THE EXACT THREE-PLAYER THRESHOLD FOR MAXIMAL MULTIPLICITY OF FEEDBACK NASH EQUILIBRIA IN THE SYMMETRIC SCALAR DISCOUNTED LINEAR-QUADRATIC GAME

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Abstract. Cavalagli, Bemporad and Zanon (IEEE Control Systems Letters **10** (2026), 919–924) bound the number of hyperbolic feedback Nash equilibria of the symmetric N-player scalar discounted linear-quadratic game $x_{t+1} = ax_t + \sum_i u_{i,t}$ and show that the maximal number $2^N - 2$ is attained when $|a| \geq (N-1)\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$, a condition that is also necessary when N is even; for odd N their proof stops with the remark that obtaining a necessary condition would mean solving their inequality (21), "which cannot be done in closed-form". We settle the smallest odd case, $N = 3$. In the aggregate coordinate $g = (k + \sigma/k)/2$ the relevant auxiliary function becomes a linear term plus a single square root, its minimisation over the negative axis reduces to the cubic $(1-c)w^3 - 3(1+c)w^2 + 4(1+c) = 0$ with $c = \gamma\sigma$, which has exactly one root $\bar{w} \in (1, 2)$ for every $c > 0$, and the threshold is $a^*(\sigma, \gamma) = 4\sqrt{\sigma}/\big((2-\bar{w})\sqrt{1+\bar{w}}\big)$. The maximal-multiplicity criterion of the source at $N = 3$ holds if and only if $a \geq a^*$; equivalently, with the cubic root eliminated, if and only if $X \geq 5c + 1$ and $(X + 2 - 2c)^3 \geq 27(1+c)^2 X$ for $X = \gamma a^2$; and $a^* < 2\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$ strictly for all parameters, so the source's sufficient condition is never necessary at $N = 3$. A root of the auxiliary function is shown to yield stable equilibria under the source's standing assumption, rational parameter points on both sides of the threshold are exhibited, and a missing factor γ in the quadratic the source displays just above its equation (13) is documented (equation (13) itself is unaffected). Every statement is verified in Lean 4 against *Mathlib*.

1. Introduction

Consider N players acting on the scalar linear system

$$x_{t+1} = a\, x_t + \sum_{i=1}^{N} u_{i,t}, \qquad J_i = \sum_{t \geq 0} \gamma^t r_i\big(\sigma_i x_t^2 + u_{i,t}^2\big), \tag{1}$$

each minimising its own discounted cost J_i over linear state feedbacks $u_{i,t} = k_i x_t$, with $a \in \mathbb{R}$, $\sigma_i, r_i > 0$ and a discount factor $\gamma \in (0, 1]$. A vector $k = (k_1, \ldots, k_N)$ is a *feedback Nash equilibrium* (FNE) when no player can lower its cost by changing its own gain, and a *stable* FNE when in addition the closed loop $a_{\text{cl}}(k) = a + \sum_i k_i$ satisfies $|a_{\text{cl}}(k)| < 1$. How many such equilibria a scalar linear-quadratic game has, and which of them are stable, is a question as old as the subject [13, 14]; the textbook account is [12], and [11] is the monograph the source cites for the discounted formulation. For the scalar game the count is sharp: at most $2^N - 1$ in the continuous-time undiscounted game, all of them realised when the open loop is unstable enough [6], with the scalar solution set studied further in [7, 10, 8, 9]; the same bound and the same asymptotic statement in discrete time [3], after the two-player case [4]; a discriminant trichotomy for two players in discrete time [5]; and, in the paper this note is about, Cavalagli, Bemporad and Zanon [1], a count for the *symmetric discounted* game (all $\sigma_i = \sigma$) through a family of auxiliary functions Φ_p, $p = 1, \ldots, N-1$.

