TURÁN GRAPHS, C_5 BLOW-UPS, AND THE FINITE-ORDER THRESHOLD FOR AVERAGE QUANTUM TRANSPORT ON TRIANGLE-FREE GRAPHS

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ABSTRACT. For a graph G of order n with adjacency matrix A let $F_G(t)$ be the average of $|(\mathrm{e}^{-\mathrm{i}tA})_{vu}|^2$ over ordered pairs of distinct vertices. Song recently determined the maximum of the dense-scaling limit of $n^2F_G(\tau/n)$ over triangle-free graphons: the balanced complete bipartite graphon is the unique maximizer exactly on $0 < \tau \le \tau_c$, where $\tau_c = 4\sin(\tau_c/2) \approx 3.79099$, and he asked whether the Turán graph $T_2(n)$ is the unique maximizer of $F_G(\tau/n)$ among triangle-free graphs of order n for all large n. No finite-order data was available. We settle the finite question exactly, in integer arithmetic, at pinned rational τ: $T_2(n)$ is the unique maximizer for $n = 5, 6, 7$ at $\tau = \frac{1}{2}, 1, 2, 3, \frac{15}{4}, \frac{19}{5}$, for $n = 6$ also at $\tau = \frac{9}{2}$, and for $n = 8$ at $\tau = \frac{1}{2}$. Since $\frac{19}{5} > \tau_c$, the finite-order transition lies strictly *above* the graphon threshold at every order tested. Above that transition the maximizer is identified and not merely beaten: at $(n, \tau) = (5, \frac{9}{2}), (5, 5), (6, 5), (7, \frac{9}{2})$ the maximum over all triangle-free graphs on n vertices is attained, to within 1.1×10^{-7}, by a blow-up of C_5 — Song's own nonbipartite competitor, which he introduces at the graphon level but never exhibits at finite order. The cell $(7, \frac{9}{2})$ lies inside the window in which his second-phase conjecture predicts a *bipartite* graphon optimum. We also prove a statement strictly stronger than the original question on the range tested: for $n = 5, 6, 7$ and $\tau \in \{\frac{1}{2}, \frac{19}{5}\}$ every triangle-free graph below the Mantel bound is strictly beaten by one with exactly one more edge, so the maximum of the transport value over the graphs with exactly k edges is strictly increasing in k. Every search is a complete exhaustion over all $2^{\binom{n}{2}}$ graphs on a labelled vertex set, and every comparison is an exact integer comparison of a truncated closed-walk series against a pinned margin that exceeds twice the largest possible truncation error by a factor of at least 10^3 — except in three places, flagged where they occur, in which a truncated value is compared exactly and with no margin instead. The searches are machine-checked in Lean 4.

1. Introduction

1.1. The functional. Let G be a finite simple graph of order $n \ge 2$ with adjacency matrix $A = A(G)$, and let $U_G(t) = \mathrm{e}^{-\mathrm{i}tA}$ be the unitary evolution of the continuous-time quantum walk on G. The entry $U_G(t)_{vu}$ is the transition amplitude from u to v at time t, and the quantity of interest is the average transition probability over ordered pairs of *distinct* vertices,

$$S_G(t) = \sum_{u \ne v} \left|U_G(t)_{vu}\right|^2, \qquad F_G(t) = \frac{S_G(t)}{n(n-1)}. \tag{1}$$

Unlike perfect state transfer, which concerns a prescribed pair of vertices, F_G aggregates over the whole graph, and one may ask which graph in a given class transports best. Write $\mathcal{T}_n$ for the class of triangle-free graphs of order n and $T_2(n) = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ for the Turán graph, the unique extremal graph in Mantel's theorem [5].

Song [9] studies F_G over $\mathcal{T}_n$ under the dense scaling $t = \tau/n$, which keeps tA of order one. Setting $\Phi_n(G, \tau) = n^2F_G(\tau/n)$, the limit along a convergent graph sequence is the graphon functional

$$\Phi_\tau(W) = \tau^2 t(K_2, W) + 2\sum_{r \ge 2} \frac{(-1)^{r+1}\tau^{2r}}{(2r)!}\, t(C_{2r}, W), \tag{2}$$

an *alternating* series in the even cycle densities whose leading term is τ^2 times the edge density. Let $M_\triangle(\tau)$ be the supremum of Φ_τ over triangle-free graphons, B_q the balanced bipartite graphon with cross-value q, and τ_c the positive solution of $\tau = 4\sin(\tau/2)$, so that $\tau_c \approx 3.790989$. Song's main theorem [9, Theorem 1.1] is that

$$M_\triangle(\tau) = 4\Big(1 - \cos\frac{\tau}{2}\Big) \qquad (0 \le \tau \le \tau_c),$$

attained, for $0 < \tau \le \tau_c$, only by B_1 up to weak isomorphism, and that B_1 is not a maximizer for any $\tau > \tau_c$. He also proves [9, Theorem 1.3] that the finite extremal values converge, $\sup_{\tau\in I} |m_n(\tau) - M_\triangle(\tau)| = O_I(n^{-1})$ on compact $I \subset [0, \tau_c]$, where $m_n(\tau) = \max_{G\in\mathcal{T}_n} \Phi_n(G, \tau)$.

1.2. **The finite question.** Convergence of the extremal *values* says nothing about the extremal *graphs*, and Song says so explicitly, in the following words.

> **Problem** [9, Problem 7.2]. *For fixed $0 < \tau < \tau_c$, is $T_2(n)$ the unique maximizer of $F_G(\tau/n)$ over $G \in \mathcal{T}_n$ for every sufficiently large n?*

His comment is that the stability theorem forces every near-maximizing sequence to converge to B_1 in cut distance, but cut distance does not detect a subquadratic number of edge changes, so a resolution needs a local comparison near $T_2(n)$. The only finite-order statement in [9] is [9, Proposition C.2]: for each $n \ge 2$ there is some $\varepsilon_n > 0$ below which $T_2(n)$ is the unique maximizer of $F_G(t)$; the constant comes from finiteness of $\mathcal{T}_n$ and carries no rate, and $\varepsilon_n = \Omega(1/n)$ is what the scaled question needs. The paper prints no finite-order data at all.

This paper supplies that data, exactly. Two features of the problem make an exact treatment possible where an interval-arithmetic treatment of a matrix exponential would be painful. First, unitarity collapses the double sum in (1) onto the diagonal (Lemma 2.1, which is [9, Lemma 2.1]), so only n numbers are needed rather than n^2. Second, the diagonal entries of $\cos(tA)$ and $\sin(tA)$ are power series whose coefficients are closed-walk counts, hence integers, so at rational τ a truncation of the series is a rational number with a fixed denominator and the comparison of two graphs becomes a comparison of two integers (Section 2). No floating-point and no interval arithmetic occurs anywhere below.

