TRIANGULAR FACES IN CONVEX RECTILINEAR DRAWINGS OF COMPLETE GRAPHS

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ABSTRACT. In a convex rectilinear drawing of K_n the n vertices are points in strictly convex position, no three of them collinear, and every edge is drawn as a straight segment; a *k-face* is a bounded face of the resulting figure, a convex polygon with k corners, and t_3 denotes the number of triangular faces. Balko, Brötzner, Klute and Tkadlec asked for the minimum of t_3 over convex drawings, over generic convex drawings, and over the regular drawing, and observed that $t_3 \geq n(n-3)$ always. We prove that $n(n-3)$ is exactly the number of faces whose closure contains a vertex of the drawing, that every such face is a triangle, and that the remaining triangular faces are in bijection with those 6-element subsets of the vertex set whose three main diagonals bound a face. Writing τ for the number of such subsets, this gives the exact splitting $t_3 = n(n-3) + \tau$, valid for every convex drawing. We then prove $\tau \geq n-5$ for $n \geq 5$, which improves the known bound to $t_3 \geq n^2 - 2n - 5$ and settles the generic minimum at $n = 5$ and $n = 6$, where it equals 10 and 19; on those two classes t_3 is in fact constant. For $4 \leq n \leq 14$ we exhibit generic drawings with exactly $3\binom{n-2}{2} + 1$ triangular faces, so the generic minimum is at most that value, and we show that the conjecture that this value is optimal is *equivalent* to the inequality $\tau \geq \binom{n-4}{2}$. All finite computations and all arithmetic identities reported here are machine-checked in Lean 4.

1. INTRODUCTION

A *rectilinear drawing* of a graph G represents the vertices by distinct points of the plane and each edge by the straight segment joining the images of its ends; throughout, as in [1], no three of the points are collinear. A rectilinear drawing D of K_n is *convex* if its n points are in convex position, and *regular* if they are the vertices of a regular n-gon. A *crossing* is a common interior point of at least two edges at which they properly cross, a *heavy crossing* is a common interior point of at least three, and D is *generic* if it has no heavy crossing. A *face* of D is a connected component of $\mathbb{R}^2 \setminus D$; exactly one face is unbounded, every bounded face is a convex polygon, the *size* of a bounded face is the number of corners of that polygon, and a bounded face of size k is a *k-face*. We write $t_k(D)$ for the number of k-faces of D, and abbreviate $t_3 = t_3(D)$.

Balko, Brötzner, Klute and Tkadlec [1] determined the possible face sizes of rectilinear drawings of complete graphs in several regimes and closed their paper with a list of open problems, of which the third reads verbatim:

Problem 1.1 ([1, Problem 3]). What is the minimum number of 3-faces in a convex drawing of K_n? What if the drawing is generic or regular?

The only bound the source states appears in the preceding paragraph: *"regarding* 3*-faces, it is simple to show that there are always at least* $n(n-3)$ *by considering the area of a convex drawing around its outer-face as long as* $n \geq 3$*, but what is the growth rate of the minimum number of* 3*-faces with respect to* n*?"* No value of the minimum is recorded there for any n.

Problem 1.1 is really three questions, and they have different statuses. The regular variant is not open: for each n there is only one regular drawing up to similarity, so its "minimum" is a single number, and that number is the sequence [5] of triangular regions of a regular n-gon with all diagonals drawn, computed by Sommars and Sommars [3] and underpinned by the intersection counts of Poonen and Rubinstein [4]. The unrestricted variant and the generic variant are genuinely different functions of n: they already differ at $n = 6$ (Proposition 9.1). This paper is about the generic variant, with the structural results of Sections 4–6 valid for all convex drawings.

Results. Our starting point is an exact splitting of the triangular faces into those that touch a vertex of the drawing and those that do not. Call a bounded face *boundary* if some drawing vertex lies on its closure and *interior* otherwise, and let $\tau(D)$ be the number of 6-element subsets S of the vertex set whose three *main diagonals* — the three chords joining antipodal points of S in the cyclic order of S — bound a face of D.

- **The structure theorem** (Theorem 4.1). For every convex drawing of K_n with $n \geq 4$, generic or not, every boundary face is a triangle, there are exactly $n(n-3)$ of them, and the interior triangular faces correspond bijectively to the 6-subsets counted by τ. Hence
$$t_3 = n(n-3) + \tau.$$
The bound of [1] is thereby identified as the boundary part of t_3, and is exactly the statement $\tau \geq 0$: everything hard about Problem 1.1 lives in τ, a count of $\binom{n}{6}$ yes/no questions.
- **An improved lower bound** (Theorem 6.3). Every generic convex drawing of K_n with $n \geq 5$ has $\tau \geq n-5$, hence
$$t_3 \;\geq\; n(n-3) + (n-5) \;=\; n^2 - 2n - 5,$$
which exceeds the bound of [1] by $n-5$.
- **Two settled values** (Theorem 6.4). The bound above is tight at $n = 5$ and $n = 6$, and there t_3 is constant: every convex drawing of K_5 has exactly 10 triangular faces and every generic convex drawing of K_6 has exactly 19. So the generic variant of Problem 1.1 is answered at $n = 5$ and $n = 6$.
- **Upper bounds** (Theorem 7.1). For every n with $4 \leq n \leq 14$ there is a generic convex drawing of K_n with integer coordinates and exactly $3\binom{n-2}{2} + 1$ triangular faces. The drawing is the "tall cup" $p_i = (i, B^i)$, which is the construction used in [1] for a different purpose. We do not prove that these values are minimal.
- **An equivalent form of the conjectured minimum** (Proposition 5.1). For a generic convex drawing of K_n, $n \geq 4$, one has $t_3 \geq 3\binom{n-2}{2} + 1$ if and only if $\tau \geq \binom{n-4}{2}$. A conjecture about all faces of the arrangement is thus reduced to one inequality about 6-subsets, whose right-hand side is a vanishing fraction of $\binom{n}{6}$.

