stops after $\mu_6(n, 2)$ and $\mu_7(n, 2) = 0$ with an ellipsis. It is written in the shape the source uses there: the factor $n(n^2 - 1)$ in front, and a numerator of degree $12 = 3 \cdot 8/2$ in total.

Proposition 5.3 (Degree and controls at $r = 8$)**.** $\Delta^{12}\mu_8(0, 2) \ne 0$ *while* $\Delta^{13}\mu_8(0, 2) = \Delta^{14}\mu_8(0, 2) = 0$*: the degree of μ_8 in n is exactly* 12*. Replacing $a_{12}^{(8)}$ by $a_{12}^{(8)} + \frac{1}{9}$ is refuted by the single value $\mu_8(12, 2)$, and truncating the sum at $V \le 11$ is refuted by the same value.*

Two structural checks. Neither of the following enters any argument above; both are consistency checks on the coefficients as displayed, and both hold.

Observation 5.4. Every right-hand side in (5)–(7) vanishes at $\ell = 1$, so $a_V(1) = 0$ for all $V \ge 2$ and $\mu_6(n, 1) = \mu_8(n, 1) = 0$, as they must be: over a one-letter alphabet $S \equiv 0$. For (6)–(7) that is eleven separate linear conditions, one on each of the eleven numerators, all satisfied.

Observation 5.5 (Asymptotic normality fixes the top coefficient). By (2), $\mu_2(n, \ell) = \frac{B}{9}n^3 + O(n^2)$. If S/σ is asymptotically standard normal — which the source asserts in its (normal) and proves there for $\ell = 2$, and which our formulas are not used to prove — then $\mu_{2k}(n, \ell) \sim (2k-1)!!\,(B/9)^k n^{3k}$, and since $\binom{n}{3k} \sim n^{3k}/(3k)!$ the top Newton coefficient must be

$$a_{3k}(\ell) = \frac{(3k)!\,(2k-1)!!}{9^k}\, B^k.$$

For $k = 1, 2, 3, 4$ the right-hand side is $\frac{2}{3}B$, $\frac{80}{3}B^2$, $\frac{22400}{3}B^3$, $\frac{68992000}{9}B^4$ — exactly the entries $a_3^{(2)}$, $a_6^{(4)}$ of Proposition 4.2 and $a_9^{(6)}$, $a_{12}^{(8)}$ of (5) and (7). (The source prints the Gaussian constant in (normal) as $1 \cdot 2 \cdot 5 \cdots (2k-1)$; $1 \cdot 3 \cdot 5 \cdots (2k-1) = (2k-1)!!$ is what is meant, and what its own binary μ_4 and σ^2 give at $k = 2$, namely 3.)

Remark 5.6 (The denominator in ℓ). Corollary 3.4 makes $\ell^V a_V(\ell)$ a polynomial of degree at most V, so $\ell^{3r/2}\mu_r(n, \ell)$ is a polynomial in ℓ. The displays (5)–(7) say more: the denominators are $\ell^2, \ell^4, \ell^6, \ell^8$ for $r = 2, 4, 6, 8$, that is ℓ^r and not $\ell^{3r/2}$. That sharper statement is an observation about the computed coefficients, made outside the formal development and not proved here; closing the gap is a small self-contained question. (The polynomials $\ell^V a_V$ are integer-valued but not, in general, integer: at $r = 6$, $V = 3$ the value is $\ell(722\ell^2 - 1800\ell + 1078)/3$.)

6. The source's own formulas, re-derived

Nothing in this section is new; it is the gate that was run before anything was claimed past the source, and we report it because one of its entries is the first verification of a published formula that has never been proved.

Theorem 6.1. *Let $\mathcal{G}$ be the grid* (1)*.*

1. *For every $(n,\ell)\in\mathcal{G}$, $\mu_2(n,\ell)$ computed from Definition 2.1 equals the right-hand side of the source's variance formula* (2)*.*
2. *For every $(n,\ell)\in\mathcal{G}$, $\mu_4(n,\ell)$ computed from Definition 2.1 equals the right-hand side of the source's fourth-moment formula* (3)*.*
3. *For every $n\le 12$, $\mu_4(n,2)$ and $\mu_6(n,2)$ computed from Definition 2.1 equal the two binary formulas* (4)*.*

Part (1) verifies a formula that the source proves. Part (2) is the one worth naming: the source states (3) "without proof, when $\ell\ge 3$", and no proof of it appears anywhere we could find, so the 49 kernel evaluations of Theorem 6.1(2) — together with a further check outside the kernel, at $\ell=2,3$ for $n\le 12$, $\ell=4$ for $n\le 9$ and $\ell=5,6,7$ for $n\le 7$ — are, as far as we know, the first verification of it from the definition. We claim no new formula there. We do observe that Theorem 3.3 reduces the missing proof, at each fixed ℓ, to a finite computation: (3) is a polynomial of degree 6 in n, so the seven values $n=0,\dots,6$ already pin it, and those are inside the grid at $\ell\le 5$ and inside the reach of the count-vector program at every ℓ.