1.3. **Results.** Throughout, a computation "over $[n]$" means an exhaustion over all $2^{\binom{n}{2}}$ graphs on the labelled vertex set $[n] = \{0, 1, \dots, n-1\}$; since every graph of order n occurs in that list, no isomorphism reduction and no completeness certificate is involved, and the price is only that the search is larger than a search over isomorphism classes.

Theorem 1.1 (Theorem 4.1)**.** *Let (n, τ) be one of the cells*

$$n \in \{5, 6, 7\},\ \tau \in \Big\{\tfrac{1}{2},\, 1,\, 2,\, 3,\, \tfrac{15}{4},\, \tfrac{19}{5}\Big\}; \qquad (n,\tau) = \Big(6, \tfrac{9}{2}\Big); \qquad (n,\tau) = \Big(8, \tfrac{1}{2}\Big).$$

Then every triangle-free graph G on $[n]$ which is not a labelled copy of $T_2(n)$ satisfies $F_G(\tau/n) < F_{T_2(n)}(\tau/n)$. In particular $T_2(n)$ is the unique maximizer of $F_G(\tau/n)$ over $\mathcal{T}_n$ at each of these 20 *cells.*

Since $\frac{19}{5} = 3.8 > \tau_c$, and since $T_2(n)$ does lose at larger τ (Theorem 4.2), the finite-order transition sits strictly above the graphon threshold at every order tested (Corollary 4.3). Nothing in [9] indicates on which side of τ_c it should fall: the sharp graphon threshold is a statement about the limit object only.

Above the transition, $T_2(n)$ is not merely beaten; the winner can be named.

Theorem 1.2 (Theorem 4.5)**.** *At each of the four cells $(n, \tau) = (5, \frac{9}{2}),\ (5, 5),\ (6, 5),\ (7, \frac{9}{2})$ the maximum of $F_G(\tau/n)$ over all triangle-free graphs G on $[n]$ is attained, to within 1.1×10^{-7}, by a blow-up of C_5: by C_5 itself at $n = 5$, by $C_5[2, 1, 1, 1, 1]$ at $n = 6$, and by $C_5[2, 2, 1, 1, 1]$ at $n = 7$. Exactly, and with no error term, that blow-up maximizes the order-K Taylor truncation of $F_G(\tau/n)$ over all triangle-free graphs on $[n]$, and it beats $T_2(n)$ by a certified margin of more than 3×10^{-5}.*

The blow-ups of C_5 are exactly Song's nonbipartite competitor W_{C_5} [9, §7.1], which he introduces at the graphon level — C_5 is the shortest odd cycle available once triangles are forbidden — and never exhibits at finite order. The finite data confirm the choice of competitor. They also show that the finite picture inside the window (τ_c, τ_5), $\tau_5 \approx 4.73387$, is not the naive reading of the graphon picture:

$\frac{9}{2} < \tau_5$, so at $(7, \frac{9}{2})$ Song's Conjecture 7.1 predicts a *bipartite* graphon optimum B_{q_τ}, whereas the finite optimum there is nonbipartite. This is finite data, not a counterexample: B_{q_τ} is a cut limit of *quasirandom* bipartite sequences, which no graph on seven vertices approximates. We return to this in Section 8.

The third result is a statement about the whole edge-count stratification, and on the range tested it is strictly stronger than the question asked.

Theorem 1.3 (Theorem 5.1)**.** *Let $n \in \{5, 6, 7\}$ and $\tau \in \{\frac{1}{2}, \frac{19}{5}\}$. Then every triangle-free graph G on $[n]$ with $e(G) < \lfloor n^2/4 \rfloor$ is strictly beaten by a triangle-free graph with exactly one more edge: there is a triangle-free H on $[n]$ with $e(H) = e(G) + 1$ whose order-K Taylor truncation of F exceeds that of G, so in particular $F_H(\tau/n) > F_G(\tau/n) - 7.1 \times 10^{-8}$. Equivalently, the maximum of that truncated value over the triangle-free graphs with exactly k edges is strictly increasing in k over the edge counts realised by triangle-free graphs.*

Because Mantel's theorem makes $T_2(n)$ the unique triangle-free graph in the top stratum, Theorem 1.3 implies the truncated form of Theorem 1.1 at the cells it covers, and it says more: no amount of structure buys back a missing edge. Where $T_2(n)$ loses, the monotonicity fails at the top strata, so Theorem 1.3 is a genuinely different phenomenon and not a restatement. The truncation in Theorems 1.2 and 1.3 is not an approximation introduced for convenience: it is the exactly computable quantity that the searches order, and Section 2 bounds its distance from F once and for all.

1.4. **Nearest precursors.** Two known extremal theorems sit close to (2), and it is worth saying why neither settles the finite question. Győri, Pach and Simonovits proved in 1991 that if H is a bipartite graph with a one-factor, then every triangle-free graph G of order n contains at most as many copies of H as $T_2(n)$ does [4]; the hypothesis has since been relaxed, by Gorovoy, Grzesik and Jaworska, to bipartite H carrying a matching of size x together with at most $\frac{1}{2}\sqrt{x-1}$ unmatched vertices [3]. Every even cycle C_{2r} qualifies, so *each individual term* of (2) is already known to be maximized by B_1. This does not transfer: Φ_τ is an alternating sum, so on the terms of negative sign maximizing $t(C_{2r}, W)$ pushes the wrong way — and that alternation is precisely what makes τ_{c} finite and the question interesting. Arman, Gunderson and Tsaturian proved that for $n \geq 141$ the balanced complete bipartite graph is the unique triangle-free graph of order n with the maximum number of cycles [1]; that is the closest existing finite-order extremal theorem in spirit, but all of its weights are $+1$. Finally, the average mixing matrix of [2] is a Cesàro average of $U_G(t) \circ \overline{U_G(t)}$ as $t \to \infty$, whereas $F_G(\tau/n)$ is evaluated at a single finite time; [9] points to it at exactly the step where it observes that the average of $|U_G(t)_{vu}|^2$ over *all* ordered pairs is identically $1/n$, so that the diagonal has to be dropped before anything can be extremal.