The values attained by the tall cups, together with the two counts that anchor them, are:

n	4	5	6	7	8	9	10	11	12	13	14
t_3, generic, best found	4	10	19	31	46	64	85	109	136	166	199
τ	0	0	1	3	6	10	15	21	28	36	45
$n(n-3)$, the bound of [1]	4	10	18	28	40	54	70	88	108	130	154
bounded faces	4	11	25	50	91	154	246	375	550	781	1079

The first row is $3\binom{n-2}{2} + 1$ and the second is $\binom{n-4}{2}$. Since $3\binom{n-2}{2} + 1$ grows like $\frac{3}{2}n^2$ while the bound of [1] grows like n^2, the growth-rate question of [1] would be answered with leading coefficient $\frac{3}{2}$ if these values were the true minima; Proposition 5.2 makes the comparison exact. We emphasise that this is conditional: what Theorem 7.1 proves is an upper bound at each $n \leq 14$, not equality.

Method and organisation. Section 2 fixes notation and records the two classical counts for a generic convex drawing. Section 3 proves an identity expressing t_3 through the sizes of the large faces. Section 4 proves the structure theorem, Section 5 the equivalence above and the exact comparison with the bound of [1], and Section 6 the improved lower bound and the two settled values. Section 7 presents the drawings that attain $3\binom{n-2}{2} + 1$ and the finite computations verifying them, Section 8 states what is left to prove and records which strengthening of it is already refuted, and Section 9 treats the unrestricted and regular variants. Section 10 describes the formal verification.

2. Preliminaries

Definition 2.1. A *convex drawing* of K_n is a rectilinear drawing D whose points $p_0, p_1, \ldots, p_{n-1}$, listed counterclockwise, are in strictly convex position with no three collinear, all $\binom{n}{2}$ segments p_ip_j

being drawn. We call these segments *chords*, refer to p_ip_{i+1} (indices modulo n) as *hull edges*, and write (p_a, p_b) for the open arc of vertices strictly between p_a and p_b in counterclockwise order.

Two facts are used throughout without further comment. First, a hull edge is uncrossed: a chord crossing p_ip_{i+1} would separate two vertices adjacent on the hull. Second, two distinct chords are never collinear, since that would place three drawing vertices on a line; consequently each side of a face lies on exactly one chord, and no drawing vertex is an interior point of a chord.

Definition 2.2. The *excess* of a drawing D is

$$E(D) \;=\; \sum_{k\geq 5}(k-4)\,t_k(D),$$

the total amount by which the faces of size at least 5 exceed size 4. We write $F(D)$ for the number of bounded faces and $S(D) = \sum_F \operatorname{size}(F)$ for the number of face corners, the sum being over bounded faces.

The following elementary characterisation is what makes t_3 a finite, purely combinatorial quantity; it is the form in which t_3 is computed in Section 7.

Lemma 2.3. *Let D be a convex drawing of K_n. A set $T \subseteq \mathbb{R}^2$ is a 3-face of D if and only if T is the interior of a triangle whose three sides lie on three distinct chords of D and whose interior meets no chord of D. The correspondence between 3-faces and the corresponding triples of chords is a bijection.*

Proof. If T is a 3-face then it is an open convex triangle whose boundary lies in D; each of its three sides lies on a single chord because two distinct chords are never collinear, the three chords are distinct for the same reason, and T misses D by the definition of a face. Conversely, such a triangle interior is open, disjoint from D and has its boundary inside D, so it is a connected component of $\mathbb{R}^2 \setminus D$, that is, a face, of size 3. The triple of chords is recovered from the face and the face from the triple, so the correspondence is bijective. Note that genericity is not needed: extra chords through a corner of T do not meet the interior of T. $\square$

Lemma 2.4. *Let D be a generic convex drawing of K_n, $n \geq 3$. Then*

$$F(D) = \binom{n}{4} + \binom{n}{2} - n + 1, \qquad S(D) = 4\binom{n}{4} + n(n-2).$$

Proof. Genericity means that every crossing is the meeting point of exactly two chords, so the crossings are in bijection with the 4-subsets of the vertex set: the arrangement has $V = n + \binom{n}{4}$ vertices. Each chord is subdivided by the crossings on it, and each crossing splits two chords, so $E = \binom{n}{2} + 2\binom{n}{4}$. The arrangement is connected, so Euler's formula gives $F(D) = E - V + 1 = \binom{n}{4} + \binom{n}{2} - n + 1$ bounded faces. For the corners: each crossing is a corner of exactly the four faces of the two crossing chords, contributing 4; each drawing vertex p_i has $n-1$ chords leaving it, in the cyclic order of the hull, so it is a corner of the $n-2$ faces lying in the sectors between consecutive chords, contributing $n-2$. Every corner of a bounded face is an arrangement vertex of one of these two kinds. $\square$

Both counts of Lemma 2.4 are classical; they are recorded here because they are the input of Theorem 3.2 and because they are recomputed, independently of the formulas, for each of the drawings of Section 7 by a face traversal of the arrangement, whose count of triangular faces agrees with the count of Lemma 2.3 in every case.

