THURSTON–BENNEQUIN-SYMMETRICAL GRAPHS: THE ODD-GRAPH CONJECTURE, THE HEAWOOD CASE, AND THE CLASSIFICATION ON TEN VERTICES

KOREA SUPERINTELLIGENCE LABS

Abstract. Chau and Shah (arXiv:2603.28487) call a finite simple graph G *TB-symmetrical* when, for every two cycle lengths $r \neq s$, the numbers of r-cycles through each edge, through each pair of adjacent edges and—with a sign recording orientation—through each pair of non-adjacent edges are a common nonnegative multiple $\rho_{r,s}$ of the corresponding numbers for s-cycles. For such G the total Thurston–Bennequin number of every Legendrian embedding is $(1+\sum_{r\neq s}\rho_{r,s})$ times the contribution of the s-cycles. They conjecture that every odd graph O_n is TB-symmetrical, prove it for $n = 2, 3$, and classify the connected pendant-free (almost-)TB-symmetrical graphs on at most nine vertices. We show that the conjecture fails at its first open case: $O_4 = K(7, 3)$ is not TB-symmetrical, the orientation condition failing between cycle lengths 6 and 7 on two explicit pairs of non-adjacent edges. Two computational statements of the source are corrected: the Heawood graph *is* TB-symmetrical (the cited code builds a Möbius ladder in its place, and that graph is not even almost-TB-symmetrical), and the 1-clique sum $K_5 \oplus^0 K_5$ is missing from the nine-vertex list. An exhaustive computation over the 9 808 209 connected pendant-free graphs on ten vertices, reported outside the formal development, finds 46 almost-TB-symmetrical graphs, 45 of them TB-symmetrical; two are new for the subject: the subdivision $S(K_4)$, a second TB-symmetrical graph with cycles of more than one length that is not obtainable from complete and complete bipartite graphs by the source's operations, and $K_{5,5}$ minus a perfect matching, a second almost- but not TB-symmetrical graph after the cube. Finally the Pappus and Desargues graphs are 3-arc transitive and not TB-symmetrical, so 3-arc transitivity does not imply TB-symmetricity, which closes the natural route to the conjecture. The (almost-)TB-symmetricity statements of the theorems are verified in Lean 4.

1. Introduction

A *Legendrian graph* is an embedding $\tilde{G}$ of a finite graph G into the standard contact 3-space $(\mathbb{R}^3, \xi_{\text{std}})$ whose edges are everywhere tangent to the contact planes. Each cycle γ of G is then a Legendrian knot with a Thurston–Bennequin number $\text{tb}(\gamma)$, and the *total* Thurston–Bennequin number of $\tilde{G}$ is $\text{TB}(\tilde{G}) = \sum_\gamma \text{tb}(\gamma)$, summed over all cycles of G; $\text{TB}_s(\tilde{G})$ denotes the same sum restricted to the cycles of length s. O'Donnol and Pavelescu [2] proved that for the complete graph K_n every $\text{TB}_r(\tilde{G})$ is a multiple of $\text{TB}_3(\tilde{G})$, and that for $K_{n,m}$ with $n \geq m \geq 3$ every $\text{TB}_{2r}(\tilde{G})$ is a multiple of $\text{TB}_4(\tilde{G})$. Their proofs are counting arguments about the abstract graph: how many r-cycles pass through an edge, through a pair of adjacent edges, and—with the two possible relative orientations kept apart—through a pair of non-adjacent edges.

Chau and Shah [1] isolated the counting condition behind these proofs. A finite simple graph is (r, s)-*TB-symmetrical* (Definition 2.1 below, which is their Definition 4.1) when there is a single $\rho \geq 0$ such that all three kinds of r-cycle counts are ρ times the corresponding s-cycle counts, the third kind being a signed count; it is *almost-*(r, s)*-TB-symmetrical* when only the first two kinds are required; and it is *(almost-)TB-symmetrical* when for every pair of distinct lengths $r, s \geq 3$ one of the two orders holds. Their main theorem (Theorem 4.5 of [1]) is that if G is TB-symmetrical and has an s-cycle then, for every Legendrian embedding,

$$\text{TB}(\tilde{G}) = \Big(1 + \sum_{r\geq 3,\ r\neq s} \rho_{r,s}(G)\Big)\ \text{TB}_s(\tilde{G}). \tag{1}$$

Everything in the present paper is about the combinatorial condition; no contact geometry is used, and (1) is quoted only to say why the condition matters.

What the source leaves open. Three things. First, its Conjecture 1.2, verbatim:

> "The odd graph O_n is TB-symmetrical for any $n \geq 2$."

The odd graph O_n has as vertices the $(n-1)$-subsets of a $(2n-1)$-set, two being adjacent when disjoint; $O_2 = K_3$ and O_3 is the Petersen graph, and these are the two cases the source settles (its Proposition 5.3(4) for the Petersen graph). Second, the classification (its Theorem 6.4) of the connected pendant-free almost-TB-symmetrical graphs stops at nine vertices. Third, the source says of the signed clause of its definition: "It is unclear to us when the third condition in Definition 4.1 is satisfied in general", and closes its Section 6 with "the Petersen graph O_3 is currently the only known graph that is TB-symmetrical and cannot be obtained from complete graphs, complete bipartite graphs, and the operations discussed in this section."

What is settled here.

(1) **Conjecture 1.2 is false at its first open case.** $O_4 = K(7,3)$ is not TB-symmetrical (Theorem 3.1). It is 3-arc transitive, hence almost-TB-symmetrical by the source's own Theorem 5.1; what fails is exactly the signed clause, between the two shortest cycle lengths 6 and 7, on two explicit pairs of non-adjacent edges. The refutation needs only the 6- and 7-cycles through one edge (9 and 36 of them).

(2) **The Heawood graph is TB-symmetrical** (Theorem 4.1), contrary to the source's Proposition 5.3(3). The discrepancy has a mechanical explanation: the Sage file the source cites for that verification, at the commit its bibliography pins, builds a 14-cycle with the seven *antipodal* chords $i \leftrightarrow i+7$, which is the Möbius ladder M_{14} and not the Heawood graph. That graph is not even almost-TB-symmetrical (Theorem 4.2). The implication diagrams of the source thereby lose their only counterexample to "4-arc transitive $\Rightarrow$ TB-symmetrical".

(3) **One graph is missing from the nine-vertex list.** The 1-clique sum $K_5 \oplus^0 K_5$ is TB-symmetrical (Theorem 5.2) and is not in the list of Theorem 6.4 of [1]. It is an instance of the source's own Lemma 6.3, so the omission is in the list, not in the method. The same enumeration settles an ambiguity in the source's definition of the path-join $\oplus^t$ (Remark 5.4).

(4) **Ten vertices.** Of the 9 808 209 connected pendant-free graphs on ten vertices exactly 46 are almost-TB-symmetrical and 45 of those are TB-symmetrical (Computation 6.1; an exhaustive machine enumeration, not part of the formal development). Two members are new for the subject and are verified individually: the subdivision $S(K_4)$ is TB-symmetrical (Theorem 6.2) and is not obtainable from complete and complete bipartite graphs by the source's operations (Proposition 6.3), a second graph of the kind the sentence quoted above calls unique to the Petersen graph; and $K_{5,5}$ minus a perfect matching is almost- but not TB-symmetrical (Theorem 6.5), the second such graph after the cube Q_3, which is $K_{4,4}$ minus a perfect matching.

(5) **3-arc transitivity does not imply TB-symmetricity.** The Pappus and Desargues graphs are 3-arc transitive and not TB-symmetrical (Theorem 7.1, Corollary 7.2), while the Petersen and Heawood graphs are. The obvious route to Conjecture 1.2—strengthening "2-arc transitive $\Rightarrow$ almost-TB-symmetrical" to "3-arc transitive $\Rightarrow$ TB-symmetrical", which would cover all odd graphs—is therefore closed.

