BOOLEAN AND DISTRIBUTIVE INTERVALS IN THE TWO BRUHAT ORDERS ON THE SYMMETRIC GROUP

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Abstract. Two exercises of Stanley, reprinted as Problems 21 and 22 of Worley's notebook of open problems, ask which intervals of the symmetric group S_n are boolean algebras in the strong Bruhat order, which are distributive lattices in the weak order, and how many there are of each. We settle the enumeration in the strong Bruhat order for $n \leq 7$: the number of boolean intervals is 1, 3, 18, 174, 2403, 44190, 1033578 for $n = 1, \ldots, 7$. These values appear not to be recorded anywhere; the most recent paper on the subject asks precisely for the proportion of Bruhat intervals that are boolean. For the weak order we correct the value printed at $n = 2$ in the notebook and in the textbook: the weak order on S_2 has three intervals and all three are distributive lattices, so the value is 3 and not 2. We then re-derive the published counts 1, 3, 16, 124, 1262, 15898, 238572 for $n \leq 7$ directly from the definitions, without using the known characterization, and we verify that characterization — an interval $[u, v]$ of the weak order is a distributive lattice if and only if $u^{-1}v$ avoids the pattern 321 — at every one of the 1899, 31711 and 672697 intervals of S_5, S_6 and S_7. Every theorem below, and each exhaustive search behind it, is machine-checked in Lean 4.

1. Introduction

Let S_n be the symmetric group on $\{1, \ldots, n\}$. It carries two classical partial orders: the *strong Bruhat order*, generated by the transpositions that increase the number of inversions, and the *weak order*, generated by the *adjacent* transpositions that do so. The intervals of these orders are among the best-studied posets in algebraic combinatorics, and two natural questions about them — which intervals are boolean algebras in the strong order, and which are distributive lattices in the weak order — appear as exercises 3.183(f) and 3.185(j) of the second edition of Stanley's *Enumerative Combinatorics*, volume 1 [4], both rated [5–], "an unsolved problem that has received little attention and may not be too difficult". They are reprinted verbatim as Problems 21 and 22 of Worley's recent notebook of open problems on tableaux [13]:

Problem 1.1 ([13, Problem 21])**.** Let S_n be the set of permutations of n ordered by the strong Bruhat order. Characterize the intervals of S_n that are boolean algebras and compute their total number.

Problem 1.2 ([13, Problem 22])**.** Let S_n^W be the set of permutations of n ordered by the weak Bruhat order. Characterize the intervals of S_n^W that are distributive lattices and compute their total number. The values for $1 \leq n \leq 8$ are 1, 2, 16, 124, 1262, 15898, 238572, 4152172.

The asymmetry between the two is the starting point of this paper. Problem 22 comes with data, and in fact both of its halves are settled in the literature, though the notebook cites neither: the characterization is Stembridge's theorem [10] on fully commutative elements, and the counting sequence is entry `A190291` of the on-line encyclopedia of integer sequences [8], contributed by Stanley in 2011 and now known through $n = 15$. Problem 21 comes with no data at all, and we have not been able to find any.

This paper does the following.

(i) The count for Problem 21, through $n = 7$. We show (Theorems 4.1, 4.2, 4.3) that the strong Bruhat order on S_n has

$$1, \quad 3, \quad 18, \quad 174, \quad 2403, \quad 44190, \quad 1033578$$

boolean intervals for $n = 1, \ldots, 7$. The corresponding numbers of *all* intervals are 1, 3, 19, 213, 3781, 98407, 3550919, the sequence `A383875` [9], so the proportion that are boolean falls from 1 at $n = 2$ to about 0.29 at $n = 7$. A search of the on-line encyclopedia — for the full sequence, for each of its tails, and for the single term 1033578 — returns nothing, and neither does a full-text search of a large corpus of arXiv preprints for consecutive pairs of these terms. Absence from a database is not a proof of novelty, and this is after all a textbook exercise whose computation is elementary; but the enumeration is explicitly left open in the most recent paper on the subject, Tenner's *Interval structures in the Bruhat and weak orders* [12], whose Question 6.1 asks what proportion of intervals in the Bruhat order are boolean.

(ii) A correction to Problem 22. The value printed for $n = 2$ is wrong. The weak order on S_2 is a two-element chain, its three intervals are $[e, e]$, $[s_1, s_1]$ and $[e, s_1]$, and each of the three is a distributive lattice; the correct value is 3 (Theorem 5.1). The error is not the notebook's: exercise 3.185(j) of [4] prints the same list, and Stanley's own encyclopedia entry `A190291` begins $1, 3, 16, 124, \ldots$.

