THE TREK-IDENTIFICATION LINEAR PROGRAM IS NOT INTEGRAL ON ITS TIGHT FACE

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Abstract. Sturma and Drton (arXiv:2507.18170) give a graphical criterion for the identifiability of parameters of a linear structural equation model with arbitrarily structured latent variables, together with a sound and complete algorithm for testing it. The algorithmic core of that test is an integer linear program $\mathrm{Lp}(G, G_1, Z, P, Y_\mathrm{a})$: a two-commodity packing of vertex-disjoint directed paths in which the two commodities leave a common super-source, share unit node capacities, and one of them is confined to a subgraph $G_1 \subseteq G$. Its value never exceeds $|Z| + |P|$, and their Theorem 4.1 says that an integer solution of that value exists exactly when the path system the criterion needs exists. They conjecture that the linear relaxation is integral on its tight face — whenever the optimal value of the relaxation equals $|Z|+|P|$, an integer optimum of that value exists — and observe that with the conjecture their identifiability algorithm, run with a bound on the size of the latent sets it searches over, would decide identifiability in polynomial time. We show that the conjecture is false. On an explicit instance with $|V| = 8$ and $|D| = 12$ the relaxation attains $|Z| + |P| = 2$ at a half-integral point, while every integer feasible point has objective at most 1 and the value 1 is attained: the integrality gap of the instance is exactly 1. The failure is localised by three controls: the worked example of the source reproduces exactly; on the same graph with $D_1 = D$ an integer optimum of value 2 exists, so the subgraph restriction and not the graph is responsible; and deleting the single arc $s_1 \to w_{11}$ already drops the optimal value below 2, so the instance is not padded. Outside the machine-checked development we report an enumeration showing that no counterexample of this shape exists on fewer than eight nodes and that there are exactly three on eight nodes up to relabelling, and a reduction from the directed two-vertex-disjoint-paths problem showing that deciding the existence of an integer solution of value $|Z| + |P|$ is NP-hard — so the conjecture would in any case have implied P = NP. Nothing here touches the criterion itself, its completeness, or the correctness of the integer program; what falls is the conjectured polynomial-time shortcut. Every statement labelled Theorem below has been formally verified in Lean 4 against *Mathlib*; the two facts reported in Section 5 are marked as lying outside that development.

1. Introduction

1.1. **Identification by trek systems.** A linear structural equation model over a directed graph on a set of variables postulates that each variable is a linear function of its parents plus noise, and the identification problem asks which entries of the parameter matrix are determined — generically, as rational functions — by the covariance matrix of the observed variables. The trek-based criteria for this problem certify identifiability by exhibiting a system of treks with no sided intersection. The half-trek criterion of Foygel, Draisma and Drton [2] requires "a system of half-treks with no sided intersection from Y to $\mathrm{pa}(v)$"; the latent-factor half-trek criterion of Barber, Drton, Sturma and Weihs [3], which works on a graph carrying the latent nodes explicitly rather than on a latent projection, requires "a system of latent-factor half-treks with no sided intersection from Y to $Z \cup \mathrm{pa}_V(v)$", with a further condition on the treks that end in Z; and the determinantal criteria of Weihs et al. [4] again require a half-trek system with no sided intersection, together with rank conditions. In each of these the search for the certificate is carried out by ordinary single-commodity maximum-flow computations with integer capacities, and the optimum of such a flow may be taken integral and read off as a family of unit-flow paths — both [2] and [3] argue in exactly that way. Integrality is free there.

Sturma and Drton [1] give a criterion of this family — the *latent-subgraph criterion* — for models whose latent structure is arbitrary, together with an algorithm that is sound and complete for it. Their certificate is more demanding than a single flow: it asks for a system of vertex-disjoint directed paths from a set of allowed source nodes to two prescribed target sets, in which the paths ending in one of the two targets are confined to a prescribed subgraph. They encode the search for such a system as an integer linear program $\mathrm{Lp}(G, G_1, Z, P, Y_\mathrm{a})$, recalled in full in Section 2, and prove ([1, Theorem 4.1]) that the program has an integer solution of value $|Z|+|P|$ exactly when the required path system exists.

1.2. **The conjecture.** The value of $\mathrm{Lp}(G, G_1, Z, P, Y_\mathrm{a})$ never exceeds $|Z| + |P|$ (Lemma 2.1), so the relevant question is always whether the value $|Z|+|P|$ is attained, and, if it is attained by the relaxation, whether it is attained integrally. The authors report ([1, Section 4.1], verbatim):

> "In all our computations, whenever the optimal value of $\mathrm{Lp}(G, G_1, Z, P, Y_a)$ was $|Z|+|P|$, it also existed an integer solution. If this is generally true, it would be sufficient to solve only the linear program, which is possible in polynomial time, instead of solving the integer linear program, which is NP-complete."

and state it as an open problem ([1, Conjecture 4.3], verbatim):

> "If the optimal value of $\mathrm{Lp}(G, G_1, Z, P, Y_a)$ is $|Z|+|P|$, then there is an integer solution."

