BOOLEAN ANTICHAINS OF INTERMEDIATE SIZE IN THE TAMARI LATTICE

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Abstract. An antichain C in a finite lattice L is *Boolean* if $\bigwedge S \vee \bigwedge S' = \bigwedge(S \cap S')$ for all $S, S' \subseteq C$; these antichains index the perfect modules over the incidence algebra of L. Garber, Goltermann, Horiatakis, König and Gottesman count the Boolean antichains of a Boolean lattice at every size, and those of a partition lattice and of a Tamari lattice at the maximal size only, and ask for a recursive formula at the intermediate sizes in the Tamari lattice. We do not give such a formula; we give the values it has to reproduce, and we locate the term of their recursion that is missing. We give the complete table of $|\mathcal{C}_k(\mathcal{T}_m)|$, the number of Boolean antichains of size k in the Tamari lattice of rank m, for all $m \leq 9$ and all k — at $m = 9$ over all 11 399 553 Boolean antichains of $\mathcal{T}_9$ — and the complete table of $|\mathcal{C}_k(\mathcal{P}_n)|$ for the partition lattice with $n \leq 6$. We then refine the Tamari table by the lattice congruence $q_m : \mathcal{T}_m \to \mathcal{T}_{m-1}$ used there, re-verify exhaustively for $m \leq 9$ their theorem that at most one member of a Boolean antichain is not the greatest element of its fibre, and count the antichains where exactly one member is not, split by whether that member lies in the fibre through the greatest element. That count is the whole of the recursion asked for; its top-fibre part meets the obvious over-count $(m-1)\,|\mathcal{C}_{k-1}(\mathcal{T}_{m-1})|$ for $k \leq 2$, as their own remark predicts, and is strictly smaller from $k = 3$ on. None of the five sequences produced here occurs in the OEIS. Every count stated here is machine-checked in Lean 4.

1. Introduction

Let L be a finite lattice with least element $\hat{0}$ and greatest element $\hat{1}$, and write $\bigwedge \emptyset = \hat{1}$. Following [1], which in turn follows Gottesman [2], a subset $C \subseteq L \setminus \{\hat{1}\}$ is a *Boolean antichain* if

$$\left(\bigwedge S\right) \vee \left(\bigwedge S'\right) \;=\; \bigwedge (S \cap S') \qquad \text{for all } S, S' \subseteq C. \tag{1}$$

Condition (1) forces C to be an antichain, and by [1, Lemma 2.1] — which is [2, Lemma 2.1.5] in the case where the ambient element is $\hat{1}$ — it says exactly that $S \mapsto \bigwedge S$ is a lattice anti-isomorphism from the power set of C onto the sublattice of L generated by C and $\hat{1}$ — whence the name. Boolean antichains are the antichains whose associated modules over the incidence algebra of L are perfect [3], and they carry part of the combinatorics behind the fractionally Calabi–Yau phenomena [5, 2, 4] that fit into a far-reaching conjecture of Chapoton [6]. We write $\mathcal{C}_k(L)$ for the set of Boolean antichains of size k in L.

Garber, Goltermann, Horiatakis, König and Gottesman [1] compute $|\mathcal{C}_k(L)|$ in three families of lattices, and in each case at one size:

- $|\mathcal{C}_k(\mathcal{B}_n)| = S(n+1, k+1)$, a Stirling number of the second kind, at *every* size k in the Boolean lattice $\mathcal{B}_n$ [1, Theorem 3.1];
- $|\mathcal{C}_{n-1}(\mathcal{P}_n)| = n^{n-2}$, Cayley's formula, at the *maximal* size only in the partition lattice $\mathcal{P}_n$ [1, Theorem 4.4];
- $|\mathcal{C}_n(\mathcal{T}_{n+1})| = \mathsf{c}_n$, the Catalan number, at the *maximal* size only in the Tamari lattice $\mathcal{T}_{n+1}$ [1, Theorem 5.4.4].

Their closing section then asks, as the first of the three questions gathered under [1, Question 6.1]:

> "Can [Proposition 5.3.3] and [Proposition 5.3.8] be used to determine a (recursive) formula for the number of Boolean antichains of size k in the Tamari lattice $\mathcal{T}_{n+1}$ for $2 \leq k < n$?" [1, Question 6.1]

Here the two bracketed references replace `\Cref`s in the original, and all cross-references into [1] in this paper — theorem, lemma, figure and question numbers alike — are to version 1 of the preprint, the only version at the time of writing.

No value of $|\mathcal{C}_k(\mathcal{T}_m)|$ at an intermediate size appears in [1]: the paper contains no table, and its only numerical data are the three closed forms above together with small worked examples, two of them in $\mathcal{T}_4$. We have found no other source that records such a value either; see Remark 4.3. The purpose of this paper is to supply the numbers, and to localise, inside the recursion that [1] sets up, exactly which term is missing.