Method. The condition of Definition 2.1 is a rank-one condition on a finite family of integer vectors (Lemma 2.3), so a graph is confirmed by computing the vectors and refuted by a nonzero 2×2 minor (Lemma 2.4). All positive and negative statements about named graphs in this paper were decided that way and are verified in Lean 4 against *Mathlib* (Section 9): the counts are *defined* as cardinalities of the sets of cycles that Definition 2.1 names, the cycle enumerator is proved to list exactly those sets without repetition, and the decision procedure is proved sound; only the final evaluations on a concrete graph are by compiled evaluation. Before anything new was claimed, the source's own Proposition 5.3(1),(2),(4),(5), its Theorem 4.6 on small complete and complete bipartite graphs, and the whole nine-vertex list of its Theorem 6.4 were re-derived from its Definition 4.1 by the same procedure (Proposition 5.1); the two disagreements found by that re-derivation are items (2) and (3) above. Three independent implementations (C, Python and the Lean development) agree on every value quoted. The exhaustive ten-vertex enumeration, the observed transfer of the property under edge subdivision, and the behaviour of the crown graphs are computations outside the formal development and are labelled as such wherever they appear.

Novelty. The term "TB-symmetrical" occurs, in a full-text search of a local arXiv e-print corpus whose mathematics shards run to the current announcement month, only in [1] itself: a single shard fires, and each of its sixty matching lines lies inside the corpus record of arXiv:2603.28487. OpenAlex resolves [1] to a work of citation count 0 and returns no citing work. None of the four integer sequences computed here—the three count sequences of Computation 6.1 and the cycle census of O_4 in Remark 3.2(ii)—is in the OEIS, whereas the sequence A004108 of the search space itself, run first as a positive control, is found. All three closures were run on 7 September 2026.

2. Preliminaries

2.1. **Cycle counts and the definition.** Throughout, G is a finite simple graph with vertex set V, $|V| = n$, and a *t-cycle* is a cycle of G of length $t \geq 3$, i.e. a set of t distinct vertices cyclically ordered so that consecutive vertices are adjacent, up to rotation and reversal. For an edge e let $f_e^{(t)}$ be the number of t-cycles containing e. For a pair $p = (e, e')$ of adjacent edges let $g_p^{(t)}$ be the number of t-cycles containing both. For a pair $p = (\{a, b\}, \{c, d\})$ of non-adjacent edges (four distinct vertices), a t-cycle through both, once oriented so that a immediately precedes b, traverses the second edge either as c then d or as d then c; let $u_p^{(t)}$ and $h_p^{(t)}$ be the numbers of t-cycles of the two kinds, and put $w_p^{(t)} = u_p^{(t)} - h_p^{(t)}$. Reversing the labelling of either edge changes the sign of $w_p^{(t)}$ for every t at once, so every statement below is independent of the labelling.

Definition 2.1 ([1, Definition 4.1])**.** Let $r, s \geq 3$ be distinct integers. G is *almost-(r, s)-TB-symmetrical* if there is a nonnegative number ρ such that

1. each edge e of G appears in f_e s-cycles and ρf_e r-cycles;
2. each pair of adjacent edges $p = (e, e')$ of G appears in g_p s-cycles and ρg_p r-cycles;

ρ is then denoted $\rho_{r,s}(G)$, with the convention $\rho_{r,s}(G) = 0$ if G has neither r- nor s-cycles. G is *(r, s)-TB-symmetrical* if moreover

3. each pair of non-adjacent edges $p = (e, e') = (\{a, b\}, \{c, d\})$ appears in u_p s-cycles and v_p r-cycles with orientation $\cdots ab \cdots cd \cdots$, and h_p s-cycles and k_p r-cycles with orientation $\cdots ab \cdots dc \cdots$, with $v_p - k_p = \rho(u_p - h_p)$.

G is *almost-TB-symmetrical* (resp. *TB-symmetrical*) if for every two distinct integers $r, s \geq 3$ it is almost-(r, s)- or almost-(s, r)-TB-symmetrical (resp. (r, s)- or (s, r)-TB-symmetrical).

In the notation above, (1)–(3) for the pair (r, s) read $f_e^{(r)} = \rho f_e^{(s)}$, $g_p^{(r)} = \rho g_p^{(s)}$ and $w_p^{(r)} = \rho w_p^{(s)}$.

Remark 2.2 (facts quoted from [1]). The following are used below and are not re-proved. Lemma 4.2: G has no r-cycle if and only if it is (almost-)(r, s)-TB-symmetrical with $\rho_{r,s}(G) = 0$ for some (equivalently every) $s \neq r$; in particular a graph all of whose cycles have the same length is TB-symmetrical. Lemma 4.3: if $\rho_{r,s}(G) \neq 0$ then $\rho_{s,r}(G) = 1/\rho_{r,s}(G)$. Theorem 4.6 (attributed to [2]): K_n and $K_{m,n}$ are TB-symmetrical for all $m, n \geq 1$. Theorem 4.7: if G is edge-transitive and (almost-)TB-symmetrical and has an s-cycle, then $\rho_{r,s}(G) = rc_r(G)/(sc_s(G))$, where $c_t(G)$ is the number of t-cycles of G. Theorem 5.1: a 2-arc transitive graph is almost-TB-symmetrical. (An s-arc is a walk $v_0, \ldots, v_s$ with $v_{i-1} \neq v_{i+1}$; G is s-arc transitive if $\mathrm{Aut}(G)$ is transitive on s-arcs, and s-arc *regular* if it is moreover regular on them.)

2.2. **Profiles, and the finite check.** For $t \geq 3$ let $V_t(G)$ be the integer vector indexed by the edges of G followed by the pairs of adjacent edges, with entries $f_e^{(t)}$ and $g_p^{(t)}$; let $W_t(G)$ be the vector indexed by the pairs of non-adjacent edges with entries $w_p^{(t)}$; and let $U_t(G) = V_t(G) \,\|\, W_t(G)$ be their concatenation. A cycle has distinct vertices, so $U_t(G) = 0$ for $t > n$.

Lemma 2.3. *G is almost-TB-symmetrical if and only if the vectors $V_t(G)$, $3 \leq t \leq n$, are pairwise proportional, and TB-symmetrical if and only if the vectors $U_t(G)$, $3 \leq t \leq n$, are pairwise proportional. When a pair is proportional the ratio can be taken nonnegative and rational.*

Proof. For fixed distinct r, s, clauses (1)–(2) say $V_r = \rho V_s$ and clause (3) says $W_r = \rho W_s$, so G is (r,s)- or (s,r)-TB-symmetrical if and only if $U_r \in \mathbb{R}_{\geq 0} U_s$ or $U_s \in \mathbb{R}_{\geq 0} U_r$. If $U_s \neq 0$ then $V_s \neq 0$ (a graph with no s-cycle has $W_s = 0$ as well), and a ρ with $U_r = \rho U_s$ is the quotient of two nonnegative integers at any coordinate where V_s is nonzero; if $U_s = 0$ then $\rho = 0$ works in the other order. Lengths $t > n$ contribute zero vectors, for which $\rho = 0$ works. The almost case is the same with V in place of U. □

Lemma 2.4 (refutation by a minor)**.** *Let $r \neq s$ be at least 3 and let i, j be two coordinates of U_t. If $U_r[i]\, U_s[j] \neq U_r[j]\, U_s[i]$ then G is neither (r,s)- nor (s,r)-TB-symmetrical, hence not TB-symmetrical; if i, j are coordinates of V_t the same conclusion holds for almost-TB-symmetricity.*

Proof. If $U_r = \rho U_s$ then $U_r[i]U_s[j] = \rho U_s[i]U_s[j] = U_r[j]U_s[i]$; the other order is symmetric. □

So a positive result costs the profiles at every length up to n, and a negative one costs two coordinates at two lengths. Every negative result below is a 2×2 minor of this kind, and the two lengths involved are always the two shortest cycle lengths of the graph or nearly so.

2.3. **The formal counterpart.** The formal development represents a graph on $\{0, \dots, n-1\}$ by a symmetric irreflexive adjacency predicate. A t-cycle through the directed edge (a, b) is represented by the list of its vertices starting $a, b, \dots$, so that each cycle through the edge $\{a, b\}$ has exactly one representative; the sets so defined are written $\mathrm{BasedCyc}(A, t, a, b)$.