1.5. **What rests on computation.** Every result above rests on a complete finite search together with one elementary analytic estimate, and we separate the two carefully. The searches are exhaustions over labelled graphs, described cell by cell in Section 3 and machine-checked in Lean 4 (Section 9). What they establish is an inequality between two *integers*: the order-K truncations of $\sum_u |U_G(\tau/n)_{uu}|^2$, cleared of denominators, differ by at least a pinned margin B. The passage from that integer inequality to the inequality between the values $F_G(\tau/n)$ is Lemma 2.3 and Corollary 2.4 below, which are proved here by hand and are not part of the formal development; they require B to exceed twice the largest possible truncation error, and at every cell where a margin was pinned the actual ratio is between 1.0×10^3 and 1.5×10^6 (Table 1). Two of the searches — the identification of the maximizer above the threshold, and the edge-count monotonicity — were run without a pinned margin; what they establish exactly is the corresponding statement about the truncated value $F^{(K)}$ of (3), which differs from F by less than 6×10^{-8} at every cell, and we state them in that form, as we do one item of Proposition 6.1. This division of labour is stated again in Section 9.

2. The functional, and an exact integer comparison

All graphs are finite and simple. For a graph G on $[n]$ we write $e(G)$ for its number of edges, $\Delta(G)$ for its maximum degree, and $(A^j)_{uu}$ for the number of closed j-walks of G at the vertex u; the latter is a nonnegative integer.

Lemma 2.1 (escape identity, [9, Lemma 2.1, eq. (12)])**.** *For every finite simple graph G of order n and every $t \in \mathbb{R}$,*

$$S_G(t) = n - D_G(t), \qquad D_G(t) := \sum_{u \in V(G)} \big|U_G(t)_{uu}\big|^2.$$

Proof. Every column of the unitary matrix $U_G(t)$ has squared Euclidean norm 1; summing the off-diagonal entries by columns gives the identity. □

Since $F_G = S_G/(n(n-1))$ and n is fixed within a cell, comparing two graphs of the same order amounts to comparing D_G with the inequality reversed: *smaller D means larger F*. Writing $C_u(t) = (\cos tA)_{uu}$ and $\Sigma_u(t) = (\sin tA)_{uu}$ we have $|U_G(t)_{uu}|^2 = C_u(t)^2 + \Sigma_u(t)^2$, and both C_u and Σ_u are power series with integer coefficients:

$$C_u(t) = \sum_{j \text{ even}} \frac{(-1)^{j/2} t^j}{j!} (A^j)_{uu}, \qquad \Sigma_u(t) = \sum_{j \text{ odd}} \frac{(-1)^{(j-1)/2} t^j}{j!} (A^j)_{uu}.$$

Truncating both after order K gives $C_u^{(K)}, \Sigma_u^{(K)}$ and $D_G^{(K)}(t) = \sum_u \big((C_u^{(K)})^2 + (\Sigma_u^{(K)})^2\big)$.

Definition 2.2. Let $n \geq 2$, $K \geq 0$ and let $\tau = p/q$ be a positive rational, so that $t = \tau/n = p/(qn)$. Put $L = (qn)^K K!$. Since $L\, t^j/j! = p^j (qn)^{K-j} K!/j!$ is a nonnegative integer for $0 \leq j \leq K$, the numbers $LC_u^{(K)}$ and $L\Sigma_u^{(K)}$ are integers, and so is

$$Q_{n,K,p,q}(G) \;:=\; \sum_{u \in [n]} \Big(\big(LC_u^{(K)}\big)^2 + \big(L\Sigma_u^{(K)}\big)^2 \Big) \;=\; L^2\, D_G^{(K)}(\tau/n) \;\in\; \mathbb{Z}.$$

The comparison of two graphs at a cell is therefore a comparison of two integers, computed from closed-walk counts by exact integer arithmetic. What remains is to bound the truncation error, and this is the one step of the development that is carried out by hand.

Lemma 2.3 (truncation bound)**.** *Let G be a graph on $[n]$, let $\tau > 0$, $t = \tau/n$, $K \geq 0$, and put $\varepsilon = \sum_{j>K} \tau^j/j!$. Then*

$$\big|D_G(t) - D_G^{(K)}(t)\big| \;\leq\; 2n\varepsilon(2+\varepsilon) \;=:\; E_0(n,\tau,K).$$

Proof. A closed j-walk at u is determined by its j steps, each of which has at most $\Delta = \Delta(G)$ choices, so $0 \leq (A^j)_{uu} \leq \Delta^j$. Since $\Delta \leq n-1$ we get $t\Delta \leq \tau(n-1)/n < \tau$, whence $|C_u - C_u^{(K)}| \leq \sum_{j>K} (t\Delta)^j/j! \leq \varepsilon$ and likewise $|\Sigma_u - \Sigma_u^{(K)}| \leq \varepsilon$. As $U_G(t)$ is unitary, $|U_G(t)_{uu}| \leq 1$, so $|C_u| \leq 1$, $|\Sigma_u| \leq 1$ and $|C_u^{(K)}| \leq 1+\varepsilon$. Hence $|C_u^2 - (C_u^{(K)})^2| = |C_u - C_u^{(K)}| \cdot |C_u + C_u^{(K)}| \leq \varepsilon(2+\varepsilon)$, and the same for Σ_u. Summing the $2n$ terms gives the claim. □

Corollary 2.4 (exact comparison)**.** *Fix a cell $(n, \tau = p/q)$ and a truncation order K, and put $E = L^2 E_0(n,\tau,K)$ with $L = (qn)^K K!$. Let G, H be graphs on $[n]$ and let B be a positive integer with $B > 2E$. If $Q_{n,K,p,q}(G) \geq Q_{n,K,p,q}(H) + B$, then $D_G(\tau/n) > D_H(\tau/n)$ and therefore*

$$F_G(\tau/n) < F_H(\tau/n).$$

Proof. By Lemma 2.3, $D_G \geq D_G^{(K)} - E_0$ and $D_H \leq D_H^{(K)} + E_0$, so $D_G - D_H \geq (Q(G) - Q(H))/L^2 - 2E_0 \geq B/L^2 - 2E/L^2 > 0$. Lemma 2.1 converts the inequality on D into the reverse inequality on S_G and hence on F_G. □

Corollary 2.4 needs a margin, and not every comparison below carries one. It is therefore worth naming the exactly computable quantity that all of the searches order, namely the order-K truncation of the transport value itself:

$$F_G^{(K)}(t) := \frac{n - D_G^{(K)}(t)}{n(n-1)}, \qquad \text{so that} \qquad \left|F_G(t) - F_G^{(K)}(t)\right| \;\le\; \eta := \frac{E_0(n,\tau,K)}{n(n-1)} \tag{3}$$

by Lemmas 2.1 and 2.3. At every cell occurring in this paper $2\eta \le 1.1 \times 10^{-7}$, and at most of them it is several orders of magnitude smaller. Comparisons made with a pinned margin $B > 2E$ are statements about F itself, by Corollary 2.4; comparisons made without one — there are two families of them, in Theorems 4.5 and 5.1 — are exact statements about $F^{(K)}$, and they transfer to F only up to 2η, as does one item of Proposition 6.1. We say which is which at every statement.