Their Theorem 2 states that, under an assumption on the roots of Φ_p (their Assumption 1), the game admits *up to*

$$M = 2\sum_{z=1}^{\hat{p}} \binom{N}{z} - \mathbf{1}_{\{N \text{ even},\ \hat{p} = N/2\}} \binom{N}{N/2} \tag{S20}$$

hyperbolic FNE, where $\hat{p}$ is the largest $p \leq \lfloor N/2 \rfloor$ for which

$$\Phi_p(\bar{k}_p)\ \text{sign}(-a) \geq 0, \qquad \Phi_p'(\bar{k}_p) = 0 \tag{S21}$$

(their (21)), and that the maximal number $2^N - 2$ is reached if

$$|a| \;\geq\; (N-1)\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma} \tag{S22}$$

(their (22)), which is necessary as well when N is even. The proof handles the even case by exhibiting the stationary point $\bar{k}_{N/2} = -\sqrt{\sigma}$ in closed form, and ends, for odd N, with the sentence

> "If N is odd, then $\Phi_{\lfloor N/2\rfloor}(\bar{k}_{\lfloor N/2\rfloor}) < \Phi_{\lfloor N/2\rfloor}(-\sqrt{\sigma})$, so the same condition remains sufficient but is no longer necessary. We remark that in order to obtain a necessary condition for case of N being odd, one should solve inequality (21), which cannot be done in closed-form."

(Statement numbers of [1] throughout are those of the published version, IEEE *Control Systems Letters* **10** (2026), 919–924, which numbers its statements consecutively; the arXiv version 1 numbers them by section, and the correspondence is Definition 2.1 → Definition 1, Proposition 2.2 → Proposition 1, Lemma 3.1 → Lemma 1, Proposition 3.2 → Proposition 2, Theorem 4.1 → Theorem 1, Assumption 4.2 → Assumption 1 and, for the multiplicity theorem, Theorem 4.3 → Theorem 2. All displayed equation numbers agree in the two versions, and so does every passage quoted below. Equations of the source that are displayed here carry the tag S followed by their number in the source, so (S21) is the source's (21); our own equations are numbered plainly.)

The smallest odd case is $N = 3$, where $\hat{p} = \lfloor 3/2\rfloor = 1$ and (S21) is a statement about one function of one variable. This paper solves it.

Results. Write $c = \gamma\sigma$ and, for $a > 0$ (the source treats $a < 0$ by symmetry), read (S21) at $p = 1$, $N = 3$ as "$\Phi_{3,1}$ takes a value ≤ 0 somewhere on the negative axis"; Section 5 explains why this is the same condition.

- **The cubic** (Theorem 4.1). The polynomial $h_c(w) = (1-c)w^3 - 3(1+c)w^2 + 4(1+c)$ has exactly one root $\bar{w} \in (1, 2)$ for every $c > 0$; it is the stationarity equation of the problem in the variable $w = \bar{k}^2/\sigma - 1$.
- **The threshold** (Theorems 5.1 and 5.2). With $a^*(\sigma,\gamma) = 4\sqrt{\sigma}/\big((2-\bar{w})\sqrt{1+\bar{w}}\big)$, the condition holds if and only if $a \geq a^*$. The minimiser sits at $\bar{k}_1 = -\sqrt{\sigma(1+\bar{w})}$, so $\bar{k}_1^2 \in (2\sigma, 3\sigma)$, strictly away from the even-N point $-\sqrt{\sigma}$.
- **The gap** (Theorem 5.3). $a^* < 2\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$ for all $\sigma, \gamma > 0$: at $N = 3$ the source's condition (S22) is sufficient and never necessary, and the interval of a on which it fails while the maximal multiplicity is reached is nonempty for every parameter choice.
- **The polynomial form** (Theorem 6.1). With $X = \gamma a^2$ the condition is $X \geq 5c + 1$ and $(X + 2 - 2c)^3 \geq 27(1+c)^2 X$ — the root $\bar{w}$ eliminated — so it is decidable in exact rational arithmetic at every rational (a, σ, γ).
- **Equilibria and controls** (Propositions 7.1 and 8.1). A root k of $\Phi_{3,1}$ yields $(k, \sigma/k, \sigma/k)$, which satisfies every player's first-order condition and is stable under the source's standing assumption $\sigma > (1-\gamma)^2/(4\gamma^2)$; and at $(\sigma,\gamma) = (1, 1)$ the condition holds at $a = 33/10$ and fails at $a = 16/5$, while at $(1, 4/5)$ it holds at $a = 17/5$ and fails at $a = 33/10$ — against the constants $2 + \sqrt{2}$ and $7/2$ of (S22).
- **A correction** (Proposition 9.1). The quadratic displayed just above equation (13) of [1] omits a factor γ in its middle coefficient; equation (13) itself is correct, and the slip is invisible when $\gamma = 1$.