Results. Throughout, $\mathcal{T}_m$ denotes the Tamari lattice of parenthesized expressions on $m+1$ letters, so that $|\mathcal{T}_m| = \mathtt{c}_m$; in the indexing of [1] it is $\mathcal{T}_m$ as well, and Question 6.1 asks for the sizes $2 \le k \le m-2$, that is, everything strictly between the two columns that are already known.

Section 4 gives the complete table of $|\mathcal{C}_k(\mathcal{T}_m)|$ for $2 \le m \le 8$ (Theorem 4.1) and the row $m = 9$ (Theorem 4.2), which runs over all 11 399 553 Boolean antichains of $\mathcal{T}_9$. Section 5 refines the table by the lattice congruence $q_m \colon \mathcal{T}_m \to \mathcal{T}_{m-1}$ of [1, Section 5.1]: writing $\varepsilon(C)$ for the number of members of C that are not the greatest element of their q_m-fibre, we confirm exhaustively for $m \le 9$ that $\varepsilon \le 1$ always and that the part with $\varepsilon = 0$ is a size-preserving copy of $\mathcal{C}_\bullet(\mathcal{T}_{m-1})$ (Proposition 5.1), so that

$$|\mathcal{C}_k(\mathcal{T}_m)| \;=\; |\mathcal{C}_k(\mathcal{T}_{m-1})| \;+\; E(m,k), \qquad E(m,k) := \#\{\, C \in \mathcal{C}_k(\mathcal{T}_m) : \varepsilon(C) = 1 \,\}, \tag{2}$$

and all the content of Question 6.1 sits in E. Theorem 5.2 computes $E(m,k)$ for $m \le 9$ and every k, together with its split into the two branches that [1] distinguishes. Against the obvious over-count $(m-1)\,|\mathcal{C}_{k-1}(\mathcal{T}_{m-1})|$ — the rightmost term of the upper bound [1, Corollary 5.3.9] — the top-fibre branch is exact for $k \le 2$ and strictly smaller from $k = 3$ on, which is the first size at which the closing-index-set condition of [1, Proposition 5.3.3] can fail. The equality at $k \le 2$ is not new: [1] already remarks that its bound is attained exactly for $k \in \{0, 1, 2\}$, and that forces equality in each of its two branches separately (Remark 5.3). What is new is the branch-by-branch values, and with them the size of the deficiency. Section 6 does the same for the partition lattice, where [1] also settles the maximal size only (Theorem 6.1).

Everything is proved by exhaustive enumeration, and every enumeration is executed inside the Lean 4 kernel; Section 8 says exactly what that means and what it does not. Section 3 states the arithmetic controls the computation was made to pass before any new number was believed, including the two-parameter positive control provided by [1, Theorem 3.1], which pins down all 21 entries of the Boolean-lattice table for $n \le 6$, and the counterexample of [1, Example 5.3.2], which certifies that the membership test implemented here is strictly stronger than the pairwise one.

2. Boolean antichains, and the three lattices

We recall what is needed and fix notation; everything in this section is from [1] unless stated otherwise.

Lemma 2.1 ([1, Remarks 2.2 and 2.3, Lemma 2.5])**.** *Let $C \subseteq L \setminus \{\hat{1}\}$ satisfy* (1)*. Then*

(1) *C is an antichain;*
(2) *$c \vee c' = \hat{1}$ for all distinct $c, c' \in C$;*
(3) *$|C| \le |\operatorname{cov}(\hat{1})|$ and $|C| \le h(L)$, where $\operatorname{cov}(\hat{1})$ is the set of coatoms and $h(L)$ the height.*

Taking $S = \{c\}$ and $S' = \emptyset$ in (1) gives $c \vee \hat{1} = \hat{1}$, which is automatic; hence every singleton in $L \setminus \{\hat{1}\}$ is a Boolean antichain and

$$|\mathcal{C}_1(L)| = |L| - 1. \tag{3}$$

Also $\mathcal{C}_0(L) = \{\emptyset\}$, so $|\mathcal{C}_0(L)| = 1$ in every lattice. If L is distributive, condition (1) collapses to the pairwise condition of Lemma 2.1(2) [1, Lemma 2.4]; in general it does not, and Proposition 3.5 below records the gap quantitatively.

The Tamari lattice. $\mathcal{T}_m$ is the set of parenthesized expressions on the letters $x_1, \dots, x_{m+1}$ — equivalently of binary trees with m internal nodes — ordered by the transitive closure of the cover relation

$$(\dots((xy)z)\dots) \lessdot (\dots(x(yz))\dots), \tag{4}$$

as in [7]. Its cardinality is the Catalan number c_m, its greatest element $\hat{1}$ is the fully right-parenthesized word, and $|\operatorname{cov}(\hat{1})| = m-1$, so by Lemma 2.1 a Boolean antichain in $\mathcal{T}_m$ has at most $m-1$ elements.