(iii) The counts for Problem 22 from the definitions. The values 1262, 15898 and 238572 at $n = 5, 6, 7$ are published; what is new here is that they are derived (Theorem 5.2) from the definition of the weak order as a reflexive-transitive closure and the definition of "distributive lattice" as "binary greatest lower bounds and least upper bounds inside the interval, with meet distributing over join", without any appeal to the characterization. The known algorithm, by contrast, counts linear extensions of inversion posets of 321-avoiding permutations and depends on the characterization for its correctness.

(iv) The characterization for Problem 22, verified. Stembridge's theorem [10], combined with the isomorphism $[u, v] \cong [e, u^{-1}v]$ of weak-order intervals [1, Prop. 3.1.6] and assembled in this form by Tenner [12, Cor. 4.4], says that $[u, v]$ is a distributive lattice if and only if $u^{-1}v$ is fully commutative, which in S_n means 321-avoiding. We prove none of this; we verify it (Theorems 6.1, 6.2) at every interval of S_n for $n \leq 7$ — 1899, 31711 and 672697 intervals at the top three ranks — and observe that the 321-avoidance side, computed alone with no distributivity test anywhere, returns the sequence of Problem 22 at each of the seven ranks (Proposition 6.4).

The characterization half of Problem 21 remains open, and we say what little we can about it in Section 7. All the counts here are obtained by exhaustive search; every search, and every lemma that says what the search is searching for, has been checked by a proof assistant, and Section 8 says exactly what that means.

2. The two orders

We write permutations in one-line notation, $w = w(1)w(2)\cdots w(n)$, and multiply so that $(wt)(i) = w(t(i))$; thus right multiplication by a transposition swaps two *positions* of the one-line notation. The inversion set and length of $w \in S_n$ are

$$\mathrm{Inv}(w) = \{(i, j) : 1 \leq i < j \leq n,\ w(i) > w(j)\}, \qquad \ell(w) = |\,\mathrm{Inv}(w)|.$$

Let t_{ab} be the transposition of $a < b$ and $s_a = t_{a,a+1}$.

Definition 2.1. The *strong Bruhat order* $\leq$ on S_n is the reflexive-transitive closure of the relation "$v = ut_{ab}$ for some $a < b$ with $\ell(u) < \ell(v)$". The *right weak order* $\leq_R$ is the reflexive-transitive closure of "$v = us_a$ for some a with $\ell(u) < \ell(v)$". For $u \leq v$ we write $[u, v] = \{x \in S_n : u \leq x \leq v\}$, and likewise in the weak order.

Since s_a is a transposition, $u \leq_R v$ implies $u \leq v$. Both orders are graded by ℓ, and the *rank* of an interval is $\ell(v) - \ell(u)$. Problem 22 says only "the weak Bruhat order" and does not pin left or right; the choice is immaterial to the counting, because $w \mapsto w^{-1}$ is an order isomorphism from the left weak order onto the right one and so carries intervals to intervals. We work throughout with the right weak order, which is also the convention of [12].

Both orders have a cheap description that we use as the computational definition, and in each case the agreement of the two descriptions is proved, for each n separately, from the two finite assertions recorded in Proposition 2.3 below.

Definition 2.2. For $w \in S_n$ and $1 \le i, j \le n$ put $c_w(i,j) = \#\{a \le i : w(a) > j\}$, and let

$$\mathrm{E}(w) = \{(a,b) : a < b,\ w^{-1}(b) < w^{-1}(a)\}$$

be the set of pairs of *values* inverted by w, that is, the pairs $a < b$ such that b precedes a in the one-line notation of w.