The mechanism is one change of coordinate. In $g = (k + \sigma/k)/2$ the radical $\sqrt{\gamma^2(k^2+\sigma)^2 + 4\gamma k^2}$ of the source becomes $2\gamma|k|\sqrt{g^2 + 1/\gamma}$, so every $\Phi_{N,p}$ is a linear function of g and $\sigma/|k|$ plus one square root (Lemma 3.1); for $N = 3$, $p = 1$ the object to minimise is $\Psi(g) = 2g - \sqrt{g^2 - \sigma} + \sqrt{g^2 + 1/\gamma}$ on $g \geq \sqrt{\sigma}$, a concave and a convex square root, and both lie on the correct side of their tangents by one line of algebra each (Lemma 3.2). No derivative is taken anywhere in the development; the cubic enters as the point where the tangent to Ψ is horizontal, and the same coordinate gives the source's even-N theorem a two-line proof (Theorem 3.3).

What is new and what is not. The objects are elementary and we are careful about attribution. Everything in Section 2 is the source's: the first-order condition, the factorisation of its Lemma 1, the family Φ_p and the identities of its equation (23), and the even-N half of its Theorem 2 (Theorem 3.3 below, whose statement is theirs and whose proof is ours; the $N = 2$ instance is Theorem 4.3 of the same authors' two-player paper [2]). Already in print, and not claimed here: $2^N - 1$ as the maximal number of equilibria and its attainment for $|a|$ large ([6] in continuous time, [3, Theorem 2] in discrete time — both existential, neither distinguishing parity, and the attainment half of [6] carries the standing hypothesis that the σ_i *differ*, so it does not cover the symmetric game, which its author treats apart, in a remark and without proof); the closed-form uniqueness boundaries of the continuous-time symmetric game, $\sqrt{\sigma}\sqrt{2N-3}$ and $-\sqrt{2N-1}\sqrt{-\sigma}$, in [9, §3.2], which are parity-independent and involve no cubic because the continuous-time branch functions carry a single radical; and the discounted $N = 2$ threshold $\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$ of [2]. What we claim is specific to *discrete* time, where the second radical $\sqrt{g^2 + 1/\gamma}$ turns the stationarity equation into a cubic in k^2: the reduction, the closed form, the exact $N = 3$ condition, the strict gap and the polynomial criterion (Theorems 4.1–6.1). The three-player discrete-time case is on record as attempted: Salizzoni, Ouhamma and Kamgarpour write "We attempted using Macaulay2 to find Gröbner bases for the system three-agent LQ games, but the solver did not seem to converge" [5]. As of 2026-09-07 the source has appeared in *IEEE Control Systems Letters* (its arXiv record is still at version 1; in everything cited or quoted here the two versions differ only in the numbering of the statements), no work citing it is indexed by Crossref or OpenAlex, and a full-text sweep of a local corpus of arXiv e-prints found the phrase "symmetric feedback Nash" in a single paper, on a resource-extraction game with threshold strategies (arXiv:2106.04860), and, among the 129 papers containing "feedback Nash equilibri", four on equilibrium counts of the scalar linear-quadratic game, namely [5, 3, 2, 1], none containing the cubic or a three-player threshold. One caveat we cannot remove: Engwerda's *Automatica* paper [8], whose Section 4 the source cites for the continuous-time symmetric case, is closed access, has no preprint version, and we could not read its Theorem 4.3. Everything around it that we could read points the same way. Its abstract announces "a complete characterization when this game will have no, one or multiple equilibria" and its concluding remarks repeat "either no, one or multiple FNE"; the source cites that very theorem, at the point where it introduces its own symmetric closed form, for the statement that "the symmetric setting always admits a symmetric FNE with a closed-form expression", which is an existence statement and not a count; and the same author's open-access recapitulation of that section, [9, §3.2] ("In Sect. 3, we recall from [12] the conditions…", its [12] being [8]), gives the same branch functions g_j, the closed-form optima $\sqrt{\sigma}\sqrt{2N-3}$ and $-\sqrt{2N-1}\sqrt{-\sigma}$, and a table whose only columns are "0", "1" and "More than 1". We therefore believe the discrete-time threshold is not there — but that is inference from the surrounding text, not inspection of the theorem, and the novelty of Theorems 4.1–6.1 carries the caveat.