The conjecture is invoked twice more. In [1, Remark 4.8], about the identifiability algorithm of [1, Section 4.3]: "If we only allow sets H_1, H_2 with $|H_1| + |H_2| \leq k$ in line 3 for fixed $k \in \mathbb{N}$, then the algorithm solves the integer linear program at most $\mathcal{O}^{2+k}\mathcal{L}^{2k}$ times. Therefore, with this restriction, the algorithm is polynomial time if Conjecture 4.3 is true." And in the discussion: "We believe that solving Conjecture 4.3, whether the program can be solved in polynomial time, is of independent interest." Apart from the paragraph quoted above, which immediately precedes the conjecture, these two are the only places in [1] where the conjecture is invoked. The sets H_1, H_2 of Remark 4.8 are the latent sets that the algorithm searches over, and $\mathcal{O}$, $\mathcal{L}$ are the observed and latent node sets of the model; we do not reproduce that algorithm and claim nothing about it beyond what its own remark says.

The conjecture is an integrality statement for a two-commodity problem, and the shape of the program is the one in which integrality classically fails. Constraint (iv) below makes the sum $f + f^1$ a single-commodity flow with unit node capacities, so the *total* throughput is governed by an integral polytope; what can be fractional is the split of that throughput between the two commodities. Two-commodity flow is the classical home of that phenomenon — see Hu [5], Rothschild and Whinston [6], and Even, Itai and Shamir [7] — but we appeal to no result from that literature, and none of the results below depends on one. The reason is that $\mathrm{Lp}(G, G_1, Z, P, Y_\mathrm{a})$ differs from the classical model in four respects at once, each visible in the statement of the program in Section 2: it is directed; its capacities (ii) and (iv) sit on nodes rather than on arcs; both commodities leave a single super-source s; and (iii) confines one of them to a subgraph. Conjecture 4.3 therefore has to be settled on its own terms.

1.3. **Results.** The conjecture is false. We exhibit an instance $\mathcal{C}$ with $|V| = 8$, $|D| = 12$, $|Z| = |P| = 1$ and $|Y_\mathrm{a}| = 2$ (Section 3) and prove:

- the optimal value of $\mathrm{Lp}(\mathcal{C})$ is exactly $|Z| + |P| = 2$, attained at an explicit half-integral point, so the hypothesis of the conjecture is satisfied (Theorem 3.1);
- every integer feasible point of $\mathrm{Lp}(\mathcal{C})$ has objective at most 1, and the value 1 is attained, so the integer optimum is exactly 1 and the integrality gap of $\mathcal{C}$ is exactly 1 (Theorem 3.2);
- hence Conjecture 4.3 of [1] fails (Theorem 3.3).

Section 4 localises the failure. The worked example of [1] reproduces in our encoding, with the integer point printed there feasible and of value 2 (Theorem 4.1); on the same graph with $D_1 = D$ there is an integer point of value 2 (Theorem 4.2), so the obstruction is the interaction of the subgraph restriction (iii) with the joint node capacity (iv) rather than the graph; and deleting the single arc $s_1 \to w_{11}$ from both D_1 and D drops the optimal value to at most $3/2$ (Theorem 4.3), so the instance is not a padded version of a smaller one.

Section 5 reports two facts established outside the machine-checked development and labelled as such: an enumeration of the minimal counterexamples of this shape, which finds none on fewer than eight nodes and exactly three on eight nodes up to relabelling; and a proof that deciding whether $\mathrm{Lp}(G, G_1, Z, P, Y_\mathrm{a})$ has an integer solution of value $|Z|+|P|$ is NP-hard, by reduction from the directed two-vertex-disjoint-paths problem. The second gives an independent, conditional reason for the conjecture to fail: had it been true, the polynomial-time solvability of the relaxation would have decided an NP-hard problem.

1.4. **What is not claimed, and what remains open.** Nothing here contradicts [1, Theorem 4.1], the latent-subgraph criterion, or the soundness and completeness of the algorithm built on it: the integer program remains a correct encoding of the certificate. What falls is the conjectured shortcut that would have replaced it by its relaxation. Two questions are left open and are stated precisely in Section 6: whether the conjecture also fails on the two-layer graphs that the reduction of [1, Section 4.2] actually constructs — our instance is provably not of that shape — and whether counterexamples exist with $|Z| + |P| \geq 3$, which our enumeration does not cover.

2. The linear program

We recall the program of [1, Section 4.1] verbatim, in its own notation.