3. The excess identity

Lemma 3.1. *Let $k_1, \ldots, k_m$ be integers with $k_i \geq 3$ for all i. Then*

$$\#\{\, i : k_i = 3 \,\} \;+\; \sum_{i=1}^{m} k_i \;=\; 4m \;+\; \sum_{i=1}^{m} \max(k_i - 4, 0).$$

Proof. Induction on m. A term with $k_i = 3$ contributes $1 + 3 = 4$ to the left and $4 + 0$ to the right; a term with $k_i \geq 4$ contributes $0 + k_i$ to the left and $4 + (k_i - 4)$ to the right. $\square$

Theorem 3.2. *Let D be a generic convex drawing of K_n, $n \geq 3$. Then*

$$t_3(D) \; = \; (n-2)^2 \; + \; E(D) \; = \; (n-2)^2 + \sum_{k \geq 5} (k-4)\, t_k(D).$$

Proof. Every bounded face is a convex polygon with at least three corners, so Lemma 3.1 applied to the sizes of the bounded faces reads $t_3(D) + S(D) = 4F(D) + E(D)$. By Lemma 2.4 and $2\binom{n}{2} = n(n-1)$,

$$4F(D) - S(D) = 4\binom{n}{4} + 4\binom{n}{2} - 4n + 4 - 4\binom{n}{4} - n(n-2) = 2n(n-1) - 4n + 4 - n^2 + 2n = (n-2)^2. \; \square$$

Corollary 3.3. *Every generic convex drawing of K_n satisfies $t_3 \geq (n-2)^2$.*

For $n > 4$ this is weaker than the bound $n(n-3)$ of [1], so it is not an improvement and we do not claim it as one. The useful half of Theorem 3.2 is the other one: *over generic convex drawings of K_n, minimising the number of triangular faces is exactly minimising the excess E.* The next identity says what value of the excess the drawings of Section 7 realise.

Proposition 3.4. *For every $n \geq 3$,*

$$(n-2)^2 + \binom{n-3}{2} \; = \; 3\binom{n-2}{2} + 1.$$

The drawings of Theorem 7.1 have no face of size at least 6 and exactly $\binom{n-3}{2}$ pentagonal faces, so their excess equals $\binom{n-3}{2}$ and Theorem 3.2 delivers the value $3\binom{n-2}{2} + 1$ with nothing left over. Consequently

$$\min\{\, t_3(D) : D \text{ a generic convex drawing of } K_n \,\} = 3\binom{n-2}{2} + 1$$

$$\iff \qquad E(D) \geq \binom{n-3}{2} \text{ for every generic convex drawing } D \text{ of } K_n.$$

Proof. The identity is elementary from $2\binom{m}{2} = m(m-1)$. The face vector of the drawings of Theorem 7.1 is computed there; the equivalence is immediate from Theorem 3.2 together with the fact that the value $3\binom{n-2}{2} + 1$ is attained at every $n \leq 14$ under consideration. $\square$

4. The structure theorem

Call a bounded face of a convex drawing *boundary* if some drawing vertex lies on its closure, and *interior* otherwise. For a 6-element subset $S = \{s_1, \ldots, s_6\}$ of the vertex set, listed in the cyclic order of the hull, the *main diagonals* of S are the three chords s_1s_4, s_2s_5, s_3s_6; they pairwise cross, and the triangle they bound is the *central triangle* of S. Let

$$\tau(D) \; = \; \#\{\, S \in \tbinom{V(D)}{6} : \text{the central triangle of } S \text{ is a face of } D \,\}.$$

Theorem 4.1. *Let D be a convex drawing of K_n with $n \geq 4$. Then*

1. *every boundary face of D is a triangle;*
2. *D has exactly $n(n-3)$ boundary faces;*
3. *the interior triangular faces of D are exactly the central triangles of the 6-subsets counted by $\tau(D)$, and distinct 6-subsets give distinct faces.*

Consequently

$$t_3(D) \; = \; n(n-3) + \tau(D).$$

Genericity is not required anywhere in the statement.

Proof. (1). Fix a vertex $p = p_i$ and write Δ_i for the triangle $p_{i-1}p_ip_{i+1}$. We claim the chords meeting $\operatorname{int}\Delta_i$ are exactly the $n-3$ chords p_ip_k with $k \neq i, i \pm 1$. Those chords do meet it, since p_{i-1} and p_{i+1} are angularly extreme in the pencil at p_i. Conversely, let c be a chord meeting $\operatorname{int}\Delta_i$ and not incident to p_i. No drawing vertex lies in $\operatorname{int}\Delta_i$, so c crosses $\partial\Delta_i$ twice; two of the three sides of Δ_i are hull edges and are uncrossed, so c would have to cross the segment $p_{i-1}p_{i+1}$ twice, which is impossible. If instead c has an end at a corner of Δ_i, say $c = p_{i-1}y$ with $y \neq p_i$, then c enters $\operatorname{int}\Delta_i$ only if y lies strictly inside the angle at p_{i-1} spanned by $p_{i-1}p_i$ and $p_{i-1}p_{i+1}$; but seen from p_{i-1} the remaining vertices appear in the hull order $p_i, p_{i+1}, p_{i+2}, \ldots$, so that angle contains no vertex. The case $c = p_{i+1}y$ is symmetric.