Proposition 2.5. *In the formal development the counts $f^{(t)}_{\{a,b\}}$, $g^{(t)}_p$ for the 2-arc $x — v — y$, and $u^{(t)}_p$ are* defined *as the cardinalities of $\mathrm{BasedCyc}(A, t, a, b)$, of the subset of $\mathrm{BasedCyc}(A, t, v, x)$ whose last vertex is y, and of the subset of $\mathrm{BasedCyc}(A, t, a, b)$ traversing c then d; $w^{(t)}_p$ is the difference of two such cardinalities; and Definition 2.1 is transcribed clause by clause with $\rho \in \mathbb{Q}$ (which is no restriction, by Lemma 2.3). The following are theorems there, proved without any compiled evaluation:*

(a) *a depth-bounded search $\mathrm{cyclesAt}(A, t, a, b)$ lists a vertex list if and only if the list is a t-cycle based at (a, b), and lists it once; consequently each of the three counts equals the length of the corresponding filtered list;*
(b) *if the profiles U_t, $3 \leq t \leq n$, computed from those lists, pass a pairwise proportionality check with a nonnegative ratio read off the profiles themselves, then G is TB-symmetrical; likewise for V_t and almost-TB-symmetricity;*
(c) *a nonzero 2×2 minor as in Lemma 2.4, on two edges, two adjacent pairs, or two non-adjacent pairs, refutes almost-TB-symmetricity, almost-TB-symmetricity, or TB-symmetricity respectively.*

The Lean names are:

(a) `mem_cyclesAt`, `nodup_cyclesAt`, `fE_eq`, `gP_eq`, `uCnt_eq`, `wD_eq`;
(b) `tbSym_of_check`, `almostTBSym_of_check`;
(c) `not_almostTBSym_of_edge_det`, `not_almostTBSym_of_arc_det`, `not_tbSym_of_nap_det`.

Their statements are in Appendix A. A positive result about a named graph is then a single compiled evaluation of the check in (b); a negative one is four compiled evaluations of counts, after which the minor is evaluated by the kernel.

2.4. **The antecedent.** The counting scheme goes back to [2]. Its Theorem 2.2 states, for a Legendrian embedding of K_n, that "$TB_r(K_n)$ is a multiple of $TB_3(K_n)$", and item (3) of its proof reads: "Non-adjacent edges do not contribute to the total writhe of the graph. The cycles containing both edges *ab* and *cd* can be paired as *...ab...cd...* and *...ab...dc....*" That is clause (3) of Definition 2.1 in the special case $u_p = h_p$. In the bipartite case (its Theorem 3.1) the two orientations are *not* balanced: a pair of non-adjacent edges of $K_{n,m}$ lies in one 4-cycle and in $(2r-3)X$ cycles of length $2r$ with $X = \frac{(m-2)!\,(n-2)!}{(m-r)!\,(n-r)!}$, split as $(r-1)X$ against $(r-2)X$ between the two orientations, so $w^{(2r)}_p = \pm X \neq 0$. Clause (3) is therefore a genuine proportionality and not a vanishing condition, and that is the shape of every positive result below (Proposition 8.1(f)).

3. The odd graph O_4

O_4 is the Kneser graph $K(7,3)$: its 35 vertices are the 3-subsets of $\{0,\dots,6\}$, adjacent when disjoint; it is 4-regular with 70 edges, of girth 6 and odd girth 7. Write abc for the subset $\{a,b,c\}$.

Theorem 3.1. *The odd graph O_4 is not TB-symmetrical. Explicitly, with the edges $e = \{012, 345\}$, $e' = \{013, 245\}$, $e'' = \{013, 246\}$ and the pairs of non-adjacent edges $p = (e, e')$, $q = (e, e'')$, labelled in the order written,*

$$w_p^{(6)} = -1, \qquad w_p^{(7)} = 0, \qquad w_q^{(6)} = 0, \qquad w_q^{(7)} = -2,$$

so the minor $w_p^{(6)} w_q^{(7)} - w_q^{(6)} w_p^{(7)} = 2$ is nonzero and no $\rho \geq 0$ satisfies clause (3) of Definition 2.1 for the pair $(6,7)$ in either order.

Proof sketch. The four values are read off the 6-cycles and the 7-cycles through the directed edge $(012, 345)$—there are 9 and 36 of them—by recording, for each, whether it traverses e' (resp. e'') as 013 then 245 (resp. 246) or the other way. Lemma 2.4 then applies. Concretely, for the order $(6,7)$ the pair p would force $-1 = \rho \cdot 0$, and for $(7,6)$ the pair q would force $-2 = \rho \cdot 0$. In the formal development the vertex list is proved by kernel evaluation to be exactly the 3-subsets of a 7-set (`kneser73_spec`), the two pairs are proved to be pairs of non-adjacent edges, the four counts are four compiled evaluations, and the minor is evaluated by the kernel. In C the same four counts take 0.02 processor-seconds. □

Theorem 3.1 refutes Conjecture 1.2 of [1] at its first open case.

Remark 3.2. (i) O_4 is 3-arc transitive, so it is almost-TB-symmetrical by Theorem 5.1 of [1]. That was not taken on trust: $\mathrm{Aut}(O_4)$ was enumerated by the backtracking of Section 7 and has order $5040 = |S_7|$, with one orbit on the 1260 3-arcs and two on the 4-arcs (a computation outside the formal development). For the odd graphs in general the source cites [9] and, at a lemma number we repeat only as the source gives it, [10, Lemma 4.1.2]. The C implementation independently confirms the proportionality of the V_t up to cycle length 20. So *only* clause (3) fails on O_4—the clause the source itself describes as unclear. Section 7 gives the structural reason. (ii) On the way, the C implementation counted the cycles of O_4 of length 6 to 16: 105, 360, 315, 840, 3024, 7560, 18270, 32760, 79380, 199584, 405720. The value $c_7 = 360 = 7!/14$ is the expected size of one S_7-orbit of 7-cycles with dihedral stabiliser, and $c_6 = 105 = 7!/48$; neither is needed for the theorem. The full cycle census of O_4 (all lengths up to 35) was not computed: from the measured growth it would cost on the order of a hundred processor-hours, and nothing here depends on it. (iii) $O_5 = K(9,4)$, on 126 vertices, was not examined. A refutation of the same shape would need only the cycles of two lengths through one edge.

4. The Heawood graph and the Möbius ladder

The Heawood graph is the incidence graph of the Fano plane, the unique $(3,6)$-cage, on 14 vertices and 21 edges; in LCF notation it is $[5,-5]^7$, i.e. the 14-cycle $0,1,\dots,13$ with the chords $i \leftrightarrow i+5$ for even i and $i \leftrightarrow i-5$ for odd i (indices mod 14). It is bipartite of girth 6 and 4-arc transitive [5]. Proposition 5.3(3) of [1] states that "the Heawood graph is 4-arc transitive, and not TB-symmetrical", and the source's two implication diagrams use it as the counterexample to "4-arc transitive ⇒ TB-symmetrical".

Theorem 4.1. *The Heawood graph is TB-symmetrical.*

Proof sketch. The profiles U_t for $3 \leq t \leq 14$ are computed and found pairwise proportional; in the formal development this is one compiled evaluation of the check of Proposition 2.5(b), on the graph given by the adjacency lists of $[5,-5]^7$. Its cycle counts are $c_6, c_8, c_{10}, c_{12}, c_{14} = 28, 21, 84, 56, 24$ (odd lengths are absent, the graph being bipartite), and the ratios found are exactly those forced by edge-transitivity and Theorem 4.7 of [1]: $\rho_{t,6} = t\, c_t/(6\, c_6) = 1, 1, 5, 4, 2$ for $t = 6, 8, 10, 12, 14$. □

The positive result is not an artefact of a vacuous clause (3): for $p = (\{0,1\},\{2,3\})$ one has $w_p^{(6)} = 2$ (Proposition 8.1(f)). The graph used was checked to be the Heawood graph in two independent ways: its adjacency lists are those of $[5,-5]^7$, and its canonical form under `nauty` [11] coincides with that of the incidence graph of the Fano plane built from the seven lines $012, 034, 056, 135, 146, 236, 245$.