3. The exhaustions

A graph on $[n]$ is encoded as one bitmask: the pair $\{i,j\}$ with $i < j$ occupies bit $\binom{j}{2} + i$, so the graphs on $[n]$ are exactly the integers below $2^{\binom{n}{2}}$. Every sweep below is a loop over that range; there is no isomorphism reduction and hence no completeness certificate to trust, at the cost of a search over labelled rather than unlabelled graphs. Triangle-freeness is tested by the triple loop over $i < j < k < n$ and membership in the labelled $T_2(n)$ class by a search over all 2^n bipartitions of $[n]$ for one of size $\lfloor n/2 \rfloor$ whose crossing pairs are exactly the edges; both tests are proved equivalent to the corresponding quantified statements, so the theorems below are statements about graphs and not about bitmasks. The closed-walk counts $(A^j)_{uu}$ for $j \le K$ are obtained by iterating $v \mapsto Av$ from each basis vector over the natural numbers, and $Q_{n,K,p,q}$ is assembled from Definition 2.2; the table of integer coefficients $p^j (qn)^{K-j} K!/j!$ that the evaluation uses is separately checked against that formula at the 23 parameter sets (K,p,q,n) of the $n \le 7$ exhaustion grid (Proposition 7.4(i)). The parameter sets occurring nowhere in that grid — the one at $n = 8$, and most of those of Section 6 — are not covered by that check, and there the tabulation is trusted rather than verified.

Table 1 lists the 24 cells settled by exhaustion, with the truncation order K, the pinned margin B, the safety factor $B/2E$ of Corollary 2.4, and the actual gap $D^{(K)}(\text{runner-up}) - D^{(K)}(T_2(n))$ in units of L^2, which is positive exactly when $T_2(n)$ wins. The margins are chosen as powers of ten for legibility; measured against the gap column they are not slack by two orders of magnitude, and at one cell that is checked in the kernel (Proposition 7.4(v)).

We record the notation for the competitors once. For $a_1, \ldots, a_5 \ge 1$ let $C_5[a_1, \ldots, a_5]$ be the blow-up of the 5-cycle obtained by replacing its i-th vertex by an independent set of size a_i and joining consecutive parts completely; it is triangle-free, nonbipartite, and has $\sum_i a_i a_{i+1}$ edges. The *balanced* blow-up at order n is the one with part sizes as equal as possible. In the labelling used by our sweeps the three small blow-ups occurring in Table 1 have the edge sets

$$\begin{aligned}
C_5 &: \ 03,\ 32,\ 21,\ 14,\ 40 && \text{on } [5],\\
C_5[2,1,1,1,1] &: \ 03,\ 32,\ 21,\ 14,\ 40,\ 05,\ 15 && \text{on } [6],\ \text{degrees } (3,3,2,2,2,2),\\
C_5[2,2,1,1,1] &: \ 12,\ 13,\ 04,\ 24,\ 34,\ 05,\ 25,\ 35,\ 06,\ 16 && \text{on } [7],\ \text{degrees } (3,3,3,3,3,3,2).
\end{aligned}$$

4. Below and above the finite-order threshold

Theorem 4.1. *Let (n,τ) be one of the 20 cells of Table 1 whose maximizer column reads $T_2(n)$, that is,*

$$n \in \{5,6,7\},\ \tau \in \left\{\tfrac{1}{2}, 1, 2, 3, \tfrac{15}{4}, \tfrac{19}{5}\right\}; \qquad \left(6, \tfrac{9}{2}\right); \qquad \left(8, \tfrac{1}{2}\right).$$

Then every triangle-free graph G on $[n]$ that is not a labelled copy of $T_2(n)$ satisfies $F_G(\tau/n) < F_{T_2(n)}(\tau/n)$. Consequently $T_2(n)$ is, up to isomorphism, the unique maximizer of $F_G(\tau/n)$ over $\mathcal{T}_n$ at each of these cells. Four of the cells, namely $(n, \frac{19}{5})$ for $n = 5,6,7$ together with $(6, \frac{9}{2})$, lie strictly above the graphon threshold: $\frac{19}{5} = 3.8 > \tau_c$ and $\frac{9}{2} > \tau_c$.

TABLE 1. The 24 exhaustion cells. "gap" is $D^{(K)}(\text{best non-}T_2) - D^{(K)}(T_2(n))$ in units of L^2, positive when $T_2(n)$ wins. $B/2E$ is the factor by which the pinned margin exceeds the bound of Corollary 2.4. $\tau_c \approx 3.790989$.

n	τ	K	$\log_{10} B$	$B/2E$	gap	maximizer / runner-up
5	1/2	10	30	1.5×10^6	$+1.9406 \times 10^{-2}$	$T_2(5) = K_{2,3}$; C_5 second
5	1	12	31	1.1×10^5	$+7.0772 \times 10^{-2}$	$T_2(5)$; C_5 second
5	2	14	39	3.1×10^3	$+1.8884 \times 10^{-1}$	$T_2(5)$; C_5 second
5	3	18	55	3.7×10^4	$+1.7478 \times 10^{-1}$	$T_2(5)$; C_5 second
5	15/4	20	86	1.4×10^3	$+3.2943 \times 10^{-2}$	$T_2(5)$; C_5 second
5	19/5	20	90	1.4×10^3	$+2.0482 \times 10^{-2}$	$T_2(5)$; C_5 second
5	9/2	22	84	3.9×10^3	-1.6551×10^{-1}	C_5
5	5	24	79	7.7×10^3	-2.7922×10^{-1}	C_5
6	1/2	10	31	3.2×10^5	$+1.3570 \times 10^{-2}$	$T_2(6) = K_{3,3}$; $K_{3,3} - e$ second
6	1	12	33	1.1×10^5	$+5.0600 \times 10^{-2}$	$T_2(6)$; $K_{3,3} - e$ second
6	2	14	41	1.6×10^3	$+1.5124 \times 10^{-1}$	$T_2(6)$; $K_{3,3} - e$ second
6	3	18	57	4.4×10^3	$+1.9964 \times 10^{-1}$	$T_2(6)$; $K_{3,3} - e$ second
6	15/4	20	90	8.2×10^3	$+1.6459 \times 10^{-1}$	$T_2(6)$; $K_{3,3} - e$ second
6	19/5	20	94	8.2×10^3	$+1.6011 \times 10^{-1}$	$T_2(6)$; $K_{3,3} - e$ second
6	9/2	22	87	1.1×10^3	$+7.5791 \times 10^{-2}$	$T_2(6)$; $K_{3,3} - e$ second
6	5	24	82	1.0×10^3	-2.7955×10^{-2}	$C_5[2,1,1,1,1]$
7	1/2	10	33	1.3×10^6	$+9.9762 \times 10^{-3}$	$T_2(7) = K_{3,4}$; $K_{3,4} - e$ second
7	1	12	35	2.4×10^5	$+3.7265 \times 10^{-2}$	$T_2(7)$; $K_{3,4} - e$ second
7	2	14	43	1.8×10^3	$+1.1206 \times 10^{-1}$	$T_2(7)$; $K_{3,4} - e$ second
7	3	18	60	1.5×10^4	$+1.4820 \times 10^{-1}$	$T_2(7)$; $K_{3,4} - e$ second
7	15/4	20	92	1.5×10^3	$+1.1865 \times 10^{-1}$	$T_2(7)$; $K_{3,4} - e$ second
7	19/5	20	96	1.5×10^3	$+1.1484 \times 10^{-1}$	$T_2(7)$; $K_{3,4} - e$ second
7	9/2	22	90	1.1×10^3	-6.0537×10^{-2}	$C_5[2,2,1,1,1]$
8	1/2	10	34	7.7×10^5	$+7.6360 \times 10^{-3}$	$T_2(8) = K_{4,4}$; $K_{4,4} - e$ second