Let $G = (V, D)$ be a directed graph, let $G_1 = (V, D_1)$ with $D_1 \subseteq D$, and let $Z, P \subseteq V$ be disjoint and $Y_\mathrm{a} \subseteq V$. The *flow graph* is

$$G_{\mathrm{flow}}(Z, P, Y_\mathrm{a}) = (V_\mathrm{f}, D_\mathrm{f}), \qquad V_\mathrm{f} = V \cup \{s, t\},$$
$$D_\mathrm{f} \;=\; D \,\cup\, \{s \to y : y \in Y_\mathrm{a}\} \,\cup\, \{z \to t : z \in Z\} \,\cup\, \{p \to t : p \in P\},$$

where s and t are two new nodes. With two weight vectors $f, f^1 \in \mathbb{R}^{D_\mathrm{f}}$, the program $\mathrm{Lp}(G, G_1, Z, P, Y_\mathrm{a})$ is

$$\text{(1)} \qquad \text{maximise} \quad \sum_{z \in Z} f^1_{zt} \;+\; \sum_{p \in P} f_{pt}$$

$$\text{(i)} \qquad \text{subject to} \quad f^1_{uv} \geq 0 \;\text{ and }\; f_{uv} \geq 0 \qquad \text{for all } u \to v \in D_\mathrm{f},$$

$$\text{(ii)} \qquad \sum_{u \in V_\mathrm{f}} f^1_{uv} = \sum_{w \in V_\mathrm{f}} f^1_{vw} \leq 1 \;\text{ and }\; \sum_{u \in V_\mathrm{f}} f_{uv} = \sum_{w \in V_\mathrm{f}} f_{vw} \leq 1 \quad \text{for all } v \in V,$$

$$\text{(iii)} \qquad f^1_{uv} = 0 \quad \text{for all } u \to v \in D \setminus D_1 \text{ and all } p \to t \in D_\mathrm{f} \text{ with } p \in P,$$

$$\text{(iv)} \qquad \sum_{u \in V_\mathrm{f}} f^1_{uv} + \sum_{u \in V_\mathrm{f}} f_{uv} \leq 1 \qquad \text{for all } v \in V.$$

A pair (f, f^1) satisfying (i)–(iv) is *feasible*; it is an *integer point* if all its coordinates are integers. We write $\mathrm{val}(G, G_1, Z, P, Y_\mathrm{a})$ for the optimal value of (1) over feasible points and $\mathrm{val}_\mathbb{Z}(G, G_1, Z, P, Y_\mathrm{a})$ for its optimum over integer feasible points. Both are attained: the zero pair is feasible, and (i) and (ii) force every coordinate of a feasible pair into $[0, 1]$, so the feasible region is a nonempty compact polytope; in particular every integer feasible point has all coordinates in $\{0, 1\}$.

The two commodities are coupled only through (iv): without it, (i)–(iii) would be two independent node-capacitated flow problems, each integral. With it, the program is a fractional relaxation of a packing of vertex-disjoint directed paths of two colours, both leaving the common super-source s, the Z-coloured ones constrained to run inside G_1.

Lemma 2.1 (the a priori bound; [1])**.** *For every feasible* (f, f^1) *the objective* (1) *is at most* $|Z| + |P|$*. Hence* $\mathrm{val}(G, G_1, Z, P, Y_\mathrm{a}) \leq |Z| + |P|$.

Proof. Let $z \in Z$. By (i) and (ii), $f^1_{zt} \leq \sum_{w \in V_\mathrm{f}} f^1_{zw} = \sum_{u \in V_\mathrm{f}} f^1_{uz} \leq 1$, and symmetrically $f_{pt} \leq 1$ for $p \in P$. Summing over Z and P gives the bound. □

The *tight face* of an instance is the face of its feasible polytope on which the objective attains the bound of Lemma 2.1; it is nonempty exactly for the instances about which the conjecture says anything. What the algorithm needs there is an *integer* point, by the following theorem of the source, which we quote verbatim and use only in Remark 3.4 and Section 5.2.

Theorem 2.2 ([1, Theorem 4.1])**.** *"There is an integer solution of* $\mathrm{Lp}(G, G_1, Z, P, Y_a)$ *with optimal value* $|Z| + |P|$ *if and only if there exists* $Y \subseteq Y_a$ *such that there is a system of directed paths with no intersection from* Y *to* $Z \cup P$ *where every path ending in* Z *only has edges in* G_1*."*

Definition 2.3 (the conjecture, as read here)**.** Following the phrasing of Theorem 2.2, an *integer solution* of an instance whose optimal value is $|Z|+|P|$ means an integer feasible point whose objective equals $|Z|+|P|$. Conjecture 4.3 of [1] is then the implication

$$\mathrm{val}(G, G_1, Z, P, Y_{\mathrm{a}}) = |Z| + |P| \implies \mathrm{val}_{\mathbb{Z}}(G, G_1, Z, P, Y_{\mathrm{a}}) = |Z| + |P|.$$

This is the only reading under which the conjecture has content: under the weaker reading "some integer feasible point exists" it holds trivially for every instance, the zero pair being feasible.