Proposition 2.3. *Fix n and suppose that*

(S1) *every length-increasing transposition step out of a permutation of $\{1,\dots,n\}$ increases the matrix c entrywise, and*
(S2) *whenever $c_u \le c_v$ entrywise and $u \ne v$, there is a transposition t with $\ell(u) < \ell(ut) \le \ell(v)$ and $c_{ut} \le c_v$ entrywise,*

and likewise (W1), (W2) with "transposition" replaced by "adjacent transposition" and the matrix comparison replaced by containment of E*. Then for $u, v \in S_n$*

$$u \le v \iff c_u(i,j) \le c_v(i,j) \ \textit{ for all } i,j, \qquad u \le_R v \iff \mathrm{E}(u) \subseteq \mathrm{E}(v).$$

The two right-hand sides are Ehresmann's criterion and the classical description of the weak order by inversion sets. Given the finite assertions, the proof is an induction on $\ell(v) - \ell(u)$: (S1) makes the criterion a consequence of the closure, and (S2) produces, from a pair satisfying the criterion, one covering step out of u towards v, which is the inductive step in the other direction. We state it in this form because it is exactly what is checked: (S1), (S2), (W1), (W2) are finite statements about S_n, verified by exhaustive evaluation at each n we use, and everything downstream of them is an ordinary proof.

3. Boolean and distributive finite posets

Let P be a finite poset. Write B_k for the lattice of all subsets of a k-element set. An element $a \in P$ is an *atom* if exactly two elements of P — necessarily a bottom element and a itself — are $\le a$; write $\mathrm{At}(P)$ for the set of atoms and, for $x \in P$, $A_x = \{a \in \mathrm{At}(P) : a \le x\}$.

Definition 3.1. P is *boolean of rank k* if it is isomorphic as a poset to B_k. P is a *distributive lattice* if every $x, y \in P$ have a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$ *in* P, and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in P$.

Both notions are intrinsic to the induced order, so "the interval $[u,v]$ is boolean" and "the interval $[u,v]$ is a distributive lattice" are statements about the subposet $[u,v]$ and not about S_n. The searches below decide them through the following two criteria, both of which are used in both directions; this is what makes the numbers counts of boolean and of distributive intervals rather than counts of intervals passing a test.

Lemma 3.2. *A finite poset P is boolean if and only if*

1. $|P| = 2^{|\mathrm{At}(P)|}$,
2. $x \mapsto A_x$ *is injective on P, and*
3. $A_x \subseteq A_y$ *implies $x \le y$.*

In that case $P \cong B_k$ with $k = |\mathrm{At}(P)|$.

Proof sketch. In B_k the atoms are the singletons, so (1)–(3) hold with A_x the isomorphic image. Conversely, transitivity gives $x \le y \Rightarrow A_x \subseteq A_y$, so with (3) the map $x \mapsto A_x$ is an order embedding of P into the subsets of $\mathrm{At}(P)$; with (2) it is injective, and with (1) it is a bijection onto all of them by counting. $\square$

Lemma 3.3. *A finite poset P is a distributive lattice if and only if for all $x, y, z \in P$ the searches for a greatest lower bound of x, y and a least upper bound of x, y in P both succeed and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

Lemma 3.3 is a triviality once one observes that antisymmetry makes greatest lower bounds and least upper bounds unique, so that a search returns *the* meet; without antisymmetry the displayed equation and the distributive law are different statements, since the latter quantifies over arbitrary elements satisfying the bound conditions.

4. Boolean intervals of the strong Bruhat order

Theorem 4.1. *The number of pairs $u \le v$ in the strong Bruhat order on S_n for which the interval $[u, v]$ is a boolean algebra is*

$$1, \quad 3, \quad 18, \quad 174, \quad 2403 \qquad (n = 1, 2, 3, 4, 5).$$

Theorem 4.2. *The strong Bruhat order on S_6 has exactly* 44190 *boolean intervals.*

Theorem 4.3. *The strong Bruhat order on S_7 has exactly* 1033578 *boolean intervals.*

The search, at $n \le 5$. All $(n!)^2$ ordered pairs (u, v) are formed; comparability is decided by Ehresmann's criterion, which Proposition 2.3 identifies with the strong Bruhat order at this n; for each comparable pair the interval is listed by scanning S_n and keeping the x with $c_u \le c_x \le c_v$; and Lemma 3.2 is applied to the resulting poset. At $n = 5$ this is 14400 pairs, of which 3781 are comparable. The whole search is evaluated exhaustively. □

The search, at $n = 6$ and $n = 7$. The naive sweep does not survive two more ranks: at $n = 7$ it would rebuild each of 3550919 intervals by two criterion evaluations per element of S_7, some 5.8×10^9 comparisons of 7×7 matrices, before any interval is tested. Three changes make it feasible, and each is proved not to change the answer (Proposition 4.4). First, the order is tabulated once: the $n! \times n!$ matrix of comparisons is computed and stored, 25.4 million entries at $n = 7$, after which each comparison is one lookup. Second, the size of an interval is computed by counting rather than by listing, and the interval is listed only when its size is a power of two; this is sound because a boolean poset of rank k has 2^k elements, and it cuts 3550919 candidate pairs to 1089070 at $n = 7$. Third, for each element x of a listed interval the set A_x of Lemma 3.2 is packed into the bits of a single integer, so that conditions (2) and (3) of that lemma become one quadratic loop of integer comparisons rather than two loops of order lookups. □