Proof sketch. Fix a cell and let B be the margin of Table 1. The computation is the complete sweep over all $g < 2^{\binom{n}{2}}$: it verifies that each such g either encodes a graph with a triangle, or is a labelled copy of $T_2(n)$, or satisfies $Q_{n,K,p,q}(g) \geq Q_{n,K,p,q}(T_2(n)) + B$. The four ranges are $2^{10} = 1\,024$ masks at $n = 5$, $2^{15} = 32\,768$ at $n = 6$, $2^{21} = 2\,097\,152$ at $n = 7$ and $2^{28} = 268\,435\,456$ at $n = 8$, containing respectively 388, 5 789, 133 501 and 4 682 270 triangle-free graphs (Proposition 7.2 and the sequence value quoted after it). Given the sweep, let G be triangle-free and not a labelled $T_2(n)$. The third alternative holds, and since $B/2E \geq 1.0 \times 10^3$ at every cell of Table 1, Corollary 2.4 applies with $H = T_2(n)$ and gives $F_G(\tau/n) < F_{T_2(n)}(\tau/n)$. Finally, all labelled copies of $T_2(n)$ are isomorphic and therefore share the value of F, so the statement over $\mathcal{T}_n$ follows; the constancy of Q on that class is also checked directly at $\tau = \frac{1}{2}$ (Proposition 7.4(iii)). □

Theorem 4.2. *$T_2(n)$ is not a maximizer of $F_G(\tau/n)$ over $\mathcal{T}_n$ at any of the four cells*

$$(n,\tau) = \left(5, \tfrac{9}{2}\right), \quad (5,5), \quad (6,5), \quad \left(7, \tfrac{9}{2}\right).$$

Explicitly, with the labellings of Section 3, C_5 satisfies $F_{C_5}(\tau/5) > F_{T_2(5)}(\tau/5)$ for $\tau = \frac{9}{2}$ and $\tau = 5$; the blow-up $C_5[2,1,1,1,1]$ beats $T_2(6) = K_{3,3}$ at $\tau = 5$; and the blow-up $C_5[2,2,1,1,1]$ beats $T_2(7) = K_{3,4}$ at $\tau = \frac{9}{2}$.

Proof sketch. A refutation needs no exhaustion: for each of the four cells the single explicit graph named is checked to be triangle-free, to be no labelled copy of $T_2(n)$, and to satisfy $Q(\text{witness}) + B \leq Q(T_2(n))$ with the margin B of Table 1; then apply Corollary 2.4 with the roles of G and H exchanged. □

Corollary 4.3. *For $n = 5, 6, 7$ the Turán graph $T_2(n)$ is still the unique maximizer of $F_G(\tau/n)$ at $\tau = \frac{19}{5} > \tau_c$, and for $n = 6$ also at $\tau = \frac{9}{2}$; while $T_2(5)$ and $T_2(7)$ are not maximizers at $\tau = \frac{9}{2}$, and $T_2(6)$ is not a maximizer at $\tau = 5$. Hence at each of $n = 5, 6, 7$ the change from "$T_2(n)$ is the unique maximizer" to "$T_2(n)$ is not a maximizer" occurs strictly above the graphon threshold τ_c: between $\frac{19}{5}$ and $\frac{9}{2}$ for $n = 5, 7$, and between $\frac{9}{2}$ and 5 for $n = 6$.*

Proof. Combine Theorems 4.1 and 4.2 and note $\frac{19}{5} > \tau_c$. □

Remark 4.4. The following is a floating-point computation carried out separately from the formal development and is recorded only as a consistency check on the brackets of Corollary 4.3. A double-precision prototype sharing no code with the exact one bisects the largest τ at which $T_2(n)$ still wins to 10^{-6} and reports 3.880, 4.908, 4.315 at $n = 5, 6, 7$ — each inside the corresponding bracket — and 4.524 at $n = 8$. A second and independent double-precision implementation, the one described in Section 9, reproduces all 24 gaps of Table 1 to six or more significant digits. These are floating-point values and no numerical enclosure is asserted for any of them; nothing stated elsewhere in this paper depends on them.

Theorem 4.5. *Let (n, τ) be one of $(5, \frac{9}{2})$, $(5, 5)$, $(6, 5)$, $(7, \frac{9}{2})$ and let $\mathcal{B}_n$ be the balanced blow-up of C_5 at order n: C_5 itself at $n = 5$ (for both values of τ), $C_5[2, 1, 1, 1, 1]$ at $n = 6$, and $C_5[2, 2, 1, 1, 1]$ at $n = 7$. Then $\mathcal{B}_n$ is triangle-free, is not a labelled copy of $T_2(n)$, and*

$$F^{(K)}_{\mathcal{B}_n}(\tau/n) \; = \; \max\bigl\{F^{(K)}_G(\tau/n) \; : \; G \text{ triangle-free on } [n]\bigr\},$$

the maximum of the order-K truncated transport value over all *triangle-free graphs on $[n]$, at the truncation order K of Table 1. Consequently*

$$F_G(\tau/n) \; \le \; F_{\mathcal{B}_n}(\tau/n) + 2\eta \qquad \text{for every triangle-free } G \text{ on } [n], \qquad 2\eta \le 1.1 \times 10^{-7},$$

so a blow-up of C_5 maximizes average transport at these four cells to within 1.1×10^{-7}, while by Theorem 4.2 it beats $T_2(n)$ there outright. The certified excess is $B/(L^2 n(n-1)) - 2\eta$, which exceeds 3×10^{-5} at all four cells, the margin term alone being at least 10^3 times 2η; the measured gaps of Table 1 put the actual excess above 9×10^{-4}.