The congruence q_m. The map $q_m \colon \mathcal{T}_m \to \mathcal{T}_{m-1}$ deletes the last letter x_{m+1} together with the parentheses that become redundant; it is a lattice congruence [1, Proposition 5.1.4], its fibres are chains [1, Lemma 5.1.1], and the fibre $[\hat{1}]$ through the greatest element has exactly m elements [1, Corollary 5.1.2]. Write $f_{m-1} \colon \mathcal{T}_{m-1} \to \mathcal{T}_m$ for the section sending a class to its greatest element; it is a lattice monomorphism [1, Section 5.2], and it carries $\hat{1}$ to $\hat{1}$, since $[\hat{1}]$ is the fibre over the greatest element of $\mathcal{T}_{m-1}$. Since q_m is surjective onto $\mathcal{T}_{m-1}$ and each fibre has exactly one greatest element, exactly $\mathsf{c}_m - \mathsf{c}_{m-1}$ elements of $\mathcal{T}_m$ fail to be the greatest element of their fibre. Following [1, Definition 5.2.1] we also write $I_v \subseteq [m-1]$ for the *closing index set* of $v \in \mathcal{T}_m$ and $k_v := \max([m-1] \setminus I_v)$, or $k_v := 0$ when $I_v = [m-1]$; these enter only in Remark 5.3 and Section 7.

The other two lattices. $\mathcal{B}_n$ is the lattice of subsets of $[n]$ under inclusion, and $\mathcal{P}_n$ the lattice of set partitions of $[n]$ ordered by refinement, with $\hat{1}$ the one-block partition. Height gives $|C| \le n$ for a Boolean antichain of $\mathcal{B}_n$ and $|C| \le n-1$ for one of $\mathcal{P}_n$; both bounds are attained, and by [1, Theorem 3.1] the first is attained $S(n+1, n+1) = 1$ time while $|\mathcal{C}_{n+1}(\mathcal{B}_n)| = 0$.

3. The enumeration and its controls

Every count below is an exhaustive enumeration, so the only mathematics that has to be trusted is the reduction that makes it feasible, and the only thing that has to be checked is that the object enumerated is the right one. This section states both.

How the search runs. Fix a lattice L on n elements together with an indexing $0, \dots, n-1$ by a linear extension of its order, so $x \le y$ implies $\operatorname{ind}(x) \le \operatorname{ind}(y)$; then $\hat{1}$ has index $n-1$ and $\hat{0}$ has index 0. For $\mathcal{T}_m$ such an indexing is free: if $w(t)$ denotes the sum, over the internal nodes of a binary tree t, of the number of leaves of the left subtree, then a rotation (4) at a node whose x has $a \ge 1$ leaves decreases w by exactly a, so sorting the c_m trees by decreasing w is a linear extension. Down-sets and up-sets are stored as bitmasks; because $\operatorname{dn}(x \wedge y) = \operatorname{dn} x \cap \operatorname{dn} y$ and $\operatorname{up}(x \vee y) = \operatorname{up} x \cap \operatorname{up} y$ in any lattice, and because the indexing is a linear extension, $x \wedge y$ is the *largest* index in the first intersection and $x \vee y$ the *smallest* index in the second. Meets and joins are therefore one bitwise AND and one bit scan each, and full $n \times n$ meet and join tables are precomputed.

The family of Boolean antichains is closed under passing to subsets, since (1) for $S, S' \subseteq D \subseteq C$ is a sub-conjunction of (1) for C. Hence a depth-first search that extends an antichain by elements of larger index only reaches every Boolean antichain exactly once, and by Lemma 2.1(2) a candidate extension c of C may be discarded as soon as $c \vee c' \ne \hat{1}$ for some $c' \in C$. What remains is the membership test.

Lemma 3.1 (reduction of the membership test). *Let $C \subseteq L \setminus \{\hat{1}\}$. Then C satisfies* (1) *for all pairs $S, S' \subseteq C$ if and only if it satisfies it for all pairs of the form $(T \sqcup A,\ T \sqcup B)$ with $C = T \sqcup A \sqcup B$ a partition of C into three (possibly empty) blocks.*

Proof. Necessity is trivial. For sufficiency, let $S, S' \subseteq C$ be arbitrary and set $T := S \cap S'$, $A := (S \setminus T) \cup \big(C \setminus (S \cup S')\big)$ and $B := S' \setminus T$. Then $C = T \sqcup A \sqcup B$, $(T \cup A) \cap (T \cup B) = T$, $S \subseteq T \cup A$ and $S' = T \cup B$. Meets reverse inclusion, so $\bigwedge S \ge \bigwedge(T \cup A)$ and $\bigwedge S' = \bigwedge(T \cup B)$, whence

$$\Big(\bigwedge S\Big) \vee \Big(\bigwedge S'\Big) \ \ge\ \Big(\bigwedge(T \cup A)\Big) \vee \Big(\bigwedge(T \cup B)\Big) \ =\ \bigwedge T \ =\ \bigwedge(S \cap S').$$

Conversely $\bigwedge S \le \bigwedge(S \cap S')$ and $\bigwedge S' \le \bigwedge(S \cap S')$, so the join is $\le \bigwedge(S \cap S')$ as well. $\square$

Lemma 3.1 is not used for the tables of Section 4 and Section 6: those rest on the literal $4^{|C|}$ test, every pair of subsets, so that no reduction stands between the definition and the number. It is used for the refined tables of Section 5, and Proposition 3.4 records that on the Tamari lattices in question the two tests return identical count vectors.