Where the source's verification went wrong. Remark 5.2 of [1] explains that the TB-symmetricity checks of Proposition 5.3 were run with the authors' Sage code, cited there as [3]. We downloaded that file from the exact commit the source's bibliography pins. Its Heawood section reads as follows (verbatim, except that an intervening `print` statement, a `# Edges` comment line, some blank lines and the middle of the 14-cycle are elided):

```
##### Heawood graph
vertices = ['0','1','2','3','4','5','6','7','8','9','A','B','C','D']
edges = [
    # Outer 14-cycle
    ('0','1'), ('1','2'), ..., ('C','D'), ('D','0'),

    # "Opposite" chords (i ↔ i+7 mod 14)
    ('0','7'), ('1','8'), ('2','9'), ('3','A'),
    ('4','B'), ('5','C'), ('6','D')
]
```

A 14-cycle together with its seven antipodal chords $i \leftrightarrow i+7$ is the Möbius ladder M_{14}: it has girth 4 and is not bipartite, whereas the Heawood graph is bipartite of girth 6. So Proposition 5.3(3) reports a computation about a different graph. That graph fails the definition at the first hurdle:

Theorem 4.2. *The Möbius ladder M_{14} is not almost-TB-symmetrical. Explicitly, the rim edge $\{0,1\}$ lies in one 4-cycle and two 6-cycles, while the chord $\{0,7\}$ lies in two 4-cycles and two 6-cycles: $f^{(4)}_{01} = 1$, $f^{(6)}_{01} = 2$, $f^{(4)}_{07} = 2$, $f^{(6)}_{07} = 2$, and $1 \cdot 2 - 2 \cdot 2 = -2 \neq 0$.*

Proof sketch. The four counts are compiled evaluations; Lemma 2.4 on the two edges at lengths 4 and 6 is then a kernel step. (The 4-cycles of M_{14} are the seven "squares" $\{i, i+1, i+8, i+7\}$, each rim edge lying in one of them and each chord in two.) □

Remark 4.3. M_{14} is vertex-transitive but not edge-transitive: by Theorem 4.2 no automorphism carries a rim edge to a chord. (The backtracking of Section 7 gives $|\operatorname{Aut}(M_{14})| = 28$, one orbit on vertices and two on edges—a computation outside the formal development, and not needed for the statement just made.) In particular M_{14} is not 1-arc transitive, so it also fails the assumption under which Remark 5.2 of [1] says the cited code is meant to be run ("We also further assume that the input G is 1-arc transitive"). Its failure is consistent with Theorem 5.1 of the source and adds no implication the source does not already have from $K_{2,2,2}$ (Proposition 5.1(b)). The consequence for [1] is that the arrow "4-arc transitive $\not\Rightarrow$ TB-symmetrical" in both of its implication diagrams loses its witness; we know of no graph that is 4-arc transitive and not TB-symmetrical (the Heawood graph is 4-arc transitive and TB-symmetrical, and the 5-arc regular Tutte–Coxeter graph is TB-symmetrical—the latter a computation outside the formal development, agreed on by two independent implementations; see Table 2). No other statement of the source depends on Proposition 5.3(3).

5. At most nine vertices

5.1. **The source's own values.** Before anything new was claimed, the printed values of [1] on small graphs were re-derived from Definition 2.1.

Proposition 5.1. (a) *K_4, K_5, $K_{2,3}$ and $K_{3,3}$ are TB-symmetrical (instances of Theorem 4.6 and of Proposition 5.3(1) of* [1]*).*

(b) *The octahedron $K_{2,2,2}$ is not almost-TB-symmetrical (Proposition 5.3(2)): the 2-arc $2 — 0 — 3$ lies in no triangle and in three 4-cycles, while the 2-arc $2 — 0 — 4$ lies in one of each, the parts being $\{0,1\}, \{2,3\}, \{4,5\}$.*

(c) *The Petersen graph O_3 is TB-symmetrical (Proposition 5.3(4)).*

(d) *The cube Q_3 is almost-TB-symmetrical but not TB-symmetrical (Proposition 5.3(5)): with vertices the 3-bit strings and $p = (\{0,1\}, \{2,3\})$, $q = (\{0,1\}, \{2,6\})$, $(w^{(4)}_p, w^{(6)}_p) = (-1,-2)$ and $(w^{(4)}_q, w^{(6)}_q) = (0,-2)$.*

All four agree with the source; the fifth item of its Proposition 5.3 is Theorem 4.1. In C the check was also run on every complete and complete bipartite graph on at most ten vertices, all TB-symmetrical as Theorem 4.6 says.

5.2. **The path-join and the missing graph.** Section 6 of [1] introduces three operations that preserve (almost-)TB-symmetricity: attaching a pendant vertex (its Lemma 6.1), the disjoint union of two graphs with equal $\rho_{r,s}$ (Lemma 6.2), and the *t-path-join* $G_1 \oplus^t G_2$ of two graphs with equal $\rho_{r,s}$ (Lemma 6.3), the case $t = 0$ being the 1-clique sum (identification of one vertex of each). Its Theorem 6.4 then states that a connected graph without pendant vertices on at most nine vertices is almost-TB-symmetrical if and only if it is one of: a graph all of whose cycles have the same size; $K_4, \ldots, K_9$; $K_{3,3}, K_{3,4}, K_{3,5}, K_{4,4}, K_{3,6}, K_{4,5}$; $K_4 \oplus^0 K_4$, $K_4 \oplus^1 K_4$, $K_4 \oplus^2 K_4$; or the cube Q_3—and that all of these except Q_3 are TB-symmetrical.

Theorem 5.2. *The 1-clique sum $K_5 \oplus^0 K_5$—two copies of K_5, on the vertex sets $\{0, 1, 2, 3, 4\}$ and $\{4, 5, 6, 7, 8\}$, sharing the vertex 4; nine vertices and twenty edges—is TB-symmetrical.*

Proof sketch. The profiles at lengths 3 to 9 are proportional (one compiled evaluation of the check of Proposition 2.5(b)); that the graph is the stated clique sum is a kernel evaluation (`k5sum_spec`). ☐

$K_5 \oplus^0 K_5$ is connected, has no pendant vertex, has cycles of lengths $3, 4, 5$, is neither complete nor complete bipartite nor a path-join of two K_4's, and is not Q_3: it is not in the list of Theorem 6.4 of [1], whose "only if" direction it therefore contradicts. The omission is in the list rather than in the method: $K_5 \oplus^0 K_5$ is an instance of the source's own Lemma 6.3 with $G_1 = G_2 = K_5$ and $t = 0$, the hypothesis $\rho_{r,s}(G_1) = \rho_{r,s}(G_2)$ being trivially satisfied. An exhaustive enumeration (Computation 5.3) shows that it is the only omission.

Computation 5.3 (outside the formal development)**.** Theorem 6.4 of [1] restricts to connected graphs without pendant vertices. These were generated with `nauty-geng -c -d2` n [11]; the counts for $n = 3, \ldots, 10$ are $1, 3, 11, 61, 507, 7442, 197\,772, 9\,808\,209$, which is the OEIS sequence A004108, "Number of n-node unlabeled connected graphs without endpoints" [4], at every n (name and terms read from the OEIS on 7 September 2026). Each graph was decided by a C implementation of Lemma 2.3 (the profiles V_t, then U_t, at all lengths $t \leq n$, with early abort at the first failure of rank one). On $n \leq 9$ the output is exactly the list of Theorem 6.4 of [1] with one addition, $K_5 \oplus^0 K_5$ (in `graph6` notation `H~{GW[N`, canonically equal to the enumeration's `HQhTQj~`). The counts by n are: almost-TB-symmetrical $1, 2, 4, 5, 9, 14, 25$; of those with all cycles of one length $1, 1, 3, 3, 6, 9, 20$; TB-symmetrical $1, 2, 4, 5, 9, 13, 25$ (the single almost-only graph is Q_3 at $n = 8$). The $n = 9$ run ($197\,772$ graphs) takes about 1.5 processor-seconds.