Verification. Every theorem, proposition and lemma below that carries a statement in Appendix A is formally verified in Lean 4 against *Mathlib*, with no compiled evaluation and no axiom beyond the three standard ones; Section 11 records the modules and the axiom print. Three things are deliberately *outside* the formal development and are labelled as such where they occur: the symbolic derivation that identifies the cubic as the stationarity equation for general (N, p) (Remark 4.2), the discriminant identity of Remark 6.2, and the exact equilibrium counts of Remark 7.2. Each of these was reproduced for this paper by exact rational computation (a hand-written multivariate polynomial class over $\mathbb{Q}$, pseudo-division, and Sturm sequences), and the text says so at each point.

2. Preliminaries

Throughout, $\sigma > 0$, $\gamma > 0$ and a are real; the source restricts to $\gamma \in (0, 1]$ and $a \neq 0$, and the formal statements carry exactly the hypotheses listed with them. For a player with gain k_i and aggregate $S_{-i} = \sum_{j \neq i} k_j$ of the other gains, the source's Proposition 1 (equation (6)) gives the first-order condition

$$f_i(k_i, k_{-i}) = -\gamma k_i^2 (a + S_{-i}) + \gamma \sigma_i (a + S_{-i}) - k_i\big(\gamma (a + S_{-i})^2 - \gamma \sigma_i - 1\big), \tag{S6}$$

and states that k is a stable FNE if $f_i = 0$ for every i and $|a_{\mathrm{cl}}(k)| < 1$. (The source's Definition 1 defines FNE through the value functions and their proof of Proposition 1 derives (S6) as the vanishing derivative of the value function of player i; we work with (S6) as the source does.) Its Lemma 1 factorises (S6) as

$$f_i = -\gamma k_i\big(S_{-i} - h_i^-(k_i)\big)\big(S_{-i} - h_i^+(k_i)\big), \qquad h_i^{\pm}(k) = \frac{\sigma_i - k(2a+k)}{2k} \pm \frac{\rho_i(k)}{2\gamma k}, \tag{S9}$$

with $\rho_i(k) = \sqrt{\gamma^2(\sigma_i + k^2)^2 + 4\gamma k^2}$ (its equation (9)); its Proposition 2 shows that a stable FNE has $S_{-i} = h_i^-(k_i)$ for every i (equation (10)) and, conversely, that an FNE with $S_{-i} = h_i^-(k_i)$ for some i is stable provided

$$\sigma_i > \frac{(1-\gamma)^2}{4\gamma^2} \tag{S11}$$

(equation (11)), which the source assumes from then on so that every FNE is stable.

In the *symmetric* game, $\sigma_i = \sigma$ for all i, the source considers profiles in which p players use a gain k and $N - p$ use its *hyperbolic partner* σ/k, and defines (its equation (19))

$$\Phi_p(k) = \frac{-2N\gamma\sigma - 2a\gamma k + \gamma(\sigma + k^2)}{2\gamma k} + \frac{p(2\gamma\sigma - 2\gamma k^2) - \rho(k)}{2\gamma k}, \qquad p = 1, \dots, N-1, \tag{S19}$$

so that $\Phi_p(k) = 0$ exactly when the optimality relation $S^* = h^-(k) + k$ of its (15a) holds with $S^* = pk + (N-p)\sigma/k$. We write $\Phi_{N,p}$ when N matters. Its equation (23) records

$$\Phi_{p+1}(k) - \Phi_p(k) = \frac{\sigma}{k} - k, \qquad \Phi_p(\sigma/k) = \Phi_{N-p}(k), \tag{S23}$$