Controls. The first control is that the lattice being searched is the intended one.

Proposition 3.2. *For $2 \le m \le 9$, the construction above produces a poset on c_m elements whose index-$(\mathsf{c}_m - 1)$ element is above all c_m elements and whose index-0 element is below all of them — so the indexing is a linear extension with $\hat{1}$ last — and in which $\hat{1}$ covers exactly $m-1$ elements. Moreover the q_m-fibre through $\hat{1}$ has exactly m elements, and exactly $\mathsf{c}_m - \mathsf{c}_{m-1}$ elements of $\mathcal{T}_m$ are not the greatest element of their fibre.*

Proof sketch. Exhaustive computation on the constructed poset for each $m \le 9$, from the rotation relation (4) closed transitively. All five statistics are predicted by [1] (the last two by [1, Corollary 5.1.2] and by surjectivity of q_m), and the constructed Hasse diagrams of $\mathcal{T}_3$ and $\mathcal{T}_4$ — 5 nodes with 5 edges, and 14 nodes with 21 edges, the greatest element joined to three below it and the least to three above — agree with the drawings of [1, Figures 4 and 6]. $\square$

The second control is on the membership test itself, and it is the strongest one available, because [1] determines the Boolean-lattice counts at every size at once.

Proposition 3.3. *For $2 \le n \le 6$ and $0 \le k \le n+1$, the number of Boolean antichains of size k in $\mathcal{B}_n$ computed from the literal definition (1) equals the Stirling number $S(n+1, k+1)$, in all 21 nonzero entries:*

$n\backslash k$	1	2	3	4	5	6
2	3	1				
3	7	6	1			
4	15	25	10	1		
5	31	90	65	15	1	
6	63	301	350	140	21	1

and $|\mathcal{C}_{n+1}(\mathcal{B}_n)| = 0$.

Proof sketch. Exhaustive enumeration for each $n \le 6$ over the 2^n-element lattice $\mathcal{B}_n$, with the literal test. The predicted values are [1, Theorem 3.1]; the triangle is [8, A008277]. A membership test that were wrong in either direction would have to reproduce all 21 entries by accident. $\square$

Proposition 3.4. *The reduced test of Lemma 3.1 and the literal test of (1) return the same count vector on $\mathcal{T}_m$ for every m with $2 \le m \le 9$.*

Proof sketch. Two independent exhaustive enumerations of the same lattices, differing only in the membership predicate; at $\mathcal{T}_9$ alone the agreement is over all 11 399 553 Boolean antichains. $\square$

The last control is negative, and it is the one that rules out a vacuous or a too-weak membership predicate. Note that a set of pairwise-joining elements need not be Boolean: this is exactly the failure that [1] exhibits by hand.

Proposition 3.5. *In $\mathcal{T}_4$ put*

$$w = (x_1((x_2(x_3x_4))x_5)), \qquad a = ((x_1(x_2x_3))(x_4x_5)), \qquad b = ((x_1x_2)(x_3(x_4x_5))).$$

Then $a \vee b = a \vee w = b \vee w = \hat{1}$, each of $\{a,b\}$, $\{a,w\}$, $\{b,w\}$ is a Boolean antichain, and $(a \wedge w) \vee (b \wedge w) < w = \bigwedge(\{a,w\} \cap \{b,w\})$, so $\{a,b,w\}$ is not. Consequently the count of subsets of $L \setminus \{\hat{1}\}$ whose elements pairwise join to $\hat{1}$ is strictly larger than $|\mathcal{C}_k(L)|$ for $L = \mathcal{T}_m$ at some size k, for each

$4 \le m \le 7$*: the two vectors are*

$m\backslash k$	1	2	3	4	5	6
4	13	21	6			
5	41	167	135	24		
6	131	1248	2182	1011	120	
7	428	9330	31997	29273	8586	720

against Table 1, while in the distributive lattices $\mathcal{B}_n$*,* $2 \le n \le 6$*, the two coincide at every size. Finally* $|\mathcal{C}_5(\mathcal{T}_6)|$ *is neither* 41 *nor* 43*, and* $\mathcal{T}_6$ *has no Boolean antichain of size* 6*.*

Proof sketch. The three expressions and the strict inequality are [1, Example 5.3.2]; that inequality, and the three pairwise statements that the source does not record, are checked here by applying the definition (1) directly to the three explicit expressions — no search, no precomputed tables — after locating them in the constructed $\mathcal{T}_4$. The two count vectors come from the same depth-first search with the membership test replaced by the pairwise condition; the coincidence on $\mathcal{B}_n$ is [1, Lemma 2.4] confirmed computationally. The last two assertions are refutations of a too-small and a too-large value of the single entry of the $\mathcal{T}_6$ row that is independently known ($|\mathcal{C}_5(\mathcal{T}_6)| = \mathsf{c}_5 = 42$), together with the assertion that the coatom bound of Lemma 2.1 is attained rather than merely valid. □