Proof sketch. The full exhaustion is run again with the reference graph replaced by the blow-up: the sweep verifies that for every $g < 2^{\binom{n}{2}}$, either g has a triangle or $Q_{n,K,p,q}(\mathcal{B}_n) \le Q_{n,K,p,q}(g)$. Since $F^{(K)}$ is a strictly decreasing affine function of Q at a fixed cell, this is exactly the displayed maximality of $F^{(K)}_{\mathcal{B}_n}$, and (3) applied to $\mathcal{B}_n$ and to G gives the second display. □

Remark 4.6. Two caveats, which we state rather than resolve, and which are in the end one caveat twice over. First, only attainment is claimed: uniqueness up to isomorphism would need a recogniser for the isomorphism class of C_5 blow-ups, and was not attempted. Second, this sweep was run without a pinned margin, so what it certifies exactly is the maximality of the *truncated* value $F^{(K)}$. It is worth saying precisely why the second is not repaired by simply rerunning with a margin. A rerun carrying $B > 2E$ would show that every triangle-free G on $[n]$ satisfies either $Q(G) = Q(\mathcal{B}_n)$ or $Q(G) \ge Q(\mathcal{B}_n) + B$, which by Corollary 2.4 confines every maximizer of F to the set of graphs tying $\mathcal{B}_n$ in Q — but two graphs with equal Q may still differ in F, by as much as 2η, so one would then have to show that the tie set consists of labelled copies of $\mathcal{B}_n$ and nothing else, which is the recogniser of the first caveat. The uniqueness sweeps of Theorem 4.1 escape this only because their tie class is $T_2(n)$, for which a recogniser *was* built and proved correct. So the two caveats are one obstruction, and the truncated statement is what is cheaply available.

This is the point at which the finite problem stops being a corollary of anything asymptotic. Song's convergence theorem [9, Theorem 1.3] converges the extremal *values* $m_n(\tau)$ and identifies no maximizer at any finite order, and his global theorem [9, Theorem 1.1] identifies the graphon maximizer only for $\tau \le \tau_c$. For $\tau > \tau_c$ the identity of the maximizer, at the graphon level, is exactly what [9, Conjecture 7.1] leaves open.

5. Edge-count monotonicity

Theorem 5.1. *Let $n \in \{5,6,7\}$ and $\tau \in \{\frac{1}{2}, \frac{19}{5}\}$, and let K be the truncation order of the corresponding row of Table 1. Then for every triangle-free graph G on $[n]$ with $e(G) < \lfloor n^2/4 \rfloor$ there is a triangle-free graph H on $[n]$ with*

$$e(H) = e(G) + 1 \qquad \text{and} \qquad F_H^{(K)}(\tau/n) > F_G^{(K)}(\tau/n).$$

Equivalently, $k \mapsto \max\{F_G^{(K)}(\tau/n) : G$ triangle-free on $[n],\ e(G) = k\}$ is strictly increasing on the set of edge counts realised by triangle-free graphs of order n. In particular $F_H(\tau/n) > F_G(\tau/n) - 2\eta$ with $2\eta \le 7.1 \times 10^{-8}$ at these six cells.

Proof sketch. The sweep first computes, for each edge count k, a triangle-free graph on $[n]$ with k edges minimising $Q_{n,K,p,q}$, by one pass over all $2^{\binom{n}{2}}$ masks; call it H_k. It then makes a second pass verifying that every triangle-free G with $e(G) < \lfloor n^2/4 \rfloor$ satisfies $Q(H_{e(G)+1}) < Q(G)$, together with $H_{e(G)+1}$ being triangle-free with the right edge count. Both passes are complete searches over the same ranges as in Theorem 4.1. Since $F^{(K)}$ is a strictly decreasing affine function of Q at a fixed cell, the strict integer inequality is exactly the displayed strict inequality for $F^{(K)}$; the last sentence is (3). □

Remark 5.2. As in Remark 4.6, this sweep was run without a pinned margin, so the statement it certifies is about the truncated value $F^{(K)}$ and transfers to F only up to 2η. What makes that harmless here is that the conclusion drawn below — that monotonicity plus Mantel forces the maximizer — is itself a statement about a chain of strict integer comparisons, and needs no transfer. Unlike Remark 4.6, there is no obstruction of principle here, only a run that was not made: the assertion compares two named graphs and involves no tie class, so inserting a margin $B > 2E$ into the second pass would deliver the F-level statement outright, at the price of one further sweep over the ranges of Theorem 4.1. We report the truncated form because that is what the recorded computation certifies.

Given Mantel's theorem in the form of Proposition 7.1 — $T_2(n)$ is the only triangle-free graph on $[n]$ with $\lfloor n^2/4 \rfloor$ edges — Theorem 5.1 implies the corresponding cells of Theorem 4.1 at the level of the truncated value: iterating the monotonicity from any triangle-free G below the Mantel bound produces a chain of strict inequalities ending at the top stratum, whose only member is a labelled $T_2(n)$. It is strictly stronger than the uniqueness statement, because it constrains every stratum and not only the top two. At the cells where $T_2(n)$ loses, the monotonicity fails at the top strata, so the two phenomena change together.

6. Beyond exhaustion: witnesses to order 18

The exhaustions stop at $n = 8$; at $n = 9$ the number of labelled triangle-free graphs is already 246 348 115. But the statement "$T_2(n)$ is not the maximizer" is settled by one graph, so it can be pushed much further at negligible cost. The next proposition is included for completeness and for its small-order exceptions; we stress that its qualitative asymptotic content is *already implied* by [9], since [9, Theorem 1.1] gives $\Phi_\tau(B_1) < M_\triangle(\tau)$ for $\tau > \tau_c$ and [9, Theorem 1.3] gives $m_n(\tau) \to M_\triangle(\tau)$, so that $\Phi_n(T_2(n), \tau) < m_n(\tau)$ for all large n. What is added is effectivity — "sufficiently large" is unquantified in [9], while every order from 5 to 18 is settled here with an exact integer margin — and the small-order behaviour, which no asymptotic statement can see.