The $n-3$ chords of the pencil therefore cut Δ_i into $n-2$ triangles with apex p_i and base on $p_{i-1}p_{i+1}$, and each of them is a face of D: its interior misses every chord and its boundary lies in D. Finally, a face G with p_i on its closure has points arbitrarily close to p_i; all points of Δ_i close enough to p_i lie in one of those $n-2$ open triangles or on a chord, so G meets one of them and hence equals it. Thus the faces whose closure contains p_i are exactly those $n-2$ triangles.

(2). Summing over the n vertices, part (1) yields $n(n-2)$ incidences (vertex, face whose closure contains it). A face whose closure contains two drawing vertices is a triangle by (1), so those two vertices are adjacent corners of it and the side joining them is an entire uncrossed chord, hence a hull edge; conversely each of the n hull edges, being uncrossed, is a side of exactly one bounded face, and that face carries the two ends of the hull edge. A face carrying three drawing vertices would have three hull edges as its sides, forcing $n = 3$. So exactly n of the incidences are duplicates, and the number of distinct boundary faces is $n(n-2) - n = n(n-3)$.

(3). Let G be an interior triangular face. Its three sides lie on three chords, by Lemma 2.3, and the three corners of G are the pairwise meeting points of those chords. A corner is not a drawing vertex, since G is interior; so the three chords pairwise cross properly, and in particular no two of them share an end, because two chords sharing an end meet only there. Their six ends $s_1 < \cdots < s_6$, in hull order, are therefore distinct. If s_1 were paired with s_k, the two arcs cut off would have $k-2$ and $6-k$ vertices of S, and each of the other two chords must have one end in each arc; this forces $k = 4$, and then $\{s_2, s_3\}$ must be matched with $\{s_5, s_6\}$ by crossing chords, i.e. s_2s_5 and s_3s_6. So the three chords are the main diagonals of S and G is the central triangle of S. Conversely, if the central triangle of a 6-subset is a face, then its corners are crossings and no drawing vertex lies on its closure — a drawing vertex on a side would be an interior point of a chord — so it is an interior triangular face. Distinct 6-subsets give distinct triangles because each side of the triangle determines its supporting chord, and the three chords determine S.

The displayed identity is the sum of (2) and (3), every triangular face being either boundary or interior. □

Remark 4.2. The hypothesis $n \geq 4$ is needed: for $n = 3$ one has $t_3 = 1$ and $n(n-3) = 0$. The identity is the exact form of the bound $t_3 \geq n(n-3)$ of [1], which in this language is the statement $\tau \geq 0$.

Corollary 4.3. *Let D be a generic convex drawing of K_n, $n \geq 4$. Then $E(D) = \tau(D) + (n-4)$.*

Proof. Subtract Theorem 4.1 from Theorem 3.2 and use $(n-2)^2 - n(n-3) = 4 - n$. □

Genericity is essential in Corollary 4.3, although not in Theorem 4.1: the hexagonal drawing of Proposition 9.1, whose three main diagonals are concurrent, has $E = 0$ and $\tau = 0$ while $n - 4 = 2$.

5. The reduction, and the exact gap

Proposition 5.1. *For every $n \geq 4$ and every generic convex drawing D of K_n:*

$$E(D) \;\geq\; \binom{n-3}{2} \quad\Longleftrightarrow\quad \tau(D) \;\geq\; \binom{n-4}{2} \quad\Longleftrightarrow\quad t_3(D) \;\geq\; 3\binom{n-2}{2} + 1.$$

Moreover, for every $n \geq 4$,

$$\binom{n-3}{2} = \binom{n-4}{2} + (n-4) \qquad \textit{and} \qquad n(n-3) + \binom{n-4}{2} = 3\binom{n-2}{2} + 1.$$

Proof. Both displayed identities are elementary consequences of $2\binom{m}{2} = m(m-1)$. The first equivalence follows from Corollary 4.3 and the first identity, the second from Theorem 4.1 and the second identity. $\square$

So the conjecture that $3\binom{n-2}{2}+1$ is the generic minimum is exactly the assertion that at least $\binom{n-4}{2}$ of the $\binom{n}{6}$ six-subsets have their three main diagonals bounding a face; the target is a vanishing fraction of the number of subsets available.

Proposition 5.2. *For every $n \geq 0$,*

$$n(n-3) \;\leq\; 3\binom{n-2}{2}+1,$$

with equality exactly at $n = 4$ and $n = 5$, and for every $n \geq 6$

$$2\Big(3\binom{n-2}{2}+1\Big) \;=\; 2\,n(n-3)+(n-6)(n-3)+2,$$

so the inequality is strict from $n = 6$ on and the gap grows like $(n-6)(n-3)/2$.

Proof. Substitute $n = m+6$ in the displayed identity and expand using $2\binom{m}{2} = m(m-1)$; the two remaining cases $n \leq 5$ are a direct check. $\square$

Since $3\binom{n-2}{2}+1 = \frac{3}{2}n^2 - \frac{15}{2}n + 10$, the drawings of Section 7 exceed the bound of [1] by a quadratic amount. If the values of Theorem 7.1 were the true minima, the growth rate asked for in [1] would have leading coefficient $\frac{3}{2}$ rather than 1; we stress again that only the upper bound is proved here.

6. A lower bound on the interior count

Throughout this section D is a generic convex drawing of K_n and v is one of its vertices, with hull neighbours p and q. Write $D-v$ for the drawing obtained by deleting v and all chords at it; it is again a generic convex drawing, of K_{n-1}, and pq is a hull edge of it. The chords of D at v are called the *pencil* at v; inside the hull of $D-v$ they are pairwise disjoint, since they meet only at v, which lies outside that hull.