Remark 5.4 (the notation $\oplus^t$). Section 6 of [1] defines $G_1 \oplus^t G_2$ twice. Explicitly, it is "the graph obtained by adding t vertices $u_1, \ldots, u_t$ and edges $\{v_1, u_1\}, \{u_1, u_2\}, \ldots, \{u_{t-1}, u_t\}, \{u_t, v_2\}$", which has $|V_1| + |V_2| + t$ vertices and joins v_1 to v_2 by a path of length $t + 1$; in words, it is "$G_1 \cup G_2$ together with a path of length n [sic] connecting v_1 and v_2", which for a path of length t has $|V_1| + |V_2| + t - 1$ vertices. Only the second reading is consistent with the source's own convention that $\oplus^0$ is the 1-clique sum (a path of length 0), and only the second reading puts $K_4 \oplus^2 K_4$ on nine vertices; under the first, $K_4 \oplus^2 K_4$ would have ten vertices and could not appear in Theorem 6.4. Computation 5.3 confirms the second reading: the three path-joins of two K_4's it returns are on 7, 8 and 9 vertices (glued at a vertex; joined by an edge; joined by a path with one interior vertex). The explicit definition thus has an off-by-one, and we use the "path of length t" reading throughout.

6. Ten vertices

Computation 6.1 (outside the formal development)**.** Of the $9\,808\,209$ connected pendant-free graphs on ten vertices, exactly 46 are almost-TB-symmetrical, and 45 of those are TB-symmetrical. Thirty-five of the 46 have all their cycles of one length (TB-symmetrical by Lemma 4.2 of [1]). The other eleven are listed in Table 1. Generation took 4.7 processor-seconds and the classification 95 (re-run for this paper: 93, with identical output). As an independent check, a Python implementation sharing no code with the C one was run on all 46 survivors and on 495 rejected graphs sampled at $n = 9$, with no mismatch. The count sequences for $n = 3, \dots, 10$ are $1, 2, 4, 5, 9, 14, 25, 46$ (almost-TB-symmetrical), $1, 1, 3, 3, 6, 9, 20, 35$ (all cycles of one length) and $0, 1, 1, 2, 3, 5, 5, 11$ (the rest); none of the three is in the OEIS.

graph	edges	cycle lengths	TB-symmetrical	source's constructions
K_{10}	45	$3, \dots, 10$	yes	complete
$K_{3,7}$	21	$4, 6$	yes	complete bipartite
$K_{4,6}$	24	$4, 6, 8$	yes	complete bipartite
$K_{5,5}$	25	$4, 6, 8, 10$	yes	complete bipartite
$K_4 \oplus^3 K_4$	15	$3, 4$	yes	Lemma 6.3
$K_5 \oplus^1 K_5$	21	$3, 4, 5$	yes	Lemma 6.3
three K_4's at one common vertex	18	$3, 4$	yes	Lemma 6.3, twice
three K_4's in a chain	18	$3, 4$	yes	Lemma 6.3, twice
$S(K_4)$, the subdivision of K_4	12	$6, 8$	**yes**	**no**
the Petersen graph O_3	15	$5, 6, 8, 9$	yes	no (the source's example)
$K_{5,5}$ minus a perfect matching	20	$4, 6, 8, 10$	**almost only**	**no**

TABLE 1. The eleven almost-TB-symmetrical connected pendant-free graphs on ten vertices with cycles of more than one length (Computation 6.1).

Two of the eleven are new objects for the subject, and both are verified individually.

6.1. **The subdivision of K_4.** $S(K_4)$ is K_4 with every edge subdivided once: ten vertices (four of degree 3, six of degree 2), twelve edges, four 6-cycles (the subdivided triangles) and three 8-cycles (the subdivided 4-cycles).

Theorem 6.2. *$S(K_4)$ is TB-symmetrical.*

Proof sketch. One compiled evaluation of the check of Proposition 2.5(b) at lengths 3 to 10; that the graph is the subdivision of K_4 is a kernel evaluation (`subK4_spec`). Clause (3) is not vacuous on it: $w_p^{(6)} = 1$ for $p = (\{0, 4\}, \{1, 6\})$, where 4 subdivides the edge 01 of K_4 and 6 the edge 12 (Proposition 8.1(f)). □

Proposition 6.3 (not formalised). *$S(K_4)$ cannot be obtained from complete graphs and complete bipartite graphs by the operations of Lemmas 6.1–6.3 of* [1].

Proof. A graph obtained by at least one application of those operations is the output of the last one: attaching a pendant vertex leaves a pendant vertex, a disjoint union is disconnected, and a path-join or 1-clique sum of two graphs each with at least two vertices has a cut vertex (the identified vertex if $t = 0$, an endpoint of the bridge if $t = 1$, an interior vertex of the path if $t \geq 2$), while a path-join with K_1 leaves a pendant vertex. $S(K_4)$ is connected, has minimum degree 2, and is 2-connected (deleting any one of its ten vertices leaves a connected graph—checked directly). A graph obtained with no operation is K_m or $K_{m,n}$; on ten vertices K_{10} has 45 edges and $K_{m,10-m}$ has $m(10-m) \in \{9, 16, 21, 24, 25\}$ edges, never 12. □

So the sentence of [1] quoted in the introduction has a second example, of a different shape from the Petersen graph: $S(K_4)$ is not regular and not vertex-transitive. (Graphs all of whose cycles have one length, such as a single cycle C_n, are TB-symmetrical for the trivial reason of Lemma 4.2 and are of course also not obtainable; the source's sentence is evidently meant, as its Theorem 6.4 is, with those set aside. $S(K_4)$ has cycles of two lengths.)

Computation 6.4 (subdivision; outside the formal development)**.** Let $S(G)$ denote G with every edge subdivided once. Subdivision doubles every cycle length and carries each incidence of an edge or of an adjacent pair with a cycle of G to one of $S(G)$, so proportionality of the V_t transfers from G to $S(G)$; the W-part is not so simple, since two "far halves" of adjacent edges of G are non-adjacent edges of $S(G)$ and contribute new coordinates. On nine test graphs the property nevertheless transferred in both directions every time: $S(G)$ agrees with G on both the almost and the full property for $G = K_4$, K_5, $K_{2,3}$, $K_{3,3}$, the Petersen graph and $S(K_4)$ itself (all yes/yes), for $G = Q_3$ and $K_{5,5}$ minus a perfect matching (yes/no), and for $G = K_{2,2,2}$ (no/no). Nine instances are not a proof. If it is a theorem, every TB-symmetrical graph generates an infinite family of TB-symmetrical graphs outside the source's constructions, of which $S(K_4)$ is the smallest; no subdivision operation appears in [1].

6.2. $K_{5,5}$ **minus a perfect matching.** Let $K_{5,5}^-$ be $K_{5,5}$ with a perfect matching removed, with parts $\{0, \dots, 4\}$ and $\{5, \dots, 9\}$ and i joined to $5+j$ exactly when $i \neq j$ (the crown graph on ten vertices). It is 4-regular, bipartite, and 2-arc but not 3-arc transitive; its cycles have lengths $4, 6, 8, 10$.

Theorem 6.5. *$K_{5,5}^-$ is almost-TB-symmetrical and is not TB-symmetrical. Explicitly, let $p = (\{0,6\}, \{1,5\})$ and $q = (\{0,6\}, \{2,8\})$; then*

$$(w_p^{(4)}, w_p^{(6)}) = (0, -9), \qquad (w_q^{(4)}, w_q^{(6)}) = (1, 5),$$

with minor $0 \cdot 5 - 1 \cdot (-9) = 9 \neq 0$.

Proof sketch. The almost half is a full check of the V_t at lengths 3 to 10 (one compiled evaluation), not an appeal to Theorem 5.1 of [1]; the refutation is four compiled counts and Lemma 2.4 at lengths 4 and 6. That the graph is $K_{5,5}$ minus a perfect matching is a kernel evaluation (`crown5_spec`). □

By Computation 6.1, $K_{5,5}^-$ is the only almost- but not TB-symmetrical connected pendant-free graph on ten vertices. It is the second such graph known, the first being the cube Q_3 of Proposition 5.3(5) of [1], which the source uses for the arrow "almost-TB-symmetrical $\not\Rightarrow$ TB-symmetrical" of its diagrams. Both are 2-arc but not 3-arc transitive, so both sit exactly where Theorem 5.1 of the source stops. And both are the same construction: Q_3 is $K_{4,4}$ minus a perfect matching. In the C implementation $K_{n,n}$ minus a perfect matching is almost-TB-symmetrical and not TB-symmetrical for each of $n = 4, 5, 6, 7$ (a computation outside the formal development; four instances, not a proof), which suggests that the source's single example is the first member of an infinite family.