4. The Tamari table

Theorem 4.1. *For* $2 \le m \le 8$ *and* $1 \le k \le m-1$*, the number* $|\mathcal{C}_k(\mathcal{T}_m)|$ *of Boolean antichains of size* k *in the Tamari lattice* $\mathcal{T}_m$ *is the entry of Table 1; moreover* $|\mathcal{C}_0(\mathcal{T}_m)| = 1$ *and* $|\mathcal{C}_k(\mathcal{T}_m)| = 0$ *for* $k \ge m$*.*

Theorem 4.2.

$$\big(|\mathcal{C}_k(\mathcal{T}_9)|\big)_{k=0}^{9} = (1,\ 4861,\ 558963,\ 3288908,\ 4632944,\ 2360907,\ 505701,\ 45838,\ 1430,\ 0).$$

$m\backslash k$	1	2	3	4	5	6	7	8
2	1							
3	4	2						
4	13	21	5					
5	41	167	106	14				
6	131	1248	1579	507	42			
7	428	9330	20870	12158	2332	132		
8	1429	71247	263823	247065	82015	10439	429	
9	4861	558963	3288908	4632944	2360907	505701	45838	1430

Table 1. $|\mathcal{C}_k(\mathcal{T}_m)|$, the number of Boolean antichains of size k in the Tamari lattice of rank m. Blank entries are 0; the $k=0$ column, omitted, is 1. The first column is $\mathsf{c}_m - 1$ by (3) and the last nonzero entry of each row is c_{m-1} by [1, Theorem 5.4.4]; every entry strictly between them is one that [1] does not determine and that we did not find recorded elsewhere (Remark 4.3). Rows $m \le 8$ are Theorem 4.1, row $m = 9$ is Theorem 4.2.

Proof sketch for Theorems 4.1 and 4.2. Both rest entirely on exhaustive computation, with the membership test written out as the literal condition (1): for a candidate extension of an antichain C by an element c, every pair of subsets $S, S' \subseteq C \cup \{c\}$ is tested, de-duplicated only by the symmetry of (1) in S and S'. The search is the depth-first traversal of Section 3, over the lattice constructed and controlled in Proposition 3.2; it visits each Boolean antichain exactly once. Its size is known exactly, being the row sum: the searches enumerate 2, 7, 40, 329, 3508, 45 251, 676 448 and 11 399 553 Boolean antichains for $m = 2, \ldots, 9$, and reject every other candidate extension that arises. Nothing is sampled and no antichain is discarded by a heuristic: the only pruning is the necessary condition of Lemma 2.1(2). The two entries of each row that are independently known — $\mathsf{c}_m - 1$ at $k = 1$ and c_{m-1} at $k = m - 1$ — come out right for every $m \leq 9$, and the whole Boolean-lattice table comes out right under the identical code path (Proposition 3.3). □

Remark 4.3. Five sequences read off Table 1 were looked up in the On-Line Encyclopedia of Integer Sequences [8] on 2026-08-29, at full length and at every prefix of length ≥ 4:

the $k = 2$ column	2, 21, 167, 1248, 9330, 71247, 558963,
the $k = 3$ column	5, 106, 1579, 20870, 263823, 3288908,
the $k = m - 2$ column	4, 21, 106, 507, 2332, 10439, 45838,
the row totals	2, 7, 40, 329, 3508, 45251, 676448, 11399553,

and the triangle read by rows. All five are absent at full length, and every prefix is a miss as well with one exception: the four terms 2, 7, 40, 329 that open the row totals occur consecutively in [8, A386209] and [8, A372348], which continue 3474 and 3550 against our 3508. Full-text searches for the individual values 558963, 3288908, 4632944, 2360907, 505701 and 11399553 return nothing in any sequence in the database. The same searches return [8, A000108] and [8, A008277] on the two positive controls, and a full-text search for "Boolean antichain" returns nothing. The entry that must not be confused with these is [8, A358562], "the number of antichains in the Tamari lattice of order n" $= 2, 3, 8, 83, 28984, 138832442543$; those are *ordinary* antichains and the numbers are unrelated to the row totals above. The two extreme columns of Table 1 do occur, as [8, A001453] ($\mathsf{c}_m - 1$) and [8, A000108], under readings that make no mention of Boolean antichains. A database this size only ever supports "not found", never "new".

Remark 4.4 (outside the formal development). An independent implementation in C, written before the formal one and differing from it in data structures, in the computation of the order and in the indexing of elements, agrees with Table 1 everywhere and continues one row further:

$$\left(|\mathcal{C}_k(\mathcal{T}_{10})|\right)_{k=0}^{9} = (\,1,\ 16795,\ 4505098,\ 40990121,\ 83309281,\\ 61023464,\ 19526769,\ 2923896,\ 198458,\ 4862\,),$$

with row total 212 498 745, in 429 seconds and 2.31 GB. Its two known entries are again correct ($16795 = \mathsf{c}_{10} - 1$, $4862 = \mathsf{c}_9$). We state this row for the record only: unlike every other number in this paper it has not been checked inside the Lean kernel, and we do not claim it at the same standard. The obstruction is cost, not method — see Section 7.