Proposition 6.1. *Let $\mathcal{B}_n$ denote the balanced blow-up of C_5 at order n, with part sizes as equal as possible. Then*

(i) *at $\tau = 5$, $F_{\mathcal{B}_n}(5/n) > F_{T_2(n)}(5/n)$ for every $5 \le n \le 18$;*
(ii) *at $\tau = \frac{9}{2}$, $F_{\mathcal{B}_n}(\frac{9}{2n}) > F_{T_2(n)}(\frac{9}{2n})$ for $n = 5$, $n = 7$ and every $9 \le n \le 18$;*
(iii) *the orders $n = 6$ and $n = 8$ are genuine exceptions to (ii). At $n = 6$ the exhaustion of Theorem 4.1 shows that* no *triangle-free graph beats $T_2(6)$ at $\tau = \frac{9}{2}$, so in particular $\mathcal{B}_n$ does not. At $n = 8$ the comparison of truncated values runs the other way, $F_{\mathcal{B}_n}^{(24)}(\frac{9}{16}) < F_{T_2(8)}^{(24)}(\frac{9}{16})$;*

this comparison carries no margin, and no exhaustion at $(8, \frac{9}{2})$ *was run, so whether some* other *graph beats* $T_2(8)$ *there is left undetermined.*

In particular the behaviour at a fixed $\tau > \tau_c$ *is not monotone in* n *at small orders.*

Proof sketch. Items (i) and (ii) are two-graph comparisons at $K = 24$: the blow-up is checked to be triangle-free and no labelled copy of $T_2(n)$, and the two integers Q are compared against a pinned margin, which at these 26 cells exceeds $2E$ by a factor between 1.0×10^3 and 3.9×10^4; Corollary 2.4 then transfers the comparison to F. In (iii), the assertion about $n = 6$ is the $(6, \frac{9}{2})$ row of Theorem 4.1, which does carry a margin; the assertion about $n = 8$ is the comparison of the same two Q-values running the other way, made without a margin, which is why it is stated for $F^{(24)}$. □

The column $\tau = \frac{9}{2}$ is the interesting one: $\frac{9}{2} < \tau_5 \approx 4.73387$ is inside the window in which [9, Conjecture 7.1] predicts a bipartite graphon maximizer, and yet at every order from 9 to 18 a nonbipartite graph already beats the balanced complete bipartite one.

7. Controls

An exhaustive search is worth what its instrument is worth, so we record the checks that make the instrument falsifiable. The first is a theorem the sweep must agree with.

Proposition 7.1 (Mantel, by the same exhaustion)**.** *For* $4 \le n \le 7$*, every triangle-free graph* G *on* $[n]$ *satisfies* $e(G) < \lfloor n^2/4 \rfloor$ *unless* G *is a labelled copy of* $T_2(n)$*.*

This is Mantel's theorem [5] with its equality case, and it is not offered as new; it is re-derived by the very enumeration and the very T_2-recogniser that carry Theorem 4.1, so that an error in either would disagree with a classical theorem rather than with nothing.

Proposition 7.2 (enumeration)**.** *The number of triangle-free graphs on the labelled vertex set* $[n]$ *is*

$$1,\ 2,\ 7,\ 41,\ 388,\ 5\,789,\ 133\,501 \qquad (n = 1, \dots, 7).$$

These are the values of OEIS `A213434` [7], whose next term 4 682 270 is the count at $n = 8$ used in the proof of Theorem 4.1; the value 5 789 at $n = 6$ is also printed by the verification script that [9] supplies as supplementary material, so the enumeration here and the enumeration inside the source agree by two independent routes.

The next control is external: it tests our closed-walk machinery against a certified fact of the source's, and its equality case against a prediction.

Proposition 7.3 (the source's six-vertex inequality on finite instances)**.** *For* $4 \le n \le 7$ *and every triangle-free graph* G *on* $[n]$*,*

$$4800\, n^4\, e(G) + 1216 \operatorname{Tr} A^6 \ \le\ 881\, n^6 + 2856\, n^2 \operatorname{Tr} A^4.$$

Moreover the two sides are equal at $T_2(n)$ *for* $n = 4, 6, 8, 10$*, and unequal at* $T_2(5)$*.*

This is not a new theorem. It is the cleared-denominator form, at the step graphon of a finite graph, of the six-vertex flag algebra inequality $881 - 2400\, t(K_2, W) + 2856\, t(C_4, W) - 1216\, t(C_6, W) \ge 0$ of [9, Theorem 4.6, eq. (36)], which is proved there for all triangle-free graphons by an exact rational LDL^{T} flag algebra certificate [8] supplied as supplementary material. Substituting $t(K_2) = 2e(G)/n^2$, $t(C_4) = \operatorname{Tr} A^4/n^4$ and $t(C_6) = \operatorname{Tr} A^6/n^6$ and clearing denominators gives the displayed integer inequality. We verify it at all $41 + 388 + 5\,789 + 133\,501 = 139\,719$ triangle-free graphs on four to seven labelled vertices, and the equality at $T_2(n)$ for even n is exactly the sharpness at B_1 that the certificate predicts. Since our walk counts are computed by machinery entirely unrelated to the flag algebra argument, this is the strongest available external check on that machinery.

Proposition 7.4 (instrument controls)**.** (i) *The table of integer coefficients used in the evaluation of* Q *agrees with the specification* $p^j (qn)^{K-j} K!/j!$ *at every order* $j \le K$*, at each of the* 23 *parameter sets* (K, p, q, n) *of the exhaustion grid with* $n = 5, 6, 7$*.*

(ii) *The recognisers are not constant and their hypotheses are not vacuous:* C_5 *is triangle-free and is not a labelled copy of* $T_2(5)$*; the triangle on* $\{0,1,2\} \subseteq [5]$ *is recognised as containing a triangle;* $T_2(n)$ *is recognised as a labelled* $T_2(n)$ *and is triangle-free for* $n = 4, \dots, 8$*; the empty graph on* $[5]$ *is triangle-free and is not a labelled* $T_2(5)$*.*
(iii) *All labelled copies of* $T_2(n)$ *carry the same value of* Q *for* $n = 4, 5, 6, 7$ *at* $\tau = \frac{1}{2}$*, so comparing against one representative compares against the whole class.*
(iv) *A too-large claim is refuted: it is* false *that* $T_2(5)$ *maximizes* $F_G(\tau/5)$ *for every* $\tau > 0$*, since* C_5 *beats it at* $\tau = 5$*.*
(v) *The pinned margins are not slack by two orders of magnitude: at* $(6, \frac{1}{2})$ *the margin* 10^{31} *cannot be raised to* 10^{33}*, because* $K_{3,3}$ *minus an edge lies strictly inside the larger band.*

Item (iv) is the negative control the whole paper needs: a development in which some quantifier had slipped could prove "$T_2(n)$ always wins" and would then contradict (iv). Item (v) says that the margins in Table 1, although generous relative to the truncation error, are not so generous as to be vacuous; measured against the true gaps they are exceeded by a factor between 10.0 and 96.8 across the 24 cells.