Define

$$\begin{aligned} M_v &= \{\text{interior triangular faces of } D \text{ whose 6-subset contains } v\},\\ N_v &= \{\text{interior triangular faces of } D-v \text{ that are met by a chord of the pencil}\},\\ M_v^{\text{good}} &= \{\Delta \in M_v : \text{the 6-subset of } \Delta \text{ contains } v, p \text{ and } q\}. \end{aligned}$$

Lemma 6.1. $\tau(D) - \tau(D-v) = |M_v| - |N_v|$.

Proof. An interior triangular face of D whose 6-subset avoids v is bounded by three chords of $D-v$ and has empty interior in D, hence in $D-v$; conversely an interior triangular face of $D-v$ met by no pencil chord is a face of D. These two classes correspond, so $\tau(D) = |M_v| + \big(\tau(D-v) - |N_v|\big)$. $\square$

Lemma 6.2. $\tau(D) \;\geq\; \tau(D-v) + |M_v^{\text{good}}|$. *In particular τ is monotone under vertex deletion.*

Proof. Define $\Psi\colon M_v \to N_v \cup \{\perp\}$ as follows. Given $\Delta \in M_v$ with 6-subset $\{s_1, \dots, s_6\}$ in hull order, let vu be the main diagonal of that subset through v, let C be the corner of Δ opposite the side carried by vu, and let G be the face of $D-v$ having C as a corner in the wedge that contains Δ. Put $\Psi(\Delta) = G$ if G is an interior triangle of $D-v$, and $\Psi(\Delta) = \perp$ otherwise.

Ψ is onto N_v. Let $G \in N_v$, an interior triangle of $D-v$ met by $m \geq 1$ pencil chords. A pencil chord meeting G enters and leaves through two of its three sides, hence cuts off one corner of G; two chords cutting off the same corner are nested, because pencil chords do not cross one another inside the hull of $D-v$, and the innermost of them bounds a triangle at that corner whose three corners are crossings of D and whose interior meets no chord. That triangle is an interior triangular face of D, its 6-subset contains v, and Ψ sends it back to G.

$M_v^{\mathrm{good}} \subseteq \Psi^{-1}(\perp)$. Suppose $\Psi(\Delta) = G \neq \perp$ and write the 6-subset of Δ as $\{s_1, \dots, s_6\}$ with $v = s_3$. Unwinding the construction, the 6-subset of G consists of s_1, s_2, s_4, s_5 together with one vertex x in the open arc (s_2, s_4) with $x \neq v$ and one vertex y in the open arc (s_5, s_1). If $\Delta \in M_v^{\mathrm{good}}$ then s_2 and s_4 are the hull neighbours p, q of v, so the open arc (s_2, s_4) is $\{v\}$ and no such x exists.

Counting, $|M_v| \geq |N_v| + |\Psi^{-1}(\perp)| \geq |N_v| + |M_v^{\mathrm{good}}|$, and Lemma 6.1 finishes the proof. Taking $|M_v^{\mathrm{good}}| \geq 0$ gives monotonicity. □

Theorem 6.3. *Every generic convex drawing D of K_n with $n \geq 5$ satisfies*

$$\tau(D) \geq n-5, \qquad \textit{hence} \qquad t_3(D) \geq n(n-3)+(n-5) = n^2-2n-5, \qquad \textit{and} \qquad E(D) \geq 2n-9.$$

Proof. The transport of the bound on τ to t_3 is Theorem 4.1, and the bound on the excess is then Corollary 4.3. For the bound on τ we induct on n, the case $n = 5$ being $\tau \geq 0$. Let $n \geq 6$. By Lemma 6.2 it suffices to exhibit, for one vertex v, an element of M_v^{good}; the induction then gives $\tau(D) \geq \tau(D - v) + 1 \geq (n - 1 - 5) + 1 = n - 5$.

Fix any vertex v with hull neighbours p, q, and list the vertices of $D' := D - v$ in hull order as $q, u_1, \dots, u_{n-3}, p$. Put $e_q = pu_1$ and $e_p = qu_{n-3}$, the short diagonals cutting off the corners q and p of D'; the corner triangles are pqu_1 and $u_{n-3}pq$, and their intersection is the face G_0 of D' incident with the hull edge pq, bounded by pq, e_q and e_p. Every pencil chord vu_t enters the hull of D' through pq, so it enters G_0, and it crosses $e_q = pu_1$ if and only if $t \geq 2$ and crosses $e_p = qu_{n-3}$ if and only if $t \leq n - 4$. Since $n \geq 6$ there is an index t with $2 \leq t \leq n - 4$, and the corresponding chord crosses both; it leaves G_0 through one of e_q, e_p.

Suppose it leaves through e_p; the other case is symmetric. Then vu_t remains inside the corner triangle pqu_1 at q, cannot cross the hull edge qu_1, and does cross e_q; so it leaves that corner triangle through e_q, inside one of the faces of D' at q. By part (1) of Theorem 4.1 applied to D', those faces are the pieces of pqu_1 cut by the chords qu_j; let Φ be the piece containing the exit, bounded by qu_{j+1}, qu_j and e_q, with $\Phi \neq G_0$. The chord enters Φ through qu_{j+1} and leaves through e_q, so inside Φ it cuts off the corner $Y = qu_{j+1} \cap e_q$, which is a crossing of D'. Among the pencil chords cutting off Y take the innermost, say $vu_{t'}$; the triangle it cuts off at Y has its three corners at crossings of D and meets no chord, so it is an interior triangular face of D, bounded by $vu_{t'}$, qu_{j+1} and pu_1. Its 6-subset is $\{v, u_{t'}, q, u_{j+1}, p, u_1\}$, six distinct vertices containing v and both of its hull neighbours; so it lies in M_v^{good}. □

The bound of Theorem 6.3 improves that of [1] by $n-5$, and it is tight at the two smallest relevant values of n, where it settles Problem 1.1 for generic drawings.