Theorem 6.2 (The source's §3 frequency table)**.** *For $\ell=2$ and $1\le n\le 7$, the number of binary words of length n with $S=t$, computed from Definition 2.1, agrees with the source's §3 display in all seven rows. For $n=7$ the counts for $t=-13,-12,\dots,13$ are*

$$0,2,0,4,0,6,0,12,0,14,0,16,0,20,0,16,0,14,0,12,0,6,0,4,0,2,0.$$

Proposition 6.3 (The score, and a correction to the source's worked example)**.** (1) *For every n, every ℓ and every word x, $S(x)=S^+(x)-S^-(x)$ with $S^\pm$ as in §1; so Definition 2.1 is the source's definition.*

(2) *The v1 e-print, in the sentence of its §1 that follows* [1, (2)]*, reads "For example, $S=5-11=-6$ for the sequence $0\,1\,1\,2\,0\,2\,1$." Counted from that same definition, the word $0\,1\,1\,2\,0\,2\,1$ has $S^+=11$ and $S^-=5$, hence $S=+6$: the two labels are exchanged.*

The exchange is not confined to the example. The source's §3 table for $n=3$ assigns the value 2 to the word $1\,0\,0$, where the definition gives -2, and the generating function $\prod_{k=1}^{n}(1+x^{\,n+1-2k})$ it derives there is the one for $S^- - S^+$. So everything the paper computes — the §1 example, the §3 table, the generating function — follows the convention $S^- - S^+$ consistently, and the outlier is the sign in its definition line [1, (2)]. We record this courteously and for completeness only, and against the v1 e-print, still the only version on 2026-09-03: by Proposition 2.2 the law of S is symmetric, the source states only even moments, and every one of them — and every formula in the present paper — is unchanged by the global sign. Nothing in [1] is affected.

7. Further questions

1. *The tenth moment.* Degree 15 in n, so the table needs $\mu_{10}(m,\ell)$ for $m\le 15$, and the count-vector program at $\ell=n=15$ has $\binom{30}{15}=1.55\cdot 10^8$ states against the $\binom{24}{12}=2.7\cdot 10^6$ that μ_8 needed. Scaling by states, transitions and moment order prices the sweep at roughly 60 CPU-hours and 70 gigabytes in our exact-arithmetic implementation, which is why it is not attempted here. A compiled implementation would be working close to the edge of machine arithmetic, but not past it: at $n=15$ one has $|S|\le\binom{15}{2}=105$, so the unnormalised tenth power sum over all 15^{15} words is at most $105^{10}\cdot 15^{15}=7.13\cdot 10^{37}$, a 126-bit number, which fits a signed 128-bit accumulator with a factor of about 2.4 to spare. The margin is thin,

and it does not by itself cover the quantities formed on the way: the recursion expands $\sum(S+d)^{10} = \sum_j \binom{10}{j} d^{10-j} \sum S^j$ with $|d| \le n-1$, and the crude bound on that expansion, $(105+14)^{10} \cdot 15^{15} = 2.49 \cdot 10^{38}$, is already past 2^{127}. An implementation at this width would have to bound the intermediates term by term, or work modularly.

(2) *The degree in ℓ* — Remark 5.6: prove $\ell^r \mu_r$ polynomial in ℓ, rather than the $\ell^{3r/2}$ the present argument gives.

(3) *The biased binary case.* The source also gives a variance and a fourth moment for $\mathbb{P}(0) = p$, $\mathbb{P}(1) = q$, and nothing higher. There $S = \sum_k (2k-n-1)x_k$ with x_k independent Bernoulli, so μ_{2m} is an explicit polynomial in the power sums of $(2k-n-1)$ and in the Bernoulli central moments; this is elementary and, as far as we know, unwritten.

(4) *Edgeworth.* The source's Edgeworth correction uses μ_4/σ^4 only, and it reports the dependence on ℓ as puzzling and in need of further numerical investigation. With μ_6 in hand the next term is available.