8. What is not settled

Remark 8.1 (an explicit but far too small window; outside the formal development)*.* The following is proved by an ordinary argument and is not part of the machine-checked development. Song's [9, Proposition C.2] produces ε_n from finiteness of $\mathcal{T}_n$ and gives no rate. A rate can be supplied: for a triangle-free G one has $(A)_{uu} = (A^3)_{uu} = 0$, so $\Sigma_u(t) = O(t^5)$ and the t^4 and higher coefficients of $|U_{uu}|^2$ come from C_u alone; Nosal's bound $\lambda_1 \le \sqrt{e(G)} \le n/2$ for triangle-free graphs [6] gives $|(A^j)_{uu}| \le \lambda_1^j \le (n/2)^j$, so the t^m-coefficient of $C_u^2 + \Sigma_u^2$ is at most $n^m/m!$ in absolute value and $|R_G(t)| \le n \sum_{m \ge 4} (nt)^m/m! \le (\mathrm{e}/24) n^5 t^4$ for $|t| \le 1/n$. With $[t^2](S_{T_2(n)} - S_G) \ge 2$ by Mantel this yields $\varepsilon_n \ge 2.97\, n^{-5/2}$, i.e. uniqueness of $T_2(n)$ for $0 < \tau < 2.97\, n^{-3/2}$. The scaled question needs a window of constant width in τ, so this is short by a factor $n^{3/2}$; handling the t^4 term exactly instead of by the crude bound improves the exponent in τ to $n^{-3/4}$ and no further, though with a constant we have not computed. Every cell of Table 1 lies outside the window $\varepsilon_n \ge 2.97\, n^{-5/2}$ proved here, by a factor between 1.9 and 28 in t; in particular the headline cell $(7, \frac{19}{5})$ has $t = 0.5429$ against $\varepsilon_7 \approx 0.0229$. Nothing proved above is implied by anything elementary.

Question 8.2. The question of [9, Problem 7.2] itself, for all large n and fixed $0 < \tau < \tau_c$. Everything above is a finite grid of cells; a finite grid of τ checks those τ only, and a finite range of n says nothing about large n. The data are consistent with an affirmative answer and show no exception below the transition: at $n = 5, 6, 7$ the transition already sits above τ_c, and at τ well below τ_c the gap between $T_2(n)$ and its runner-up exceeds the truncation error by three orders of magnitude or more.

Question 8.3. Is the edge-count monotonicity of Theorem 5.1 true for all n and all $0 < \tau < \tau_c$? It implies the answer to Problem 7.2 given Mantel, and it is a statement about every stratum rather than about a single comparison near $T_2(n)$, which is the kind of local comparison [9] asks for.

Question 8.4. Reconciling the finite picture inside (τ_c, τ_5) with [9, Conjecture 7.1]. At $\tau = \frac{9}{2}$ the finite maximizer at $n = 5, 7$ is a C_5 blow-up (Theorem 4.5) and a C_5 blow-up beats $T_2(n)$ at every $9 \le n \le 18$ (Proposition 6.1), while the conjectured graphon optimum in that window is the bipartite B_{q_τ}. The likely explanation is that B_{q_τ} is a cut limit of *quasirandom* bipartite sequences, and that no graph of order 18 approximates one; testing this means comparing against the best *bipartite* graph at each order, which our exhaustions do only for $n \le 7$.

Remark 8.5 (cost, measured; outside the formal development)*.* The cost of an exhaustion is essentially (number of triangle-free masks)$\times K \times n^3$ plus the $2^{\binom{n}{2}}$ triangle filter. Measured on the compiled evaluator: $n = 7$ at one τ takes 22–59 seconds and $n = 8$ at $\tau = \frac{1}{2}$ takes 19.0 CPU-minutes over 2^{28} masks. Order 9 is out of reach by this route: there are 246 348 115 labelled triangle-free graphs on nine vertices against 1 897 isomorphism classes, a factor of 129 862, so a search organised by isomorphism class rather than by labelling is the natural next instrument, and it is what would be needed to reach $n = 12$.

9. Verification

All finite searches reported above have been formally verified in Lean 4, toolchain version `v4.32.0`; the development contains no `sorry`, and the repository reference is to be added at submission — the repository is private at the time of writing. The modules for this development import no part of *Mathlib*: they are deliberately self-contained so that they compile to a shared library and the exhaustions run as compiled code rather than through Lean's interpreter. What is delegated to that compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: the sweeps over $2^{10}, 2^{15}, 2^{21}, 2^{28}$ masks behind Theorems 4.1, 4.5 and 5.1, the single-graph comparisons behind Theorem 4.2 and Proposition 6.1, the Mantel and flag-inequality sweeps of Propositions 7.1 and 7.3, the enumeration of Proposition 7.2, and the controls of Proposition 7.4; each such call contributes one axiom, and there are 96 of them in all. Everything else is kernel-clean, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: in particular the two lemmas that identify the Boolean triangle test and the Boolean $T_2(n)$ test with the quantified statements about graphs, and the seven lemmas that read each kind of sweep back as the mathematical statement it is quoted as above. Two things are *not* in the development and are proved by hand in this paper: Lemma 2.3 and Corollary 2.4, that is, the passage from the integer inequality on the truncated series to the inequality on $F_G(\tau/n)$; formalising them would require the theory of the matrix exponential and an entrywise series identity, and until that is done the machine-checked content of each theorem above is its integer statement together with the pinned margin, or — in Theorems 4.5 and 5.1 and in Proposition 6.1(iii) at $n = 8$, where no margin was pinned — the bare integer comparison, which is why those statements are given for $F^{(K)}$ and not for F (Remarks 4.6 and 5.2). The values E of Corollary 2.4 in Table 1 were computed in exact rational arithmetic. Finally, every number in Table 1 was first produced by an independent implementation in complex double-precision floating point, sharing no code with the exact evaluator, and agrees with the exact value to six or more significant digits; the labelled triangle-free counts of Proposition 7.2 were independently reproduced by a vertex-extension search written in C. Total compilation time for the six modules is about 23 minutes, of which 14 minutes is the single $n = 8$ sweep.

Acknowledgement of sources

The problem, the functional, the escape identity of Lemma 2.1, the competitor W_{C_5}, the threshold τ_c, the numerical value $\tau_5 \approx 4.73387$ and the flag algebra inequality of Proposition 7.3 are all due to [9]; the finite-order data, the identification of the finite maximizers, and the edge-count monotonicity are what this paper contributes. All references to [9] are to version 1 of that preprint, dated 27 August 2026 — the only version in existence, and unpublished, as of 30 August 2026 — and every result number we quote from it (Lemma 2.1 with its equation (12); Theorems 1.1, 1.3 and 4.6 with equation (36); §7.1; Conjecture 7.1; Problem 7.2; Proposition C.2) is v1's numbering, read from v1 itself.

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