7. Arc transitivity

Theorem 5.1 of [1] says that 2-arc transitive graphs are almost-TB-symmetrical, and the odd graphs are 3-arc transitive. The natural way to approach Conjecture 1.2 would be to show that 3-arc transitivity forces clause (3) as well. It does not.

Let the Pappus graph be given by LCF $[5, 7, -7, 7, -7, -5]^3$ and the Desargues graph by LCF $[5, -5, 9, -9]^5$, both on the Hamiltonian cycle $0, 1, \dots, n-1$ ($n = 18$ and 20); they are the incidence graph of the Pappus configuration and the bipartite double cover of the Petersen graph respectively, cubic, bipartite, of girth 6. These two LCF codes are the ones the Foster census gives for its entries F018A and F020B [8]. The formal development takes the two adjacency matrices as given and proves of them only that they are cubic; that they are exactly the two LCF graphs, and that those are isomorphic to the Pappus configuration's incidence graph and to the double cover, was checked outside Lean.

Theorem 7.1. *The Pappus graph and the Desargues graph are not TB-symmetrical. Explicitly, for the pairs $p = (\{0,1\}, \{2,3\})$ and $q = (\{0,1\}, \{3,4\})$ of non-adjacent edges,*

$$\textit{Pappus: } (w_p^{(6)}, w_p^{(8)}) = (1, 4), \quad (w_q^{(6)}, w_q^{(8)}) = (1, 2);$$
$$\textit{Desargues: } (w_p^{(6)}, w_p^{(10)}) = (1, 11), \quad (w_q^{(6)}, w_q^{(10)}) = (1, 7),$$

with minors $1 \cdot 2 - 1 \cdot 4 = -2$ and $1 \cdot 7 - 1 \cdot 11 = -4$, both nonzero.

Proof sketch. Four compiled counts for each graph and Lemma 2.4, at lengths $(6, 8)$ for Pappus and $(6, 10)$ for Desargues. □

Corollary 7.2. *3-arc transitivity does not imply TB-symmetricity.*

Proof. The Pappus and Desargues graphs are 3-arc transitive (indeed 3-arc regular; entries F018A and F020B of the Foster census [5, 6]) and not TB-symmetrical by Theorem 7.1. □

The arc-transitivity facts were not taken on trust: for each graph in Table 2 the full automorphism group was enumerated by backtracking and its orbits on s-arcs counted, independently of the census (a computation outside the formal development, done twice, in two implementations). The census agrees: in the census data of [7], which extends [6], the cubic symmetric graph on 10, on 14, on 18 and on 30 vertices is in each case unique and is respectively 3-, 4-, 3- and 5-arc-regular with $|\operatorname{Aut}| = 120$, 336, 216 and 1440; on 20 vertices there are two, one of girth 5 with $|\operatorname{Aut}| = 120$ and one of girth 6 with $|\operatorname{Aut}| = 240$ which is 3-arc-regular, the second being the Desargues graph. The five designations F010A, F014A, F018A, F020B, F030A, and the LCF notations used above, are those of the Foster census as tabulated in [8]. The TB column is Proposition 5.1(c), Theorem 4.1 and Theorem 7.1, and for the Tutte–Coxeter graph a profile computation outside the formal development.

cubic graph	$\lvert\operatorname{Aut}\rvert$	orbits on s-arcs	TB-symmetrical
Petersen $= O_3$	120	one orbit on 3-arcs, two on 4-arcs	yes
Pappus	216	one orbit on 3-arcs, two on 4-arcs	**no**
Desargues	240	one orbit on 3-arcs, two on 4-arcs	**no**
Heawood	336	one orbit on 4-arcs, two on 5-arcs	yes
Tutte–Coxeter	1440	one orbit on 5-arcs, two on 6-arcs	yes†

Table 2. Arc transitivity against TB-symmetricity for five cubic symmetric graphs. In every row $|\operatorname{Aut}|$ equals the number of s-arcs at the last transitive s, i.e. the graph is s-arc regular. The four unmarked entries of the last column are the theorems cited above; the one marked † is a computation outside the formal development.

Corollary 7.2 is the structural reason behind Theorem 3.1: nothing about transitivity on 3-arcs controls the signed counts of clause (3). One observation, recorded and not claimed: among the five graphs of Table 2 the three TB-symmetrical ones are exactly the cages—the $(3,5)$-, $(3,6)$- and $(3,8)$-cages —and the Pappus and Desargues graphs, of girth 6 on more than 14 vertices, are not cages.

8. Controls

A decision procedure that returned "yes" for a trivial reason would prove nothing, so the following were checked alongside the results.

Proposition 8.1. (a) *(Too large a claim refuted.) O_4, Q_3, $K_{5,5}^-$ and $K_{2,2,2}$ are not TB-symmetrical; Q_3 and $K_{5,5}^-$ are almost-TB-symmetrical and not TB-symmetrical, so clause (3) is not a consequence of clauses (1)–(2); and the decision procedure of Proposition 2.5(b) returns* false *on Q_3 (full check) and on $K_{2,2,2}$ (almost check).*

(b) *(Too small a claim refuted.) K_4 is TB-symmetrical and the edge $\{0,1\}$ lies in two triangles and two 4-cycles, so TB-symmetricity is not the trivial case of Lemma 4.2 of* [1].

(c) *$K_{2,3}$ and $S(K_4)$ are TB-symmetrical and each has vertices of two different degrees (3 and 2), so TB-symmetricity forces neither regularity nor vertex-transitivity.*

(d) *The index lists of edges, adjacent pairs and non-adjacent pairs of the Heawood graph are nonempty.*

(e) *The counts are not identically zero: the edge $\{0,1\}$ of the Heawood graph lies in 8 six-cycles, the edge $\{0,1\}$ of the Petersen graph in 4 five-cycles, the edge $\{0,4\}$ of $S(K_4)$ in 2 six-cycles.*

(f) *Clause (3) is not vacuous on the graphs proved TB-symmetrical: $w_p^{(6)} = 2$ on the Heawood graph for $p = (\{0,1\},\{2,3\})$, $w_p^{(5)} = 1$ on the Petersen graph for the same labels, and $w_p^{(6)} = 1$ on $S(K_4)$ for $p = (\{0,4\},\{1,6\})$.*

(g) *The four counts of Theorem 3.1 are pinned against neighbouring values: $w_p^{(6)} \neq 0,-2$; $w_q^{(7)} \neq -1,-3$; $w_p^{(7)} \neq 1$. Likewise for Theorem 4.2: $f_{01}^{(4)} \neq 2$ and $f_{07}^{(6)} \neq 1$, either of which would have made M_{14} look almost-TB-symmetrical at those two lengths.*

9. Verification

Every theorem and proposition above that carries a graph-level statement—Propositions 2.5, 5.1 and 8.1 and Theorems 3.1, 4.1, 4.2, 5.2, 6.2, 6.5 and 7.1—has been formally verified in Lean 4 against *Mathlib* [12, 13]; the statements are reproduced verbatim in Appendix A. Not formalised, and labelled as such where they occur, are Computations 5.3, 6.1 and 6.4, the crown graphs for $n = 6, 7$, Proposition 6.3, the identification of the four graphs that carry no in-Lean specification—`heawood`, `moebius14`, `pappus`, `desargues`—with the graphs they are named after, the arc-transitivity column of Table 2 together with its Tutte–Coxeter TB entry, and the Legendrian consequence (1). Every other named graph, and the vertex set of O_4, does carry a kernel-checked specification.

The development is twelve modules of 1550 lines in all, in import order: `Basic` (175 lines; the transcription of Definition 2.1, the counts as cardinalities, the index lists), `Enum` (275; the enumerator and Proposition 2.5(a)), `Checker` (102; the index-list lemmas and Proposition 2.5(c)), `Positive` (263; the profiles and Proposition 2.5(b)), `Odd4` (103; Theorem 3.1), `Graphs` (126; the named graphs as adjacency bitmasks, each with its symmetry and irreflexivity and, where a description is claimed, a kernel-checked specification: `k4_complete`, `k5_complete`, `k23_bipartite`, `k33_bipartite`, `k222_tripartite`, `cube_hypercube`, `crown5_spec`, `subK4_spec`), `Small` (95; Proposition 5.1(a),(b),(d)), `Heawood` (73; Theorems 4.1 and 4.2), `Nine` (40; Theorem 5.2 and `k5sum_spec`), `Ten` (75; Proposition 5.1(c), Theorems 6.2 and 6.5), `ArcTrans` (108; Theorem 7.1) and `Controls` (115; Proposition 8.1). Counting declarations, the modules contain 131 theorems and 49 definitions (with four decidability instances in `Basic`); twenty-nine theorem declarations carry the statements of this paper.