Proposition 4.4. *Fix n and suppose that the tabulated list of permutations enumerates S_n, each once, with the stated matrices c_w, and that the stored $n! \times n!$ table is the entrywise comparison of those matrices. Then the count returned by the accelerated search equals the number of pairs $u \le v$ with $[u, v]$ boolean, as defined in Sections 2 and 3.*

Proof sketch. Four steps. The packed atom signatures compute conditions (2) and (3) of Lemma 3.2 exactly, because bit i of the signature of x is the assertion $a_i \le x$, so equality of signatures is agreement on all atoms and containment of signatures is implication. The tabulated comparison of count matrices is Ehresmann's criterion. The boolean test is invariant under reordering the list of elements of an interval and transports along any injective relabeling that preserves the order — which is what lets the accelerated search work with array indices while the definition works with one-line notations. Finally, the power-of-two prefilter discards only non-boolean intervals, by condition (1) of Lemma 3.2. The two hypotheses are finite assertions about the tables, verified by evaluation and not assumed. □

Proposition 4.5. *At $n = 3, 4, 5$ the accelerated search of Proposition 4.4 returns* 18*,* 174 *and* 2403*, in agreement with the direct sweep of Theorem 4.1.*

Proposition 4.5 is a genuine cross-check and not a tautology: the two routes share no step. They enumerate S_n differently, represent the order differently (stored bits against recomputed matrix comparisons), test booleanness differently (packed signatures against direct quantification over atoms), and loop differently; only Proposition 4.4 forces their outputs to agree. Independently of all of this, the values through $n = 7$ were reproduced by three programs written for the purpose, including one written without sight of the others, with matching distributions by rank; at $n = 6$ and $n = 7$ the rank distributions are

$$720,\ 3708,\ 9234,\ 13335,\ 11244,\ 5019,\ 894,\ 36$$

and

$$5040,\ 33984,\ 111910,\ 223704,\ 286108,\ 231484,\ 111064,\ 27484,\ 2736,\ 64.$$

Two published theorems, recovered. Restricting the same search to special shapes recovers two counts that are in the literature, which is the sharpest available check on the definition itself.

Proposition 4.6. *The number of $w \in S_n$ for which the lower interval $[e, w]$ is boolean is $1, 2, 5, 13, 34$ for $n \le 5$, and the number of boolean intervals whose bottom element has length one is $8, 40, 160$ for $n = 3, 4, 5$.*

The first list is F_{2n-1}, the odd-indexed Fibonacci numbers (`A001519` [5]): Tenner [11, Thm. 4.3] proves that $[e, w]$ is boolean exactly when w avoids both 321 and 3412, and her Corollary 4.4 credits the resulting count to Fan and West. The second list is the sequence of row sums of Table 4 of [12], whose Theorem 5.3 evaluates in closed form the number $t(n, k)$ of boolean intervals whose bottom is the transposition s_k. Both are reproduced exactly.

Negative controls. An enumeration is only as good as the evidence that it is not enumerating something else. We record what was refuted.

Proposition 4.7. *The whole of S_3, which is an interval of the strong Bruhat order, is not a boolean algebra, and is not a distributive lattice in the weak order either. Moreover the naive guess "an interval of rank k is boolean if and only if it has 2^k elements" is true for $n \le 4$ and false for $n = 5$: exactly 2415 intervals of S_5 have $2^{\ell(v)-\ell(u)}$ elements, against 2403 boolean ones.*

The location of that counterexample is worth stating plainly: the size condition and booleanness agree on all 213 intervals of S_4 and part company only at $n = 5$, among 3781 intervals, which is past the range where hand-checking is comfortable. This is the argument for computing rather than guessing, and it is also the mistake a formalization would most naturally make.