and its Theorem 2 is the count (S20)–(S22) of the introduction, valid under its Assumption 1 ("for every $p \in [N-1]$ the scalar equation $\Phi_p(k) = 0$ admits two admissible roots, each associated with a hyperbolic pair $(k, \sigma/k)$", with the two branches coinciding when N is even and $p = N/2$). The source also notes that the change of variables $\tilde a = a\sqrt{\gamma}$, $\tilde\sigma_i = \gamma\sigma_i$, $\tilde k_i = k_i\sqrt{\gamma}$ turns the discounted game into an undiscounted one; accordingly all conditions below depend on (a, σ, γ) only through $X = \gamma a^2$ and $c = \gamma\sigma$ once a is measured in units of $\sqrt{\sigma}$.

The formal development transcribes these objects literally. In Appendix A, `foc a σ γ ki Smi` is (S6) with $\sigma_i = \sigma$, `rad σ γ k` is $\rho(k)$, `hMinus` and `hPlus` are $h^{\mp}$, `PhiPaper n p a σ γ k` is (S19) verbatim and `Phi` the same function with its numerator collected (the two are proved equal, lemma `PhiPaper_eq`, untagged); N and p are *real* parameters there, which is what lets Theorem 3.3 be stated without a parity hypothesis.

Proposition 2.1 (the source's identities, formalised)**.** *Let $\gamma > 0$ and $k \neq 0$.*

(i) *If $\rho(k)^2 = \gamma^2(k^2+\sigma)^2 + 4\gamma k^2$, then $f(k,S) = -\gamma k\,(S - h^-(k))\,(S - h^+(k))$ for every S (Lemma 1 of* [1]*).*
(ii) *$\Phi_{N,p+1}(k) - \Phi_{N,p}(k) = \sigma/k - k$ (first identity of* (S23)*).*
(iii) *If $\sigma > 0$, $\Phi_{N,p}(\sigma/k) = \Phi_{N,N-p}(k)$ (second identity of* (S23)*).*
(iv) *$\Phi_{N,p}(k) = h^-(k) + k - \big(pk + (N-p)\sigma/k\big)$ (the derivation of* (S19) *from (15a)).*

Proof. All four are rational identities after clearing $2\gamma k$ (formal: `foc_factor`, `Phi_step`, `Phi_hyper`, `Phi_eq_hMinus`). In (i) the hypothesis on $\rho(k)^2$ is stated explicitly rather than derived, so that the identity is a statement about the formal square root only through its square; it holds for the actual ρ since the radicand is nonnegative. In (iii) the radical obeys $\rho(\sigma/k) = (\sigma/k^2)\rho(k)$. Nothing here is new. ☐

3. The aggregate coordinate

For $k = -t < 0$ put

$$g(t) = \frac{t^2 + \sigma}{2t} = \frac{1}{2}\Big(t + \frac{\sigma}{t}\Big), \qquad t > 0, \tag{2}$$

so that $g(t) \geq \sqrt{\sigma}$ by the arithmetic–geometric mean inequality, with equality at $t = \sqrt{\sigma}$, and $g(t)^2 - \sigma = \big((t^2 - \sigma)/(2t)\big)^2$. The whole analysis rests on the observation that the source's radical is a function of

g: $\rho(t) = 2\gamma t\sqrt{g(t)^2 + 1/\gamma}$ for $t > 0$, an identity whose square is a polynomial identity (formal: `rad_eq_gg`).

Lemma 3.1 (Φ in the aggregate coordinate)**.** *For $\sigma, \gamma, t > 0$ and all real N, p, a,*

$$\Phi_{N,p}(-t) = (2p-1)\,g(t) + (N-2p)\,\frac{\sigma}{t} + \sqrt{g(t)^2 + 1/\gamma} - a.$$

Proof. Substitute $\rho(-t) = \rho(t) = 2\gamma t\sqrt{g(t)^2 + 1/\gamma}$ into (S19) and clear denominators (formal: `Phi_neg`). This substitution is not in the source; it is a one-line change of variable, and it is the step that makes the rest elementary. ☐