3. The counterexample

3.1. **The instance.** Let $\mathcal{C} = (G, G_1, Z, P, Y_{\mathrm{a}})$ be given by

$$V = \{s_1, s_2, w_{11}, w_{12}, w_{21}, w_{22}, z, p\}, \qquad Z = \{z\}, \quad P = \{p\}, \quad Y_{\mathrm{a}} = \{s_1, s_2\},$$

$$D_1 = \{\, s_1 \to w_{11},\ w_{11} \to w_{12},\ w_{12} \to z,\ s_1 \to w_{21},\ w_{21} \to w_{22},\ w_{22} \to z \,\},$$

$$D \setminus D_1 = \{\, s_2 \to w_{11},\ w_{11} \to w_{21},\ w_{21} \to p,\ s_2 \to w_{12},\ w_{12} \to w_{22},\ w_{22} \to p \,\},$$

so $|V| = 8$, $|D| = 12$, $|D_1| = 6$ and $|Z| + |P| = 2$. The graph is acyclic, with topological order $s_1, s_2, w_{11}, w_{12}, w_{21}, w_{22}, z, p$; the flow graph $G_{\mathrm{flow}}(Z, P, Y_{\mathrm{a}})$ has the twelve arcs of D together with $s \to s_1$, $s \to s_2$, $z \to t$ and $p \to t$, sixteen arcs in all.

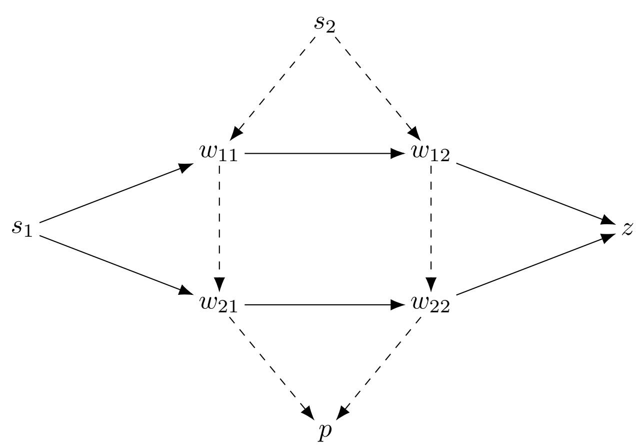

Figure 1. The instance $\mathcal{C}$. Solid arcs are D_1, dashed arcs are $D \setminus D_1$; $Z = \{z\}$, $P = \{p\}$, $Y_{\mathrm{a}} = \{s_1, s_2\}$. The super-source s and the sink t are not drawn. The node w_{ij} is the meeting point of the i-th path to z and the j-th path to p; the 2×2 crossing pattern is what no integral choice can avoid.

3.2. **The relaxation attains the a priori bound.**

Theorem 3.1. *Let (f, f^1) be defined on the flow graph of $\mathcal{C}$ by*

$$f^1 = \tfrac{1}{2} \text{ on each of the six arcs of } D_1, \qquad f^1_{ss_1} = f^1_{zt} = 1, \qquad f^1 = 0 \text{ elsewhere,}$$

$$f = \tfrac{1}{2} \text{ on each of the six arcs of } D \setminus D_1, \qquad f_{ss_2} = f_{pt} = 1, \qquad f = 0 \text{ elsewhere.}$$

Then (f, f^1) is feasible for $\mathrm{Lp}(\mathcal{C})$ and its objective is 2, while every feasible point of $\mathrm{Lp}(\mathcal{C})$ has objective at most 2. Consequently

$$\mathrm{val}(\mathcal{C}) = |Z| + |P| = 2,$$

and the hypothesis of Conjecture 4.3 holds for $\mathcal{C}$.

Proof sketch. Feasibility is a finite check on a fixed rational point: sixteen nonnegativity constraints for each of f and f^1; the conservation and unit-capacity constraints (ii) at each of the eight nodes of V, for each commodity; the seven instances of (iii) — the six arcs of $D \setminus D_1$ and the arc $p \to t$ — on which f^1 indeed vanishes; and the eight joint capacities (iv), each of which is tight, every node of V carrying total weight exactly 1. The value of the objective is $f^1_{zt} + f_{pt} = 1 + 1 = 2$. The upper bound is Lemma 2.1 for this instance, i.e. (ii) at z and at p. All of these are machine-checked; see Section 7. The point was also found independently, as the vertex returned by an exact-rational simplex implementation (Bland's rule, fraction arithmetic, no floating point) run on $\mathrm{Lp}(\mathcal{C})$ from scratch, which reports optimal value 2. □

3.3. **The integer optimum is one.**

Theorem 3.2. *Every integer feasible point (f, f^1) of $\mathrm{Lp}(\mathcal{C})$ has objective at most 1, and the value 1 is attained — for instance by f equal to 1 on each arc of $s \to s_1 \to w_{21} \to p \to t$ and 0 elsewhere, with $f^1 = 0$. Hence*

$$\mathrm{val}_{\mathbb{Z}}(\mathcal{C}) = 1 = \mathrm{val}(\mathcal{C}) - 1,$$

and the integrality gap of $\mathcal{C}$ is exactly 1.