Proposition 4.8. *At $n = 6$ and $n = 7$: the counts 44190 and 1033578 fall short of $6!^2 = 518400$ and $7!^2 = 25401600$, so not every interval is boolean; they exceed $6! = 720$ and $7! = 5040$, so the singleton intervals $[u, u]$ are not all of them; the copy of S_3 inside S_6 and inside S_7 is an interval of rank 3 with six elements, hence not boolean, and it really is an interval, so the previous assertion is not vacuous; the order is not the discrete one; and the counts differ from each of their two neighbours.*

Beyond $n = 7$. The same enumeration, run outside the verified setting, gives 29757420 boolean intervals in S_8; we state this as a computation and not as a theorem. Its two forced columns agree with what they must: the rank-0 term is $8! = 40320$, and the rank-1 term, 341136, is the number of edges of the Hasse diagram of the Bruhat order on S_8, the corresponding term of `A002538` [6]. The count of boolean lower intervals at $n = 8$ is $610 = F_{15}$, as Tenner's theorem requires. The obstruction to going further inside the verified setting is storage rather than time: the tabulated order at $n = 8$ has $(8!)^2 \approx 1.6 \times 10^9$ entries.

5. Distributive intervals of the weak order

Theorem 5.1. *The number of pairs $u \le_R v$ in the right weak order on S_n for which $[u, v]$ is a distributive lattice is*

$$1, \quad 3, \quad 16, \quad 124 \qquad (n = 1, 2, 3, 4).$$

In particular the value at $n = 2$ is 3, not 2.

Proof of the case $n = 2$, by hand. The weak order on $S_2 = \{e, s_1\}$ is a chain with two elements. Its intervals are $[e, e]$, $[s_1, s_1]$ and $[e, s_1]$; the first two are one-element posets and the third is a two-element chain, and all three are distributive lattices. Hence the number is 3. The value 2 printed in [13, Problem 22] and in [4, Ex. 3.185(j)] is an error. It is worth noting that Stanley's own encyclopedia entry `A190291` [8] has $a(2) = 3$, so the notebook has faithfully reproduced a misprint. □

The search, at $n \leq 4$. All $(n!)^2$ ordered pairs are formed, comparability is decided by containment of the sets E of Proposition 2.3, and Lemma 3.3 is applied to each of the resulting intervals; at $n = 4$ there are 151 comparable pairs. The value at $n = 2$ is thus obtained a second time, independently of the argument above, and a third time in Proposition 5.4. □

Theorem 5.2. *The right weak order on S_5, S_6 and S_7 has exactly* 1262*,* 15898 *and* 238572 *intervals that are distributive lattices.*

These three values are the published terms $a(5)$, $a(6)$, $a(7)$ of `A190291` [8], a sequence contributed by Stanley in 2011 and extended to $n = 15$ by Corti [3]; no new value is claimed. What Theorem 5.2 adds is the derivation. The published route to these numbers goes through the characterization of Section 6 and counts linear extensions of inversion posets; the route here tests the definition of Section 3 on each interval, and so is an independent check of the sequence rather than an application of the theorem that explains it.

The search. All $(n!)^2$ ordered pairs are formed and comparability is decided by containment of the value-inversion sets E, which Proposition 2.3 identifies with the right weak order at this n; the numbers of comparable pairs are 1899, 31711 and 672697 for $n = 5, 6, 7$. For each comparable pair the interval is listed and Lemma 3.3 is applied. Applied literally, that lemma re-searches for a meet or a join at each step of a triple loop, at a cost of order m^5 in the size m of the interval; since the whole weak order on S_5 is itself an interval with $m = 120$, this alone stops the computation at $n = 4$. The remedy is to tabulate the meet and the join once per rank, after which the test is a triple loop of table lookups, of order m^3; summed over all intervals this is about 1.1×10^6 steps at $n = 5$, 1.3×10^8 at $n = 6$ and 2.1×10^{10} at $n = 7$. The correctness of the tables is itself checked, by the identity of Proposition 5.3. □

Proposition 5.3. *Fix n and suppose that the tabulated list of permutations enumerates S_n once each with the stated inversion sets; that the stored order table is containment of those sets; that D_x and U_x are the down-set and the up-set of x; that*

$$D_x \cap D_y = D_{x \wedge y} \qquad \text{and} \qquad U_x \cap U_y = U_{x \vee y}$$

for the tabulated meet and join; and that the stored order is antisymmetric. Then the count returned by the accelerated search equals the number of pairs $u \leq_R v$ with $[u, v]$ a distributive lattice.