Theorem 6.4. *Every convex drawing of K_5 has exactly 10 triangular faces, and every generic convex drawing of K_6 has exactly 19. In particular the minimum number of triangular faces of a generic convex drawing of K_n equals 10 for $n = 5$ and 19 for $n = 6$, and these are the values $3\binom{n-2}{2} + 1$.*

Proof. For $n = 5$ there is no 6-subset, so $\tau = 0$ and Theorem 4.1 gives $t_3 = 5 \cdot 2 = 10$; note that a convex drawing of K_5 is automatically generic, since three pairwise crossing chords need six distinct ends.

For $n = 6$ there is exactly one 6-subset, namely the whole vertex set, and we claim its central triangle is always a face when D is generic. A chord meeting the interior of that triangle would have to cross two of its three sides, since no drawing vertex lies inside it. But every chord other than the three main diagonals shares an end with two of them and so crosses at most the third; and each main diagonal contains a side of the triangle rather than crossing the interior. Hence $\tau = 1$ and $t_3 = 6 \cdot 3 + 1 = 19$. Genericity is used only to know that the central triangle is nondegenerate: when the three main diagonals are concurrent the triangle collapses, $\tau = 0$, and $t_3 = 18$ (Proposition 9.1). □

7. Drawings with few triangular faces

For an integer $B \geq 2$ let $\mathrm{cup}_B(n)$ be the drawing with vertices

$$p_i = (i, B^i), \qquad i = 0, 1, \dots, n - 1.$$

These points lie on a strictly convex increasing chain, so they are in convex position with no three collinear, and the listing is counterclockwise. The same construction, with each new point placed "sufficiently high above" the previous ones, is the drawing used in [1] to prove that for every n there is a generic convex drawing of K_n with no k-face for $k \geq 6$. The base B is what buys "sufficiently high", and it has to grow with n; Proposition 7.5(i) shows that this is not an artefact of the analysis.

Theorem 7.1. *For each n with $4 \leq n \leq 14$, put $B = 2$ for $n \leq 10$, $B = 3$ for $11 \leq n \leq 13$ and $B = 4$ for $n = 14$. Then $\mathrm{cup}_B(n)$ is a generic convex drawing of K_n with exactly*

$$t_3 \;=\; 3\binom{n-2}{2} + 1$$

triangular faces, the values being $4, 10, 19, 31, 46, 64, 85, 109, 136, 166, 199$ for $n = 4, \ldots, 14$. Consequently the minimum number of triangular faces of a generic convex drawing of K_n is at most $3\binom{n-2}{2} + 1$ for those n. Moreover each of these drawings has no face of size at least 6 and has exactly $\binom{n-3}{2}$ pentagonal faces, so its excess is $\binom{n-3}{2}$; by Theorem 4.1 its value of τ is $\binom{n-4}{2}$.

Proof. The statement is a finite assertion about eleven explicit integer point sets, and it is established by exhaustive computation with exact integer arithmetic, verified in Lean 4 (Section 10). For each of the eleven drawings the computation performs the following finite searches.

(a) The point set is a convex drawing. All $\binom{n}{3}$ triples of points are tested for collinearity and for a counterclockwise turn, which certifies strictly convex position in the given cyclic order.

(b) The drawing is generic. All $\binom{\binom{n}{2}}{3}$ triples of chords are tested; a triple is rejected if its three members pairwise cross and share a common point. At $n = 14$ there are $\binom{91}{3} = 121\,485$ triples.

(c) The triangular faces are counted twice, independently. The first computation enumerates the same triples of chords and, for each, decides by Lemma 2.3 whether the three chords bound a triangle whose interior meets no chord; disjointness of a segment from an open triangle is decided by a separating-axis test on four axes — the three sides and the segment itself — which is complete because the Minkowski difference of a triangle and a segment has exactly those edge directions. The second computation builds the planar subdivision: the arrangement vertices are the n drawing vertices together with the $\binom{n}{4}$ crossings, carried as reduced homogeneous integer triples so that equality of points is equality of triples; each chord is split at the arrangement vertices lying on it; the half-edges at each arrangement vertex are sorted by angle; and the faces are traversed. At $n = 14$ the arrangement has 1015 vertices, 2093 edges and 1079 bounded faces. This second computation returns the whole face-size vector, from which t_3, the largest face size, the pentagon count and the excess are read off. The two counts of t_3 agree at every one of the eleven drawings, and the number of bounded faces and the number of face corners produced by the traversal agree with the classical values of Lemma 2.4 in every case. □

Remark 7.2. The sequence $3\binom{n-2}{2} + 1 = 4, 10, 19, 31, 46, \ldots$ is the sequence of centred triangular numbers [7]; we know of no recorded connection between that sequence and drawings or arrangements.

The same computation reproduces three published statements of [1] on these drawings. They are recorded here because they are independent tests of the face traversal, not as new results.

Proposition 7.3. *For the eleven drawings of Theorem 7.1: the largest face size is 5 at every n with $5 \leq n \leq 14$ and 3 at $n = 4$, in accordance with* [1, Theorem 3]*; there is no 5-face at $n = 4$ and at least one at every n with $5 \leq n \leq 14$, in accordance with* [1, Theorem 2]*; and there is no 4-face at $n = 4, 5$ while there are three at $n = 6$, in accordance with the proposition on 3- and 4-faces of* [1].