Proof sketch. Attainment is one more finite check on an explicit point. The upper bound is a statement about *all* integer feasible points, not about a finite list of candidates: the constraints (i)–(iv) of $\mathrm{Lp}(\mathcal{C})$, instantiated at the sixteen arcs and the eight nodes of V, are seventy-nine linear inequalities and equations in the 32 integer unknowns $(f_e, f^1_e)_{e\in D_\mathrm{f}}$, and the assertion is that they imply $f^1_{zt} + f_{pt} \le 1$. This is a quantifier-free implication in linear integer arithmetic, and it is discharged as such, by a decision procedure for that fragment, with no enumeration and no case list supplied by hand (Section 7).

Independently, the bound was confirmed by exhaustive search in exact arithmetic. Every integer feasible point has all coordinates in $\{0, 1\}$ (Section 2), so the search space is $\{0, 1\}^{32}$; (ii) and (iii) for f^1 leave exactly three admissible f^1 — the zero vector and the two paths $s \to s_1 \to w_{11} \to w_{12} \to z \to t$ and $s \to s_1 \to w_{21} \to w_{22} \to z \to t$ — and for each of them all 2^{16} vectors f were checked. The maximum objective over all integer feasible points is 1. This computation is an independent check on the machine-checked bound, not the ground of it. □

Theorem 3.3. *Conjecture 4.3 of* [1] *is false. For the instance $\mathcal{C}$ the optimal value of $\mathrm{Lp}(\mathcal{C})$ equals $|Z| + |P| = 2$ and is attained at a rational feasible point, while every integer feasible point of $\mathrm{Lp}(\mathcal{C})$ has objective strictly less than $|Z| + |P|$.*

Proof. Combine Theorems 3.1 and 3.2: the hypothesis of the conjecture holds for $\mathcal{C}$ by the first, and its conclusion fails by the second, since $1 < 2$. □

Remark 3.4 (why the instance works). The machine-checked proof of Theorem 3.2 argues directly from (i)–(iv) and uses nothing about paths. The combinatorial reason, which is what guided the search, runs through Theorem 2.2 instead. The D_1-paths from Y_a to z are exactly $s_1 \to w_{11} \to w_{12} \to z$ and $s_1 \to w_{21} \to w_{22} \to z$, because s_2 has no outgoing D_1-arc and $z \notin Y_\mathrm{a}$. Both begin at s_1, so a D-path to p that is vertex-disjoint from one of them must begin at s_2, i.e. start $s_2 \to w_{11}$ or $s_2 \to w_{12}$. A path starting $s_2 \to w_{11}$ meets the first Z-path at w_{11}, so it must avoid $\{s_1, w_{21}, w_{22}, z\}$; from w_{11} its options are w_{21}, which is forbidden, and w_{12}, from which the only continuations are z and w_{22}, both forbidden. A path starting $s_2 \to w_{12}$ meets the first Z-path at w_{12}, and from w_{12} the options z and w_{22} are again both forbidden. So no vertex-disjoint system exists, and by Theorem 2.2 no integer solution of value 2 exists. Fractionally, the difficulty dissolves: each of the two Z-paths and each of the two P-paths carries weight $1/2$, every node of V is loaded to exactly 1, and (iv) is tight everywhere. The pattern is visible in Figure 1: the four support paths form a 2×2 grid in which every Z-path meets every P-path, so that halving is exactly what makes the two colours fit.

4. Controls

The three statements of this section are not new results; they are checks that Theorems 3.1–3.3 say what they appear to say.

4.1. **The encoding reproduces the source's own numbers.** Before an encoding of Lp is used to contradict anything, it should reproduce the published values of the program it claims to encode.

Theorem 4.1 (the example of [1, Example 4.2, Figure 4(a)])**.** *Let $\mathcal{C}_0$ be the instance of* [1, Example 4.2]*:* $V = \{y_1, y_2, v, w, z, p\}$*,* $D_1 = \{y_1 \to v,\ y_1 \to w,\ y_2 \to v,\ v \to z,\ v \to p\}$*,* $D \setminus D_1 = \{y_2 \to p,\ w \to z\}$*,* $Z = \{z\}$*,* $P = \{p\}$*,* $Y_{\mathrm{a}} = \{y_1, y_2\}$*. Then $\mathcal{C}_0$ is a well-formed instance with $|Z| + |P| = 2$, and the integer point displayed in* [1] *—* $f^1_{sy_1} = f^1_{y_1 v} = f^1_{vz} = f^1_{zt} = 1$ *and* $f_{sy_2} = f_{y_2 p} = f_{pt} = 1$*, all other coordinates* 0 *— is feasible with objective* 2*, exactly as stated there: "Then the tuple* (f, f^1) *is integer-valued and maximizes* $\mathrm{Lp}(G, G_1, Z, P, Y_a)$ *since* $f^1_{zt} + f_{pt} = 2 = |Z| + |P|$*."*