5. The fibre refinement, and where the recursion is missing

The route of [1] to the maximal size goes through the congruence q_m. Its [1, Proposition 5.3.3] says that a Boolean antichain of $\mathcal{T}_m$ of size $k \leq m - 1$ — which by Lemma 2.1(3) is every one — has the form $\{f_{m-1}(c_1), \ldots, f_{m-1}(c_{k-1}), w\}$, so that at most one member is not the greatest element of its q_m-fibre. Writing, for a Boolean antichain C of $\mathcal{T}_m$,

$$\varepsilon(C) := \#\{c \in C : c \neq f_{m-1}(q_m(c))\},$$

we re-ran the enumeration carrying ε as a statistic.

Proposition 5.1. *For $2 \leq m \leq 9$: every Boolean antichain C of $\mathcal{T}_m$ has $\varepsilon(C) \leq 1$; and for every k the number of Boolean antichains of size k with $\varepsilon = 0$ equals $|\mathcal{C}_k(\mathcal{T}_{m-1})|$, the map f_{m-1} being a size-preserving bijection from $\mathcal{C}_k(\mathcal{T}_{m-1})$ onto that part.*

Proof sketch. Exhaustive: the full (k, ε) table is computed for each $m \leq 9$ and every entry with $\varepsilon \geq 2$ is zero, over 725 585 Boolean antichains for $m \leq 8$ and over a further 11 399 553 at $m = 9$. The $\varepsilon = 0$ column of $\mathcal{T}_m$ is then compared entrywise with the count vector of $\mathcal{T}_{m-1}$ from Table 1; at $m = 9$ it reads $1, 1429, 71247, 263823, 247065, 82015, 10439, 429, 0, 0$, the $\mathcal{T}_8$ row verbatim. Both halves are established in general in [1]; what the computation adds is the numerical form, which is what (2) needs. In detail: $\varepsilon \leq 1$ is the first assertion of [1, Proposition 5.3.3], which says that a

Boolean antichain of size $k \leq m-1$ in $\mathcal{T}_m$ has the form $\{f_{m-1}(c_1), \ldots, f_{m-1}(c_{k-1}), w\}$. For the second half, if $\varepsilon(C) = 0$ then C is of that form with w a fibre maximum, so the second alternative of [1, Definition 5.3.1] — which puts w in $[\hat{1}] \setminus \{\hat{1}\}$, all of whose members are non-maxima — is excluded and $\{c_1, \ldots, c_{k-1}, q_m(w)\}$ is a Boolean antichain of $\mathcal{T}_{m-1}$ mapped onto C by f_{m-1}. Conversely f_{m-1} maps Boolean antichains to Boolean antichains, being an injective lattice map that fixes $\hat{1}$, so (1) transports along it. The refined tables use the reduced membership test of Lemma 3.1, whose agreement with the literal one on these very lattices is Proposition 3.4. □

Proposition 5.1 turns the question into (2), so what has to be counted is $E(m,k)$, the antichains with exactly one member off its fibre maximum. That member w either lies in the fibre $[\hat{1}]$ through the greatest element or does not, which is precisely the case distinction in the definition of *almost Boolean* in [1, Definition 5.3.1]. Write $E^{\mathrm{top}}(m,k)$ for the number of $C \in \mathcal{C}_k(\mathcal{T}_m)$ having exactly one member in $[\hat{1}] \setminus \{\hat{1}\}$.

Theorem 5.2. *For $2 \leq m \leq 9$ and every k, the numbers $E(m,k)$ and $E^{\mathrm{top}}(m,k)$ are the entries of Table 2. Moreover*

$$E^{\mathrm{top}}(m,k) = (m-1)\,|\mathcal{C}_{k-1}(\mathcal{T}_{m-1})| \qquad \text{for } k \in \{1,2\},$$
$$E^{\mathrm{top}}(m,k) < (m-1)\,|\mathcal{C}_{k-1}(\mathcal{T}_{m-1})| \qquad \text{for } 3 \leq k \leq m-1,$$

for every m in that range.

$E(m,k)$ $m\backslash k$	1	2	3	4	5	6	7	8
2	1							
3	3	2						
4	9	19	5					
5	28	146	101	14				
6	90	1081	1473	493	42			
7	297	8082	19291	11651	2290	132		
8	1001	61917	242953	234907	79683	10307	429	
9	3432	487716	3025085	4385879	2278892	495262	45409	1430

$E^{\mathrm{top}}(m,k)$ $m\backslash k$	1	2	3	4	5	6	7	8
2	1							
3	2	2						
4	3	12	5					
5	4	52	68	14				
6	5	205	647	353	42			
7	6	786	5502	5942	1720	132		
8	7	2996	45253	85842	45099	8030	429	
9	8	11432	371440	1159681	979700	302394	36400	1430

Table 2. Top: $E(m,k)$, the Boolean antichains of size k in $\mathcal{T}_m$ with exactly one member off its q_m-fibre maximum. Bottom: $E^{\mathrm{top}}(m,k)$, those of them whose off-maximum member lies in the fibre through $\hat{1}$. Blank entries are 0. By (2) the first table added to the $\mathcal{T}_{m-1}$ row of Table 1 gives the $\mathcal{T}_m$ row.