Proposition 7.4. *For each n with $6 \leq n \leq 12$, the value of τ for the drawing of Theorem 7.1 computed by selecting, among the triples of chords bounding a triangular face, those whose three chords pairwise cross, agrees with the value computed by testing each of the $\binom{n}{6}$ six-subsets to see whether its three main diagonals bound a face. At $n = 12$ the second search runs over 924 six-subsets and both computations return $28 = \binom{8}{2}$.*

The next proposition collects the checks that make the preceding computations informative rather than decorative.

Proposition 7.5. (1) The height of the cup matters. $\mathrm{cup}_2(11)$ *is a generic convex drawing of* K_{11} *with the same number of bounded faces and the same number of face corners as* $\mathrm{cup}_3(11)$*, but with* 119 *triangular faces instead of* 109*; its excess is* 38 *against* 28 *and its* τ *is* 31 *against* 21*. So the value computed is not independent of the base, and "sufficiently high" in* [1, Theorem 3] *does real work. What goes wrong is not the appearance of a larger face: the face-size vector of* $\mathrm{cup}_2(11)$ *is* $(t_3, t_4, t_5) = (119, 218, 38)$*, with no face of size at least* 6*, so the ten extra units of excess are ten extra pentagons.*

(2) The identity away from the extremal case. *Two near-regular integer drawings, at* $n = 7$ *with a* 7*-face and at* $n = 9$ *with faces of sizes* 6 *and* 9*, satisfy Theorem 3.2 in the form* $35 = 25+10$ *and* $90 = 49+41$*; their values of* τ *are* 7 *and* 36 *against the* 3 *and* 10 *of the cups. In particular the excess term is exercised on drawings that genuinely have large faces, and the near-regular drawings are far from minimal.*

(3) The counts of Lemma 2.4 discriminate. *The non-generic hexagonal drawing of Proposition 9.1 has* 24 *bounded faces against the generic value* 25*, and* 78 *face corners against the generic value* 84*; so passing both counts is informative.*

8. What remains

By Proposition 5.1, the conjecture that $3\binom{n-2}{2} + 1$ is the generic minimum is exactly $\tau \geq \binom{n-4}{2}$. Since $\binom{n-4}{2} - \binom{n-5}{2} = n - 5$, an induction on n from the base $\tau = 0$ at $n = 5$ needs only the following increment, and needs it at a single vertex.

Conjecture 8.1. For every generic convex drawing D of K_n, $n \geq 5$, and every vertex v,

$$\tau(D) \;\geq\; \tau(D - v) + (n - 5).$$

Proposition 8.2. *The inequality of Conjecture 8.1 holds at every vertex of* $\mathrm{cup}_B(n)$ *for every* n *with* $6 \leq n \leq 11$*, and it is an equality at every vertex of every one of those drawings, so the constant* $n-5$ *cannot be improved. It fails for the non-generic hexagonal drawing of Proposition 9.1, where* $\tau = 0$*; so the genericity hypothesis is not vacuous.*

Proof. A finite computation: for each of the six drawings and each of its n vertices, τ is recomputed for the drawing with that vertex deleted and compared with τ of the whole drawing. The computations are as in Theorem 7.1; the inequality, and the failure on the hexagon, are verified in Lean 4, and the equality was read off the same evaluations. □

Beyond these drawings, Conjecture 8.1 was tested on 9 581 randomly generated convex drawings with $7 \leq n \leq 11$, computing $\tau(D) - \tau(D - v)$ at every vertex: there was no violation, and the smallest increment observed was exactly $n-5$ at each n. This sampling, and every other sampled count quoted in this section, was carried out not in Lean but by the separate implementation in C described in Section 10; it is evidence, not part of the formal development.

By Lemma 6.2, Conjecture 8.1 at a vertex v follows from $|M_v^{\mathrm{good}}| \geq n - 5$, and the induction only needs one good vertex:

Conjecture 8.3. Every generic convex drawing of K_n, $n \geq 5$, has a vertex v with $|M_v^{\mathrm{good}}| \geq n - 5$: at least $n - 5$ interior triangular faces have v and both hull neighbours of v among the six vertices of their 6-subset.

Conjecture 8.3 implies Conjecture 8.1 at that vertex, hence $\tau \geq \binom{n-4}{2}$, hence the conjectured generic minimum. It was tested on 2 850 randomly generated convex drawings with $7 \leq n \leq 11$ without a violation, and there the smallest value of $\max_v |M_v^{\mathrm{good}}|$ over the sample was exactly $n - 5$ at each n, so the constant $n - 5$ cannot be raised. Extending the same sweep to $7 \leq n \leq 13$, 2 997 drawings in all, again produced no violation; sharpness is quoted only for $n \leq 11$ because at $n = 12, 13$ too few drawings were sampled for that smallest value to come down to $n - 5$, it being 9 and 10 there.

Remark 8.4. The natural strengthening, "$|M_v^{\mathrm{good}}| \geq n-5$ for *every* vertex v", is false, already at $n = 7$. An explicit violator is the convex heptagon with vertices

$$(882, -5319), \quad (-7341, -5671), \quad (-9000, -5959), \quad (-8794, -7731),$$
$$(-8268, -9000), \quad (-4907, -8960), \quad (9000, -7180),$$

whose values of $|M_v^{\mathrm{good}}|$ are $2, 2, 2, 1, 2, 1, 2$, with minimum 1 while $n - 5 = 2$. The value 1 occurs at every n from 7 to 12 that was sampled, so this is not an accident of small n; note that the maximum on this drawing is $2 = n - 5$, so Conjecture 8.3 survives on it. A proof must therefore choose the vertex. This drawing and its per-vertex counts were computed and re-verified by the implementation in C of Section 10, not inside the Lean development.