8. Verification

Every statement above that carries a theorem, proposition, corollary or remark number, with the exceptions named next, has been formalised and machine-checked in Lean 4 [7] (toolchain `leanprover/lean4:v4.32.0`) against *Mathlib* [8] at its tag `v4.32.0`, commit `81a5d257`, in a development of eight files and 742 lines in which S, the moments and the closed forms are defined exactly as in Definition 2.1 and (5)–(7), and the moments are always evaluated by summing over the full word space $\{0, \dots, \ell-1\}^n$ — never from a recursion, a generating function or a previously verified formula. The exceptions are of two kinds. First, Theorem 3.3 (equivalently Theorem 1.3) and its Corollary 3.4: of these, Step 3 — Lemma 3.1, the only step with content beyond bookkeeping — is formalised, while Steps 0, 1, 2 and 4 and the corollary are written out above as ordinary mathematics. Second, Observations 5.4 and 5.5 and Remark 5.6, which are consistency checks on the computed coefficients and are labelled as such where they occur; nothing depends on those three. Accordingly the "for every n and every ℓ" halves of Theorems 4.1 and 5.1 are conditional on Theorem 3.3 and Corollary 3.4 and on the off-kernel evaluations reported in their proofs, and it is the grid statements that are machine-checked outright.

Two levels of trust are involved and it is worth separating them. Checked by the Lean kernel alone, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, are Proposition 2.2 in both its forms (the reversal identity and the vanishing of every odd moment, for all n and all ℓ), Lemma 3.1 in both its forms, and Proposition 6.3(1). Everything else is a finite evaluation carried out by Lean's compiled evaluator (`native_decide`) and therefore depends, in addition, on one evaluation axiom of its own: the grid evaluations behind Theorems 6.1, 4.1 and 5.1 (49 parameter pairs for each of $\mu_2, \mu_4, \mu_6, \mu_8$, and 13 values of n for each binary formula), the arithmetic identities of Proposition 4.2 and Corollary 5.2, the forward differences of Propositions 3.5 and 5.3, the controls of Proposition 4.4 and Remarks 3.2 and 4.3, and the two computations of Proposition 6.3(2) and Theorem 6.2. That evaluator is the only trusted step. There is no `sorry` anywhere in the development and it declares no axiom of its own.

The grid (1) is bounded by evaluation cost — ℓ^n words per point, each scored over $\binom{n}{2}$ position pairs — and not by the mathematics. Independently of Lean, two exact rational evaluators written separately (direct enumeration, and a dynamic program over count vectors) agree with each other and with all the formulas above at every point where they overlap, which is 402 values of μ_2, μ_4, μ_6 at $\ell \le 10$ and 176 values of μ_8 at $\ell \le 12$, together with spot checks at $\ell = 13, 17, 25$. The repository is private; repository reference to be added at submission.

Acknowledgement of the source

This paper exists because [1] poses the question, computes the two moments that can be computed by hand, and says plainly which calculation it did not do and which method it could not find. Its §1 also draws the distinction from the classical ties-fixed theory that makes the question well posed.

References

[1] P. Lindqvist, *A test against trend in random sequences*, arXiv:1906.00752 [math.ST], 2019. Cited throughout as the source. All quotations, section numbers and equation numbers above are from the v1 e-print, which on 2026-09-03 is still the only version and still carries no journal reference; its only DOI is the one arXiv assigns automatically, `doi:10.48550/arXiv.1906.00752`. The mnemonic names (var), (four) and (normal) are labels of its LaTeX source, and (2), (4), (5), (6) and (10) are the numbers those labels and the characteristic function print as. Its section numbering, used above, is §1 Introduction, §2 The Basic Results, §3 The Probability Function, §4 Approach to Normality, §5 Binary Sequences not Equiprobable, §6 The Variance (with General ℓ), §7 Edgeworth's Approximation. Its arXiv comment field reads: "This is essentially a revision of REPORT-MAT-A234, Helsinki University of Technology 1986. (Some details are from REPORT-MAT-A131 (1978).)"

[2] P. Lindqvist, *A test against trend in random sequences*, Report-Mat-A234, Helsinki University of Technology, 1986, pp. 1–23. Cited as [L2] in [1]; the report is not digitised and we could not obtain a copy, so this entry reproduces the bibliography entry of [1], page range included.

[3] P. Lindqvist, *On a statistical "trend-test"*, Report-Mat-A131, Helsinki University of Technology, 1978, pp. 1–13. Cited as [L1] in [1], which says of it, in the same paragraph of its §6 quoted above, "Corresponding formulae for S^+ are given in [L1]". Not digitised; as for [2], this entry reproduces the bibliography entry of [1], page range included.

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[6] H. Silverstone, *A note on the cumulants of Kendall's S-distribution*, Biometrika **37** (1950), no. 3–4, 231–235; `doi:10.1093/biomet/37.3-4.231`.

[7] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, Cham, 2021, pp. 625–635; `doi:10.1007/978-3-030-79876-5_37`.

[8] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; `doi:10.1145/3372885.3373824`.