Proof sketch. A finite check on an explicit integer point, of the same shape as in Theorem 3.1; machine-checked. The same exact-rational simplex reports optimal value 2 for $\mathcal{C}_0$. □

4.2. **The subgraph restriction, not the graph, is responsible.**

Theorem 4.2. *Let $\mathcal{C}' = (G, G, Z, P, Y_{\mathrm{a}})$ be the instance obtained from $\mathcal{C}$ by setting $D_1 = D$, everything else unchanged. Then $\mathcal{C}'$ has an integer feasible point of objective $|Z| + |P| = 2$, namely f equal to* 1 *on the arcs of $s \to s_1 \to w_{21} \to p \to t$ and f^1 equal to* 1 *on the arcs of $s \to s_2 \to w_{11} \to w_{12} \to z \to t$, all other coordinates* 0*. With Lemma 2.1 this gives* $\mathrm{val}_{\mathbb{Z}}(\mathcal{C}') = \mathrm{val}(\mathcal{C}') = 2$*.*

Proof sketch. The two paths are vertex-disjoint, so the displayed point satisfies (iv); the remaining constraints are a finite check, machine-checked. The point uses the arc $s_2 \to w_{11}$, which lies in $D \setminus D_1$ and is therefore forbidden to f^1 in $\mathcal{C}$ by (iii). □

So the obstruction of Theorem 3.2 is not the graph G: it is the interaction of the subgraph restriction (iii) with the joint node capacity (iv). Removing either restores an integer optimum of value $|Z| + |P|$.

4.3. **The instance is not padded.**

Theorem 4.3. *Let $\mathcal{C}^-$ be obtained from $\mathcal{C}$ by deleting the single arc $s_1 \to w_{11}$ (from both D_1 and D). Then every feasible point of* $\mathrm{Lp}(\mathcal{C}^-)$ *has objective at most* $3/2$*; in particular* $\mathrm{val}(\mathcal{C}^-) < |Z| + |P| = 2$*, so $\mathcal{C}^-$ does not satisfy the hypothesis of Conjecture 4.3.*

Proof sketch. Linear-programming duality on a fixed instance: a nonnegative combination of the constraints (i)–(iv) of $\mathrm{Lp}(\mathcal{C}^-)$ bounds the objective by $3/2$. The multipliers were found automatically and the resulting inequality is machine-checked. □

Remark 4.4 (the other deletions; outside the machine-checked development)*.* The exact-rational simplex mentioned above was run on all twelve instances obtained from $\mathcal{C}$ by deleting one arc of D. The optimal values are $3/2$ for $s_1 \to w_{11}$, $w_{11} \to w_{12}$, $w_{12} \to z$, $s_1 \to w_{21}$, $w_{12} \to w_{22}$ and $w_{22} \to p$; $5/3$ for $w_{21} \to w_{22}$, $w_{22} \to z$, $s_2 \to w_{11}$, $w_{11} \to w_{21}$ and $w_{21} \to p$; and $7/4$ for $s_2 \to w_{12}$. Every one is strictly below 2, so $\mathcal{C}$ is arc-minimal: no proper subgraph of it satisfies the hypothesis of the conjecture. Deleting any one of the six internal nodes likewise drops the value (to 1 for s_1, s_2, w_{12}, w_{21} and to $3/2$ for w_{11}, w_{22}). Only the first of these eighteen values, the one in Theorem 4.3, is part of the formally verified development; the other seventeen are computations reported here and were obtained in exact rational arithmetic. The value $7/4$ is worth noting on its own: the optimum of that deleted instance is attained at a vertex with a coordinate of denominator 4, so nothing in this family of programs forces half-integrality.

5. Two facts established outside the formal development

The two subsections that follow are *not* part of the machine-checked development. The first is an enumeration carried out by a program; the second is an ordinary mathematical argument that has not been formally verified. Neither is needed for Theorems 3.1–3.3.

5.1. **How small a counterexample can be.** Call a counterexample *of half-integral* $2+2$ *shape* if $|Z| = |P| = 1$ and the relaxation attains $|Z| + |P| = 2$ at a half-integral point supported on two paths to z and two paths to p, each of weight $1/2$; the instance $\mathcal{C}$ is of this shape. For such a counterexample, deleting every arc and every node that carries no weight, and every element of Y_{a} that starts no support path, leaves a counterexample on at most as many nodes: the half-integral point stays feasible, and deleting arcs, nodes or allowed sources can only destroy integral solutions. Hence the minimal counterexamples of this shape have D equal to the union of the four support paths and Y_{a} equal to the set of their four starting nodes, and those can be enumerated: one enumerates the admissible unions of two paths to z and two paths to p subject to the node-multiplicity condition (each node carried by at most two of the four paths, which is exactly feasibility of the half-integral point for (iv)), and for each tests every pair consisting of a simple D_1-path from Y_{a} to z and a simple D-path from Y_{a} to p for vertex-disjointness. The result, with the first Z-path normalised by relabelling so that it runs through the internal nodes in index order:

$\lvert V\rvert$	4	5	6	7	8
instances of this shape	1	102	4,302	182,133	9,437,133
counterexamples	0	0	0	0	30

The thirty counterexamples at eight nodes fall into exactly three classes up to relabelling. So eight nodes is the minimum for this shape, and there are exactly three counterexamples there; all three have $|D_1| = 6$, $|D| = 12$ and $|Y_{\mathrm{a}}| = 2$, and $\mathcal{C}$ is one of them. The whole table costs under a minute on one core. The enumeration covers $|Z| = |P| = 1$ and half-integral optima with two support paths per commodity only; it says nothing about $|Z| + |P| \geq 3$ or about optimal points with other denominators. Searches of the On-Line Encyclopedia of Integer Sequences for the counts $1, 102, 4302, 182133, 9437133$ and for the subword $102, 4302, 182133$ returned nothing.

5.2. **The decision problem is NP-hard.** The following argument is an ordinary mathematical proof and has *not* been machine-checked; it is independent of Sections 3–4 and is reported because it gives a second, conditional reason why the conjecture had to fail.

Proposition 5.1 (not machine-checked)**.** *There is a polynomial-time many-one reduction from the directed two-vertex-disjoint-paths problem — given a digraph $H = (V_H, A)$ and four distinct vertices s_1, t_1, s_2, t_2, decide whether H contains vertex-disjoint directed paths $s_1 \rightsquigarrow t_1$ and $s_2 \rightsquigarrow t_2$ — to the problem of deciding whether a given instance $(G, G_1, Z, P, Y_{\mathrm{a}})$ admits an integer solution of value $|Z| + |P|$. Taking from the literature that the former is NP-complete* [8]*, the latter problem is therefore NP-hard, and if Conjecture 4.3 of* [1] *were true, then* $\mathrm{P} = \mathrm{NP}$.

Proof. Given H and the four vertices s_1, t_1, s_2, t_2, put

$$V := V_H, \quad D := A, \quad D_1 := A \setminus \{\text{arcs leaving } s_2\}, \quad Z := \{t_1\}, \quad P := \{t_2\}, \quad Y_{\mathrm{a}} := \{s_1, s_2\}.$$

This is a polynomial-time transformation. By Theorem 2.2, an integer solution of value $|Z| + |P| = 2$ exists if and only if there are vertex-disjoint paths from a subset of $\{s_1, s_2\}$ to $\{t_1, t_2\}$ whose t_1-path uses only arcs of D_1.

($\Leftarrow$) Given vertex-disjoint $\pi\colon s_1 \rightsquigarrow t_1$ and $\sigma\colon s_2 \rightsquigarrow t_2$ in H, the path π avoids s_2, since s_2 lies on σ; hence π uses no arc leaving s_2 and is a D_1-path. So $\{\pi, \sigma\}$ is a system of the required kind.

($\Rightarrow$) In such a system the two paths are vertex-disjoint, so they start at two distinct elements of $Y_{\mathrm{a}} = \{s_1, s_2\}$. The path ending at t_1 cannot start at s_2: as $s_2 \neq t_1$ its first arc would leave s_2 and hence not lie in D_1. So it starts at s_1, the other starts at s_2 and ends at t_2, and the two are vertex-disjoint paths of the required kind in H.

For the consequence: the optimal value of the relaxation can be computed in polynomial time, so if it always equalled the integer optimum on the tight face, the test "val $= |Z| + |P|$" would decide an NP-hard problem in polynomial time. □

The source's own remark that the integer program is "NP-complete" is about integer programming in general, not about this decision problem in particular: the next sentence of [1] on the subject, immediately after Conjecture 4.3, is "However, note that despite NP-completeness, there are very efficient methods for solving integer linear programs, such as cutting-plane methods or branch-and-bound methods". What Proposition 5.1 adds is that the specific problem behind Lp is itself hard, which is why the conjectured shortcut was always improbable. We emphasise that Theorem 3.3 does not depend on it: the refutation is unconditional.