Proof sketch. Exhaustive, by the depth-first search of Theorem 4.1 run with two $\{0,1\}$-weightings of the elements of $\mathcal{T}_m$ — the indicator of "not the greatest element of its q_m-fibre" and the indicator of $[\hat{1}] \setminus \{\hat{1}\}$ — accumulating the full (k, ε) table rather than the count vector. The searches are again over all 725 585 Boolean antichains for $m \leq 8$ and all 11 399 553 at $m = 9$, and use the reduced membership test of Lemma 3.1, cross-checked against the literal one by Proposition 3.4. The two displayed relations are then finitely many integer comparisons against the entries of Table 1: at $m = 9$,

$$8 = 8 \cdot 1, \qquad 11\,432 = 8 \cdot 1429, \qquad 371\,440 < 8 \cdot 71\,247 = 569\,976,$$

and so on along the row. □

Remark 5.3. The quantity $(m-1)\,|\mathcal{C}_{k-1}(\mathcal{T}_{m-1})|$ counts the pairs consisting of a Boolean antichain of size $k-1$ in $\mathcal{T}_{m-1}$ and one of the $m-1$ non-greatest elements of $[\hat{1}]$, with the closing-index-set condition of [1, Proposition 5.3.3] ignored; it is the rightmost term of the upper bound [1, Corollary 5.3.9]. The sentence following that corollary reads, verbatim, "Equality holds for the cases $k \in \{0, 1, 2\}$. In all other cases, it can be verified that this bound is not tight." The two groups of terms in the bound count disjoint classes of almost Boolean antichains — the rightmost those with a member in $[\hat{1}]$, the others those projecting to a Boolean antichain of size k in $\mathcal{T}_{m-1}$ — and each group separately dominates the corresponding part of $|\mathcal{C}_k(\mathcal{T}_m)|$, so equality in the total forces equality in each branch. The equality at $k \leq 2$ in Theorem 5.2 is therefore [1]'s, not ours. What Theorem 5.2 adds is the individual branch: the source's remark says only that the *total* bound is not tight for $k \geq 3$, and does not say which branch loses; the table shows that the top-fibre branch does, and by how much. The threshold is where one expects it: $k = 3$ is the first size at which a Boolean antichain contains two distinct elements $f_{m-1}(c_i)$, so it is the first size at which the condition "at most one i with $k_w \in I_{f_{m-1}(c_i)}$" can be violated.

Remark 5.4. Two facts can be read off Table 2 directly. At the maximal size $k = m - 1$ the three quantities $|\mathcal{C}_{m-1}(\mathcal{T}_m)|$, $E(m, m-1)$ and $E^{\mathrm{top}}(m, m-1)$ all equal c_{m-1}: every Boolean antichain of maximal size has exactly one member off its fibre maximum and that member is in the top fibre, which is the shape that the valid-labelling bijection of [1, Section 5.4] exploits. And no Boolean antichain ever has two members in $[\hat{1}] \setminus \{\hat{1}\}$, as it cannot: all of those are non-maxima, and $\varepsilon \leq 1$.

6. The partition lattice

The same gap exists in the partition lattice, where [1, Theorem 4.4] settles the maximal size $k = n-1$ only.

Theorem 6.1. *For $3 \leq n \leq 6$ and $1 \leq k \leq n-1$, the number $|\mathcal{C}_k(\mathcal{P}_n)|$ of Boolean antichains of size k in the partition lattice $\mathcal{P}_n$ is*

$n \backslash k$	1	2	3	4	5
3	4	3			
4	14	45	16		
5	51	620	670	125	
6	202	9750	24050	11595	1296

with $|\mathcal{C}_0(\mathcal{P}_n)| = 1$ and $|\mathcal{C}_k(\mathcal{P}_n)| = 0$ for $k \geq n$.