Proof sketch. The displayed identity is exactly the assertion that the tabulated $x \wedge y$ is a greatest lower bound of x and y in the whole of S_n: $z \leq x$ and $z \leq y$ if and only if $z \leq x \wedge y$. Read that way it costs a quadratic number of set intersections instead of the cubic quantifier it replaces, which is the point of stating it in this form. Two further steps take it to intervals. First, the global meet of two elements x, y of an interval $[u, v]$ lies in $[u, v]$ — it is below $x \leq v$, and above u because u is a common lower bound of x and y — and a global greatest lower bound lying in a subset is a greatest lower bound there; likewise for joins. Second, antisymmetry makes meets and joins unique, so the distributive law in the form quantified over arbitrary bounds collapses to the single equation the triple loop tests, as in Lemma 3.3. As in Proposition 4.4, the test transports along an order-preserving injective relabeling and is blind to the order in which the interval is listed. □

Proposition 5.4. *At $n = 2, 3, 4$ the accelerated search of Proposition 5.3 returns* 3*,* 16 *and* 124*, in agreement with the direct sweep of Theorem 5.1.*

Again the two routes share no step: a different enumeration of S_n, inversion bitmasks against explicit sets of value pairs, tabulated meets and joins against repeated searches, and a different loop. In particular the corrected value 3 at $n = 2$ is obtained twice, by routes with nothing in common.

Proposition 5.5. *At $n = 5, 6, 7$: the counts 1262, 15898 and 238572 are smaller than $5!^2 = 14400$, $6!^2 = 518400$ and $7!^2 = 25401600$, so not every interval is a distributive lattice; at $n = 5$ the sharper comparison holds, $1262 < 1899$, the number of pairs that are actually comparable, so the refutation does not rest on incomparable pairs; the counts exceed $5! = 120$, $6! = 720$ and $7! = 5040$, so the singleton intervals are not all of them; the copy of the whole of S_3 — the hexagon, which contains a pentagon N_5 — sits inside each of S_5, S_6, S_7 as a weak-order interval and is not a distributive lattice, and it really is an interval; the order is not the discrete one; the counts differ from each of their two neighbours; and they strictly increase with n.*

6. The characterization in the weak order

Say that $w \in S_n$ *avoids* 321 if there are no $i < j < k$ with $w(i) > w(j) > w(k)$. The characterization half of Problem 22 is settled in the literature, though the notebook gives no reference for it: Stembridge [10, Thm. 2.2] proved that the principal order ideal $[e, w]$ of the weak order is a distributive lattice exactly when w is fully commutative, Björner and Brenti [1, Prop. 3.1.6] give the isomorphism $[u, v] \cong [e, u^{-1}v]$ of weak-order intervals, and Tenner [12, Cor. 4.4] states the combination; for unsigned permutations, fully commutative means 321-avoiding. We prove none of this. We check it.

Theorem 6.1. *For $u \leq_R v$ in S_4, the interval $[u, v]$ is a distributive lattice if and only if $u^{-1}v$ avoids 321.*

Theorem 6.2. *The same holds in S_5, in S_6 and in S_7: for every one of the 1899, 31711 and 672697 pairs $u \leq_R v$, the interval $[u, v]$ is a distributive lattice if and only if $u^{-1}v$ avoids 321.*

The search. For each comparable pair, the interval is tested for distributivity exactly as in Section 5, and $u^{-1}v$ is tested for 321-avoidance by a triple loop over positions; the two answers are compared. The pattern test adds only a few percent to the cost of the distributivity test it accompanies. The conventions have to match for the statement to be the intended one, and they do: $[u, v]$ is taken in the right weak order, s_a swaps positions, and $u^{-1}v$ sends i to the position of $v(i)$ in u. □

Proposition 6.3. *Fix n and suppose the tables satisfy the hypotheses of Proposition 5.3 and that the tabulated comparison of the two predicates returns agreement on every comparable pair of indices. Then for all $u, v \in S_n$ with $u \leq_R v$, the interval $[u, v]$ is a distributive lattice if and only if $u^{-1}v$ avoids 321; and the list of pairs $u \leq_R v$ with $[u, v]$ distributive is equal, as a list, to the list of pairs $u \leq_R v$ with $u^{-1}v$ 321-avoiding.*

The point of Proposition 6.3 is that the object actually evaluated is an equality between two loops over array indices, while what it is allowed to conclude is a statement about the reflexive-transitive closure of Section 2 and about greatest lower bounds inside the interval. The equality of lists in the second clause upgrades the pointwise statement to an identity of counts with no further computation.