6. What remains open

6.1. **The instances the algorithm actually builds.** Conjecture 4.3 is stated in [1] for arbitrary $(G, G_1, Z, P, Y_\mathrm{a})$, and Theorem 3.3 refutes it as stated. The reduction of [1, Section 4.2], on which the identifiability algorithm of its Section 4.3 rests, does not, however, feed Lp arbitrary instances. Given the graph $G = (V, D)$ of the model on observed and latent nodes, with $D_\mathrm{lat} \subseteq D$ the edges of its *latent subgraph* — those whose tail is a latent node — it builds a two-layer graph $G^\mathrm{lp} = (V \cup V', D^\mathrm{lp})$ with V' a disjoint copy of V and

$$D^\mathrm{lp} = \{v \to h : h \to v \in D_\mathrm{lat}\} \cup \{v \to v' : v \in V\} \cup \{u' \to v' : u \to v \in D\},$$

$$D^\mathrm{lp}_\mathrm{lat} = \{v \to h : h \to v \in D_\mathrm{lat}\} \cup \{v \to v' : v \in V\} \cup \{u' \to v' : u \to v \in D_\mathrm{lat}\},$$

the subgraph playing the role of G_1 being the one spanned by $D^\mathrm{lp}_\mathrm{lat}$. The instance handed to the program is $\mathrm{Lp}(G^\mathrm{lp}, G^\mathrm{lp}_\mathrm{lat}, Z, P, Y_\mathrm{a})$ with Z, P and Y_a all subsets of the *unprimed* layer V: in [1, Corollary 4.4] they are the sets of the criterion, and in the algorithm of [1, Section 4.3] they are, for the node v being solved, a subset Z of the allowed nodes, the observed parents of v, and the allowed sources Y_a, all of them observed nodes of the model.

Our instance is provably not of that shape, and the check is one line: in G^lp every node is an endpoint of an arc $v \to v'$ lying in $D^\mathrm{lp}_\mathrm{lat}$, so every node has either an outgoing arc in the subgraph (if it is unprimed) or an incoming arc (if it is primed). The node s_2 of $\mathcal{C}$ has neither: its two outgoing arcs $s_2 \to w_{11}$ and $s_2 \to w_{12}$ lie in $D \setminus D_1$, and its in-degree in G is 0. Since s_2 is exactly the node that forces both Z-paths to leave s_1, the counterexample does not transfer, and the question

> *does Conjecture 4.3 fail on the instances* $(G^\mathrm{lp}, G^\mathrm{lp}_\mathrm{lat}, Z, P, Y_\mathrm{a})$ *that the reduction of* [1, Section 4.2] *constructs?*

is open. It is the question with algorithmic consequences: if the answer is yes, the conditional polynomial-time claim of [1, Remark 4.8] fails where it is actually used; if no, the algorithm survives with a restricted conjecture, and that restriction is then the statement worth proving. The search space is not large — mixed graphs on four nodes already give $|V \cup V'| = 8$ — and the enumeration of Section 5.1, with an exact linear-programming solver behind it, is the natural instrument.

6.2. **Larger demands, and a possible repair.** The enumeration of Section 5.1 is confined to $|Z| = |P| = 1$ and to half-integral optima with two support paths per commodity. Whether counterexamples exist with $|Z| + |P| \geq 3$, and whether the minimum is still eight nodes once optimal points with other denominators are allowed, are both open; the corresponding enumerations are several orders of magnitude larger and were not attempted.

Finally, the failure in $\mathcal{C}$ has a single identifiable cause: the two Z-paths are forced to leave the same allowed source. All three eight-node counterexamples share this feature. A conjecture repaired along these lines — for instance, that an integer solution exists whenever the optimal value is $|Z| + |P|$ *and* the Z-commodity can be routed from $|Z|$ distinct allowed sources — is untested, and we state it only as the direction the evidence points in, not as a claim.

7. Verification

Every statement labelled Theorem in Sections 3 and 4 has been formally verified in Lean 4 against *Mathlib* [9, 10]; the six theorems there correspond to ten formal statements in the development, a single theorem here being assembled from separate formal statements of well-formedness, feasibility, the value of the objective, and the upper bound. The program (i)–(iv) is transcribed once, over an arbitrary commutative ring with a partial order, so that the very same predicate expresses feasibility for the linear program (over $\mathbb{Q}$) and for the integer program (over $\mathbb{Z}$); this matters, because the conjecture is precisely the implication between the two, and the two systems must not be separate transcriptions. Feasibility of the explicit points in Theorems 3.1, 3.2, 4.1 and 4.2 is checked by normalising the arc sums; the upper bounds in Theorems 3.1 and 4.3 are linear consequences of finitely many constraints, found automatically; and the bound of Theorem 3.2, the one statement that quantifies over infinitely many points, is obtained by instantiating (i)–(iv) at the sixteen arcs and eight nodes, expanding the sums, and handing the resulting quantifier-free implication in 32 integer unknowns to a decision procedure for linear integer arithmetic — so it holds for all integer points, and rests on no enumeration. No proof uses compiler-level evaluation of a Boolean, and none is left incomplete: each of the ten statements depends only on the three standard axioms of the system (propositional extensionality, choice, and soundness of quotients). One practical note for anyone reproducing this: kernel-level evaluation is unusable on rational certificates here, because normalisation of Lean's rational numbers passes through a well-founded greatest-common-divisor computation that does not reduce inside the kernel, so the rational checks must be done by normalisation tactics instead. The source of the development is currently held in a private repository: repository reference to be added at submission.

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