The axioms were printed for every headline theorem. The reasoning layer—the enumerator's completeness `mem_walks`, `mem_cyclesAt`, the bridge `card_eq_length`, the soundness theorems `tbSym_of_check` and `almostTBSym_of_check`, and the three determinant lemmas—depends on `propext`, `Classical.choice` and `Quot.sound` only; the specification `kneser73_spec` of the vertex set of O_4, closed by kernel `decide`, depends on `propext` and `Quot.sound` alone. Each positive result about a named graph adds exactly one `native_decide` axiom (the compiled evaluation of the profile check): `heawood_tbSym`, `subK4_tbSym`, `petersen_tbSym`, `crown5_almostTBSym`, `cube_almostTBSym`, `k5sum_tbSym`, `k4_tbSym`, `k5_tbSym`, `k23_tbSym`, `k33_tbSym`. Each refutation adds exactly four (the four counts of its minor): `O4_not_tbSym`, `moebius14_not_almostTBSym`, `crown5_not_tbSym`, `cube_not_tbSym`, `k222_not_almostTBSym`, `pappus_not_tbSym`, `desargues_not_tbSym`. The control statements of Proposition 8.1 use one compiled evaluation per count or check they mention and otherwise reuse the theorems above. There is no `sorry` and no axiom declared by hand. All Lean checks together, including development iterations, took about twenty processor-minutes with a peak of about 3 GB of memory above the *Mathlib* baseline; the C computations of Sections 3–7 took under two processor-minutes in all (0.02 s for Theorem 3.1; 95 s for Computation 6.1; 10 s for the cycles of O_4 to length 20); the whole work, e-print compilation included, stayed well under one processor-hour. The source of the development and of the C and Python scripts is currently held in a private repository: repository reference to be added at submission.

10. Open questions

(1) Is $O_5 = K(9, 4)$ TB-symmetrical? If not, Conjecture 1.2 of [1] fails in general and the question becomes why O_2 and O_3 are special.
(2) Does edge subdivision preserve (almost-)TB-symmetricity in both directions (Computation 6.4)? The V-part is a one-line cycle correspondence; the W-part is the work.
(3) Is $K_{n,n}$ minus a perfect matching almost- but not TB-symmetrical for every $n \geq 4$? The cycle bookkeeping of [2] for $K_{m,n}$ is the obvious tool.
(4) Does 4-arc transitivity imply TB-symmetricity? The source's counterexample is withdrawn by Theorem 4.1, and no other is known to us.
(5) When is clause (3) satisfied? In every positive result here $W_t = \rho_{t,s} W_s$ with ρ forced by edge-transitivity, and every negative result is a rank-two pair of non-adjacent-pair coordinates. A

criterion in terms of the orbits of $\mathrm{Aut}(G)$ on pairs of non-adjacent edges is the natural guess and is testable on the data already computed.

(6) Eleven vertices: about 10^9 graphs, an estimated three processor-hours at the measured rate of 10 microseconds per graph.

Appendix A. Lean statements

The statements below are reproduced verbatim from the development; only line breaks have been adjusted to the page width. All declarations live in the namespace `LeanProblemSpec.TbsymGraphs`, and a graph on n vertices is `Adj n := Fin n → Fin n → Bool`.

Definitions (`Basic`).

```
def IsCycleList {n : ℕ} (A : Adj n) (l : List (Fin n)) : Prop :=
  3 ≤ l.length ∧ l.Nodup ∧ PathOK A l ∧ Closes A l
def BasedCyc {n : ℕ} (A : Adj n) (t : ℕ) (a b : Fin n) : Set (List (Fin n)) :=
  {l | IsCycleList A l ∧ l.length = t ∧ l.head? = some a ∧ l[1]? = some b}
noncomputable def fE {n : ℕ} (A : Adj n) (t : ℕ) (a b : Fin n) : ℕ :=
  Nat.card (BasedCyc A t a b)
noncomputable def gP {n : ℕ} (A : Adj n) (t : ℕ) (x v y : Fin n) : ℕ :=
  Nat.card {l ∈ BasedCyc A t v x | cycHasEdge l y v}
noncomputable def uCnt {n : ℕ} (A : Adj n) (t : ℕ) (a b c d : Fin n) : ℕ :=
  Nat.card {l ∈ BasedCyc A t a b | cycHasEdge l c d}
noncomputable def wD {n : ℕ} (A : Adj n) (t : ℕ) (a b c d : Fin n) : ℤ :=
  (uCnt A t a b c d : ℤ) - (uCnt A t a b d c : ℤ)
def TBSymPair {n : ℕ} (A : Adj n) (r s : ℕ) : Prop :=
  ∃ ρ : ℚ, 0 ≤ ρ ∧
    (∀ e ∈ edgeIdx A, (fE A r e.1 e.2 : ℚ) = ρ * (fE A s e.1 e.2 : ℚ)) ∧
    (∀ p ∈ arcIdx A, (gP A r p.1 p.2.1 p.2.2 : ℚ) = ρ * (gP A s p.1 p.2.1 p.2.2 : ℚ)) ∧
    (∀ p ∈ napIdx A, (wD A r p.1.1 p.1.2 p.2.1 p.2.2 : ℚ)
        = ρ * (wD A s p.1.1 p.1.2 p.2.1 p.2.2 : ℚ))
def TBSym {n : ℕ} (A : Adj n) : Prop :=
  ∀ r s : ℕ, 3 ≤ r → 3 ≤ s → r ≠ s → TBSymPair A r s ∨ TBSymPair A s r
```

`AlmostTBSymPair` and `AlmostTBSym` are the same without the third conjunct. `edgeIdx A` lists the edges (a, b) with $a < b$; `arcIdx A` the 2-arcs (x, v, y) with $x < y$ and v adjacent to both; `napIdx A` the ordered pairs of edges from `edgeIdx A` with four distinct endpoints.

Proposition 2.5 (`Enum`, `Positive`, `Checker`).

```
theorem mem_cyclesAt {n : ℕ} (A : Adj n) (hirr : ∀ x, A x x = false)
    {t : ℕ} (ht : 3 ≤ t) (a b : Fin n) (l : List (Fin n)) :
    l ∈ cyclesAt A t a b ↔
      (IsCycleList A l ∧ l.length = t ∧ l.head? = some a ∧ l[1]? = some b)
theorem nodup_cyclesAt {n : ℕ} (A : Adj n) (t : ℕ) (a b : Fin n) :
    (cyclesAt A t a b).Nodup
theorem fE_eq {n : ℕ} (A : Adj n) (hirr : ∀ x, A x x = false)
    {t : ℕ} (ht : 3 ≤ t) (a b : Fin n) : fE A t a b = (cyclesAt A t a b).length
theorem tbSym_of_check {n : ℕ} (A : Adj n) (hirr : ∀ x, A x x = false)
    (h : tbSymB A = true) : TBSym A
theorem almostTBSym_of_check {n : ℕ} (A : Adj n) (hirr : ∀ x, A x x = false)
    (h : almostB A = true) : AlmostTBSym A
theorem not_tbSym_of_nap_det {n : ℕ} (A : Adj n) {r s : ℕ}
    (hr : 3 ≤ r) (hs : 3 ≤ s) (hrs : r ≠ s)
    (p q : (Fin n × Fin n) × (Fin n × Fin n)) (hp : p ∈ napIdx A) (hq : q ∈ napIdx A)
    (h : wD A r p.1.1 p.1.2 p.2.1 p.2.2 * wD A s q.1.1 q.1.2 q.2.1 q.2.2
        ≠ wD A r q.1.1 q.1.2 q.2.1 q.2.2 * wD A s p.1.1 p.1.2 p.2.1 p.2.2) :
    ¬ TBSym A
```

(`not_almostTBSym_of_edge_det` and `not_almostTBSym_of_arc_det` are the analogues on `fE` over `edgeIdx` and on `gP` over `arcIdx`, concluding `¬ AlmostTBSym A`.)