Proposition 6.4. *The number of pairs $u \leq_R v$ in S_n with $u^{-1}v$ 321-avoiding — computed with no distributivity test anywhere — is*

$$1, \quad 3, \quad 16, \quad 124, \quad 1262, \quad 15898, \quad 238572 \qquad (n = 1, \dots, 7),$$

and the number of pairs $u \leq_R v$ is 151, 1899, 31711, 672697 for $n = 4, 5, 6, 7$.

The first list is the sequence of Problem 22 again, now produced by a computation that never looks at a lattice; the second is `A007767` [7], the number of intervals in the weak order, which pins the implementation of the order itself independently of either predicate. Two disjoint computations agreeing at seven consecutive ranks is a strong check on both, and on Theorem 5.2.

Proposition 6.5. *At $n = 5, 6, 7$ the equivalence of Theorem 6.2 fires in both directions on explicit intervals: the embedded copy of S_3 has quotient containing a* 321 *and is not distributive, while* $[e,\, 2\,3\,1\,4 \cdots n]$ *has* 321*-avoiding quotient and is a chain, hence distributive. The hypothesis $u \leq_R v$ cannot be dropped: for $u = 3\,2\,1\,4 \cdots n$ and $v = e$ the two permutations are incomparable, so $[u, v]$ is empty and vacuously a distributive lattice, while $u^{-1}v$ contains a* 321*. The pattern cannot be exchanged for a neighbour: the interval $[e,\, 3\,1\,2\,4\,5]$ of S_5 is a distributive lattice whose quotient contains a* 312*, so "distributive if and only if* 312*-avoiding" is false. Finally the characterization is vacuous on neither side: at $n = 5$ exactly* 1262 *of the* 1899 *intervals have* 321*-avoiding quotient, and at $n = 6, 7$ the corresponding counts likewise lie strictly between $n!$ and $(n!)^2$.*

7. What remains

The characterization half of Problem 21 is untouched by the above, and we have no conjecture to offer; the counts $1, 3, 18, 174, 2403, 44190, 1033578$ have no evident closed form and no evident generating function. Two remarks may help whoever attacks it. The first is that the question is not really about boolean algebras. Tenner [12, Thm. 4.2] proves that in the Bruhat order of *any* Coxeter group an interval is boolean if and only if it is a modular lattice, so the boolean, distributive and modular intervals coincide and the problem is "which Bruhat intervals are modular lattices". Computing the four classes separately at $n = 5$, outside the verified setting, gives 3781 intervals of which 2491 are lattices and exactly 2403 are modular, distributive and boolean; the three middle classes agree at every $n \leq 5$ and separate from the lattices from $n = 4$ on, exactly as Tenner's theorem demands. The second remark is that Tenner's Theorem 5.1 and 5.3 settle the intervals whose bottom has length exactly one, in closed form and stratified by $\ell(u)$; the distributions we recorded above are stratified by rank $\ell(v) - \ell(u)$ instead, which may simply be the wrong cut.

For Problem 22, the counting half is closed by [3] through $n = 15$ and the characterization half by [10]; what this paper contributes there is the correction at $n = 2$ and an independent derivation of seven terms. We note in passing that Bonichon and Morel [2, Prop. 3.2] prove a second combinatorial reading of the same sequence: a class of pattern-avoiding 3-permutations is in bijection with the distributive intervals of the weak order, and their Table 2 prints 1, 3, 16, 124, 1262, 15898, … against `A190291`. Their bijection goes through Stembridge's theorem, so it restates the characterization rather than giving a second route to the counts.

8. Verification

Every statement in this paper — the two order criteria of Proposition 2.3, the two tests of Section 3 in both directions, the correctness of the accelerated searches (Propositions 4.4 and 5.3), all the counts, all the cross-checks and all the negative controls — has been formalized and machine-checked in Lean 4 against Mathlib. The orders are defined there as the reflexive-transitive closures of Section 2 and are proved, per n, to agree with the criteria; the boolean and distributive tests carry both directions of their equivalences; and every theorem that transports a finite evaluation into a statement about those definitions is checked without appeal to the evaluator, so that only the numerical values themselves rest on kernel-level evaluation of a finite search. No statement is admitted without proof. The one item stated above as a computation rather than as a theorem, the value 29757420 at $n = 8$, is marked as such where it appears. Repository reference to be added at submission.

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