Proof sketch. Exhaustive enumeration with the literal test (1), over $\mathcal{P}_n$ built from the restricted growth strings of $[n]$ ordered by refinement and indexed by decreasing block count, which is a linear extension. The two known columns come out right for every n: the last nonzero entry of each row is $n^{n-2} = 3, 16, 125, 1296$ by [1, Theorem 4.4], and the $k = 1$ column is $|\mathcal{P}_n| - 1 = B_n - 1$ by (3), where B_n is a Bell number. The five entries strictly between them — 620, 670, 9750, 24050, 11595 — are the ones [1] leaves open. Neither of the two rows $51, 620, 670, 125$ and $202, 9750, 24050, 11595$, nor either of the two intermediate columns $3, 45, 620, 9750$ and $16, 670, 24050$, nor the triangle read by rows occurs in [8] (searched 2026-08-29; the $k = 1$ column is of course there, being the Bell numbers less one). □

7. What is not settled

The missing term. Equation (2) together with Proposition 5.1 reduces Question 6.1 to a formula for $E(m, k)$, and Theorem 5.2 says where such a formula must come from: the top-fibre branch is a strict sub-count of $(m-1)|\mathcal{C}_{k-1}(\mathcal{T}_{m-1})|$ from $k = 3$ on, and the deficiency is what the closing index sets I_v of

[1, Definition 5.2.1] control. Refining Table 2 by I_v — that is, counting the pairs $(\{c_1, \ldots, c_{k-1}\}, w)$ by the number of i with $k_w \in I_{f_{m-1}(c_i)}$ — is a finite computation of the same shape as the ones here and is the natural next step; we have not done it. We also found no closed form for any column: fitting a hypergeometric ratio $a_{m+1}/a_m = P(m)/Q(m)$ with $\deg P, \deg Q \leq 3$ over the rationals succeeds on the maximal-size diagonal — returning $(4m-2)/(m+1)$, the Catalan recurrence, which is [1, Theorem 5.4.4] and a control on the fit — and fails on the columns $k \leq 5$, on the diagonals $k = m-2$ and $k = m-3$, and on the row totals. With eight or nine terms per sequence this is weak evidence and no more.

Larger m. Both remaining rows are blocked by cost rather than by method. The figures in this paragraph are measurements of, and extrapolations from, the programs described in Remarks 4.4 and 8.1; they say nothing about the mathematics. For $\mathcal{T}_{10}$ the search is about 54 times the $\mathcal{T}_9$ search; Remark 4.4 gives the row from an ordinary compiled program, but reproducing it inside the Lean kernel would take some three hours in a single declaration and we have not run it. For $\mathcal{T}_{11}$ the $n \times n$ meet and join tables alone would occupy $2 \cdot 58786^2 \cdot 4 = 27.6$ GB, and without them each of the roughly 7.7×10^{11} join tests becomes a 919-word bitmask scan. Reaching $m = 11$ therefore wants a different algorithm — meets and joins computed directly on bracketing vectors, or an actual recursion — and not a bigger table.

The other questions of [1]**.** [1, Question 6.2] asks whether the poset of Boolean antichains of maximal size in $\mathcal{T}_m$, ordered by inclusion of the generated order ideals, is isomorphic to the Tamari order, to the non-crossing partition lattice, or to the Dyck path order. That is again a finite question at each m, and the enumeration here already produces the antichains; only the comparison of their order ideals is missing. We have not carried it out.

8. Verification

Every statement labelled Proposition or Theorem in Sections 3–6 has been checked in Lean 4 (toolchain `v4.32.0`), in the directory `LeanProblemSpec/TamariBoolean/` of a repository that is otherwise built against *Mathlib*; the files of this development import only Lean core, and are free of `sorry`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: thirteen declarations, each depending on `propext`, `Quot.sound`, one `native_decide` axiom and — in all but one of the thirteen — `Classical.choice`, and on nothing else. The evaluation core — the lattice constructions, the meet and join tables, the two membership tests and the depth-first search — imports nothing at all, so it compiles to a shared library that is loaded with `--load-dynlib`; the object code and the interface file are emitted by a single compiler invocation, so they cannot drift apart. The three files take 36.8 s, 197 s and 517 s of wall time at peak resident sizes of 1.63, 1.89 and 1.91 GB. Independently, every number in this paper was first produced by an implementation in C, written before the Lean one and differing from it in data structures, in how the order is computed and in how elements are indexed; the two agree on every value they both produce, which is everything except the row of Remark 4.4. Lemma 3.1 is proved by hand above and is not part of the formal development; it is used only for the tables of Section 5, and Proposition 3.4 checks its consequence directly on those lattices.

Remark 8.1 (outside the formal development). Three cost observations may save a reader a repetition. First, the reduced test of Lemma 3.1 performs 8.1 times fewer join tests than the literal one (8.7×10^8 against 7.1×10^9 at $m = 9$) and is about 15 times faster in C, yet inside Lean the two cost the same (231 s against 226 s at $m = 9$), its per-test cost being correspondingly higher; so the headline tables can rest on the literal definition at no price. Second, the meet and join tables are 34% of the C bill at $m = 9$ but only a few percent of the Lean one — three searches sharing one pair of tables cost 517 s against about 3×197 s for three that each build their own — so in Lean essentially the whole bill is the search. Third, a trap: `native_decide` counts heartbeats against its own evaluation, so the same statement that passes at the default budget for $m = 8$ in 7 seconds fails for $m = 9$ after 740 seconds with a message reading `timeout at isDefEq`, which looks like an elaboration blow-up and is not one. The diagnosis is the smaller instance passing at the default; the fix is to disable the heartbeat limit.

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