Remark 8.5. The proof of Theorem 6.3 produces, for each of the $n-5$ pencil chords vu_t with $2 \leq t \leq n-4$, an interior triangular face at the point where the chord leaves one of the two corner triangles. What it does not produce is $n - 5$ *distinct* such faces: two pencil chords can leave the same corner triangle inside the same face at q and cut off the same corner Y, and of a nested pair only the innermost bounds a face. The missing step is therefore a spreading statement — for a good choice of v, the $n - 5$ cut-off corners are distinct — which is exactly Conjecture 8.3.

9. The unrestricted and the regular variants

Proposition 9.1. *The hexagonal drawing H with vertices*

$$(2, 0),\ (1, 2),\ (-1, 2),\ (-2, 0),\ (-1, -2),\ (1, -2)$$

is a convex drawing of K_6 which is not generic — its three main diagonals meet at the origin — and it has 24 bounded faces and 18 triangular faces. Consequently the minimum number of triangular faces over all convex drawings of K_6 is at most 18, strictly less than the value 19 that is the minimum over generic convex drawings (Theorem 6.4): the unrestricted and generic variants of Problem 1.1 are different functions of n. Moreover H is the image of the regular hexagon under $(x, y) \mapsto (x, 2y/\sqrt{3})$, a linear map of positive determinant, hence it has the same arrangement combinatorics; so its counts 18 and 24 are those of the regular hexagon, verifying [5] and [6] at $n = 6$.

Proof. The counts are a finite computation on an explicit integer point set, performed by the two independent methods described in the proof of Theorem 7.1 and verified in Lean 4. A linear map of positive determinant preserves the orientation of every triple of points, hence preserves every incidence and crossing pattern of the drawing, hence the whole face vector. □

Remark 9.2. Theorem 3.2 accounts for the difference exactly. Making the three main diagonals concurrent removes one bounded face, adds six face corners and destroys all three pentagons, and $t_3 = 4F - S + E$ turns $(-1, -6, -3)$ into $4(-1) - (-6) + (-3) = -1$.

Remark 9.3. Theorem 4.1 can be checked against published data. For the regular n-gon R_n the number of triangular regions is [5], namely $1, 4, 10, 18, 35, 56, 90, 120, 176, 276, 377$ for $n = 3, \ldots, 13$; the theorem predicts that $t_3(R_n) - n(n-3) = \tau(R_n) \geq 0$ for $n \geq 4$, and indeed the differences are $0, 0, 0, 7, 16, 36, 50, 88, 168, 247$ for $n = 4, \ldots, 13$, non-negative throughout and zero exactly at $n = 4, 5, 6$. (At $n = 3$ the difference is 1, as it must be, the theorem requiring $n \geq 4$.) These numbers were computed by other authors, by another method, in 1998 [3, 4]. The comparison also shows that the regular drawing is far from minimal: at $n = 7$ and $n = 9$ it carries 35 and 90 triangular faces against the 31 and 64 of Theorem 7.1, and by $n = 14$ it carries nearly two and a half times as many. A further consistency check: for odd n the regular drawing is generic [4], so its total number of bounded regions must equal the generic value of Lemma 2.4, and [6] gives $50, 154, 375, 781$ at $n = 7, 9, 11, 13$, agreeing with $\binom{n}{4} + \binom{n}{2} - n + 1$.

10. Verification

Every finite computation reported above — the eleven drawings of Theorem 7.1 with their face vectors, the verifications of Proposition 7.3, the two computations of τ in Proposition 7.4, the deletion sweeps of Proposition 8.2, the control drawings of Proposition 7.5 and the hexagon of Proposition 9.1 — and every arithmetic identity used to combine them — Lemma 3.1, Theorem 3.2 given the counts of Lemma 2.4, Proposition 3.4, Proposition 5.1, Proposition 5.2 and the transport of a lower bound on τ to one on t_3 — has been formally verified in Lean 4 against *Mathlib*, with all geometric predicates evaluated in exact integer arithmetic and the arrangement points carried as reduced homogeneous integer triples; the arithmetic statements are proved for all n and depend on no axioms beyond propositional extensionality and quotient soundness, while each per-drawing statement additionally depends on the compiler-evaluation axiom generated for that drawing's single evaluation, and the development contains no incomplete proofs. Three things are, by contrast, *not* part of the formal development and are proved here by hand: the two geometric bridges of Lemma 2.3 and of the face traversal (that the length of a face's boundary walk equals the number of corners of the polygon, which holds because every arrangement vertex has degree at least three and so no arrangement vertex lies in the relative interior of a side of a face), the structure theorem (Theorem 4.1) and the lower bound (Lemma 6.2, Theorem 6.3), for which the formal development has no notion of face; what is machine-checked about them is that their conclusions hold at every drawing listed above, by two independent computations, and the random sampling quoted in Section 8 was carried out by a separate implementation in C using exact 128-bit integer arithmetic. That implementation also reproduced $t_3 = n(n-3) + \tau$ without a failure on at least 10 450 randomly generated convex drawings with $6 \leq n \leq 12$; extended to $6 \leq n \leq 14$ the same check recorded 10 488 drawings and again no failure. Repository reference to be added at submission.

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