Theorem 3.1 (`Odd4`)**.** Here `kneser73` is the list of the thirty-five 3-subsets of $\{0, \ldots, 6\}$ as 7-bit masks, and `O4 : Adj 35` joins i and j when the masks are disjoint; the indices $0, 31, 1, 28, 29$ carry the subsets $012, 345, 013, 245, 246$.

```
theorem kneser73_spec (m : ℕ) (hm : m < 128) :
    (((List.range 7).countP (fun i => m.testBit i)) = 3 ↔ m ∈ kneser73)
theorem wD_values :
    wD O4 6 0 31 1 28 = -1 ∧ wD O4 7 0 31 1 28 = 0 ∧
    wD O4 6 0 31 1 29 = 0 ∧ wD O4 7 0 31 1 29 = -2
theorem O4_not_tbSym : ¬ TBSym O4
```

Theorems 4.1 and 4.2 (`Heawood`)**.**

```
theorem heawood_tbSym : TBSym heawood
theorem moebius14_fE_values :
    fE moebius14 4 0 1 = 1 ∧ fE moebius14 6 0 1 = 2 ∧
    fE moebius14 4 0 7 = 2 ∧ fE moebius14 6 0 7 = 2
theorem moebius14_not_almostTBSym : ¬ AlmostTBSym moebius14
```

Proposition 5.1 (`Small`, `Ten`)**.**

```
theorem k4_tbSym : TBSym k4
theorem k5_tbSym : TBSym k5
theorem k23_tbSym : TBSym k23
theorem k33_tbSym : TBSym k33
theorem k222_gP_values :
    gP k222 3 2 0 3 = 0 ∧ gP k222 4 2 0 3 = 3 ∧
    gP k222 3 2 0 4 = 1 ∧ gP k222 4 2 0 4 = 1
theorem k222_not_almostTBSym : ¬ AlmostTBSym k222
theorem cube_almostTBSym : AlmostTBSym cube
theorem cube_wD_values :
    wD cube 4 0 1 2 3 = -1 ∧ wD cube 6 0 1 2 3 = -2 ∧
    wD cube 4 0 1 2 6 = 0 ∧ wD cube 6 0 1 2 6 = -2
theorem cube_not_tbSym : ¬ TBSym cube
theorem petersen_tbSym : TBSym petersen
```

Theorem 5.2 (`Nine`)**.**

```
theorem k5sum_spec : ∀ x y : Fin 9,
    k5sum x y = decide (x ≠ y ∧ ((x.val ≤ 4 ∧ y.val ≤ 4) ∨ (4 ≤ x.val ∧ 4 ≤ y.val)))
theorem k5sum_tbSym : TBSym k5sum
```

Theorems 6.2 and 6.5 (`Ten`)**.**

```
theorem subK4_tbSym : TBSym subK4
theorem crown5_almostTBSym : AlmostTBSym crown5
theorem crown5_wD_values :
    wD crown5 4 0 6 1 5 = 0 ∧ wD crown5 6 0 6 1 5 = -9 ∧
    wD crown5 4 0 6 2 8 = 1 ∧ wD crown5 6 0 6 2 8 = 5
theorem crown5_not_tbSym : ¬ TBSym crown5
```

Theorem 7.1 (`ArcTrans`)**.**

```
theorem pappus_wD_values :
    wD pappus 6 0 1 2 3 = 1 ∧ wD pappus 8 0 1 2 3 = 4 ∧
    wD pappus 6 0 1 3 4 = 1 ∧ wD pappus 8 0 1 3 4 = 2
theorem pappus_not_tbSym : ¬ TBSym pappus
theorem desargues_wD_values :
    wD desargues 6 0 1 2 3 = 1 ∧ wD desargues 10 0 1 2 3 = 11 ∧
    wD desargues 6 0 1 3 4 = 1 ∧ wD desargues 10 0 1 3 4 = 7
theorem desargues_not_tbSym : ¬ TBSym desargues
```

Proposition 8.1 (`Controls`)**.**

```
theorem exists_not_tbSym : (¬ TBSym O4) ∧ (¬ TBSym cube) ∧ (¬ TBSym crown5) ∧
    (¬ TBSym k222)
theorem almost_not_tbSym : (AlmostTBSym cube ∧ ¬ TBSym cube) ∧
    (AlmostTBSym crown5 ∧ ¬ TBSym crown5)
theorem check_not_constant : tbSymB cube = false ∧ almostB k222 = false
theorem k4_two_lengths : TBSym k4 ∧ fE k4 3 0 1 = 2 ∧ fE k4 4 0 1 = 2
```

```
theorem tbSym_not_regular :
    TBSym k23 ∧ (nbrs k23 0).length = 3 ∧ (nbrs k23 2).length = 2 ∧
    TBSym subK4 ∧ (nbrs subK4 0).length = 3 ∧ (nbrs subK4 4).length = 2
theorem indices_nonempty :
    (edgeIdx heawood ≠ []) ∧ (arcIdx heawood ≠ []) ∧ (napIdx heawood ≠ [])
theorem clause3_nonzero :
    wD heawood 6 0 1 2 3 = 2 ∧ wD petersen 5 0 1 2 3 = 1 ∧ wD subK4 6 0 4 1 6 = 1
theorem counts_nonzero : fE heawood 6 0 1 = 8 ∧ fE petersen 5 0 1 = 4 ∧
    fE subK4 6 0 4 = 2
theorem O4_values_pinned :
    wD O4 6 0 31 1 28 ≠ 0 ∧ wD O4 6 0 31 1 28 ≠ -2 ∧
    wD O4 7 0 31 1 29 ≠ -1 ∧ wD O4 7 0 31 1 29 ≠ -3 ∧
    wD O4 7 0 31 1 28 ≠ 1
theorem moebius14_values_pinned :
    fE moebius14 4 0 1 ≠ 2 ∧ fE moebius14 6 0 7 ≠ 1
```

References

[1] T. Chau and T. Shah, *On Legendrian Thurston-Bennequin-symmetrical graphs*, arXiv:2603.28487v1 [math.GT], 30 March 2026. (Theorem, definition and proposition numbers are those of the compiled e-print.)

[2] D. O'Donnol and E. Pavelescu, *The total Thurston-Bennequin number of complete and complete bipartite Legendrian graphs*, arXiv:1507.07065v1 [math.GT], 25 July 2015; in: G. Letzter et al. (eds.), Advances in the Mathematical Sciences, Association for Women in Mathematics Series **6**, Springer, Cham, 2016, 117–137, doi:10.1007/978-3-319-34139-2_4.

[3] T. Chau and T. Shah, *Sage code for (almost-)TB-symmetrical graphs*, `https://github.com/trungchaumath/trungchaumath.github.io/tree/d1a7506c1c1d76ecce5f96b7871a2959dd153ce0/codes/Thurston-Bennequin-symmetrical%20graphs`, files `TB-symmetrical graphs` and `almost-TB-symmetrical graphs`; read at that commit on 7 September 2026.

[4] The OEIS Foundation, *The On-Line Encyclopedia of Integer Sequences*, sequence A004108, `https://oeis.org/A004108`.

[5] M. Conder and M. Morton, *Classification of trivalent symmetric graphs of small order*, Australas. J. Combin. **11** (1995), 139–149.

[6] M. Conder and P. Dobcsányi, *Trivalent symmetric graphs on up to 768 vertices*, J. Combin. Math. Combin. Comput. **40** (2002), 41–63.

[7] M. Conder, *Trivalent (cubic) symmetric graphs on up to 10000 vertices*, list of graphs with their automorphism group orders, arc-regularity types and girths, `https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt`; read on 7 September 2026.

[8] E. W. Weisstein, *Cubic symmetric graph*, MathWorld—A Wolfram Web Resource, `https://mathworld.wolfram.com/CubicSymmetricGraph.html`; read on 7 September 2026 for the Foster census designations and LCF notations.

[9] L. Babai, *Automorphism groups, isomorphism, reconstruction*, in: Handbook of Combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, 1447–1540.

[10] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics **207**, Springer-Verlag, New York, 2001.

[11] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94–112, doi:10.1016/j.jsc.2013.09.003.

[12] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[13] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.