Two cases of Lemma 3.1 are used. For $p = N/2$ the σ/t term vanishes and $\Phi_{N,N/2}(-t) + a = (N-1)g + \sqrt{g^2 + 1/\gamma}$ is increasing in $g \geq \sqrt{\sigma}$; that is Theorem 3.3. For $N = 3$, $p = 1$, since $\sigma/t = g - \sqrt{g^2 - \sigma}$ when $t \geq \sqrt{\sigma}$ and $\sigma/t = g + \sqrt{g^2 - \sigma} \geq g - \sqrt{g^2 - \sigma}$ when $0 < t < \sqrt{\sigma}$, minimising $\Phi_{3,1}(-t)$ over $t > 0$ is minimising

$$\Psi(g) = 2g - \sqrt{g^2 - \sigma} + \sqrt{g^2 + 1/\gamma}, \qquad g \geq \sqrt{\sigma}, \tag{3}$$

and every $g \geq \sqrt{\sigma}$ is realised by $t = g + \sqrt{g^2 - \sigma}$. We also introduce, as an *expression* and never as a derivative,

$$\Psi'(g) := 2 - \frac{g}{\sqrt{g^2 - \sigma}} + \frac{g}{\sqrt{g^2 + 1/\gamma}}. \tag{4}$$

Lemma 3.2 (tangent lines)**.** *Let $\sigma, \gamma > 0$.*

(i) *For $g \geq \sqrt{\sigma}$ and $g_0 > \sqrt{\sigma}$, $\Psi(g) \;\geq\; \Psi(g_0) + \Psi'(g_0)\,(g - g_0)$.*
(ii) *For every $t > 0$, $\Psi(g(t)) \leq \Phi_{3,1}(-t) + a$.*
(iii) *For every $g_0 \geq \sqrt{\sigma}$, $\Phi_{3,1}\big(-(g_0 + \sqrt{g_0^2 - \sigma})\big) = \Psi(g_0) - a$.*

Proof. (i) is the sum of two one-line inequalities, proved in the development as untagged lemmas: $x \mapsto \sqrt{x^2 + c}$ lies above its tangents because $xx_0 + c \leq \sqrt{x_0^2 + c}\sqrt{x^2 + c}$ (Cauchy–Schwarz; formal: `sqrt_add_tangent`), and $x \mapsto \sqrt{x^2 - \sigma}$ lies below its tangents for $x, x_0 \geq \sqrt{\sigma}$ because $(xx_0 - \sigma)^2 - (x^2 - \sigma)(x_0^2 - \sigma) = \sigma(x - x_0)^2 \geq 0$ (formal: `sqrt_sub_tangent`); dividing by $\sqrt{x_0^2 + c}$, resp. $\sqrt{x_0^2 - \sigma}$, and adding $2g$ gives (i) with Ψ' as in (4) (formal: `Psi_tangent`). (ii) is Lemma 3.1 with $\sigma/t \geq g(t) - \sqrt{g(t)^2 - \sigma}$ (formal: `Psi_le_Phi`); (iii) is Lemma 3.1 at $t = g_0 + \sqrt{g_0^2 - \sigma}$, where $g(t) = g_0$ and $\sigma/t = g_0 - \sqrt{g_0^2 - \sigma}$ (formal: `Phi_eq_Psi`). ☐

Theorem 3.3 (the even case; Theorem 2 of [1], even half)**.** *Let $\sigma, \gamma > 0$ and let $n \geq 1$ be real.*

(i) *There is $k < 0$ with $\Phi_{n,n/2}(k) \leq 0$ if and only if $(n-1)\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma} \leq a$.*
(ii) *For all real n, c, v,*

$$\begin{aligned} &c(v+1)^2\Big(\big(1 - \tfrac{n}{2}\big)v + \big(n - \tfrac{n}{2} - 1\big)\Big)\Big(\big(n - \tfrac{n}{2}\big) - \tfrac{n}{2}v\Big) + v\Big(\big(1 - 2\cdot\tfrac{n}{2}\big)v + \big(2n - 2\cdot\tfrac{n}{2} - 1\big)\Big)^2 \\ &\qquad = (v-1)^2\Big(c\,\tfrac{n}{2}\big(\tfrac{n}{2} - 1\big)(v+1)^2 + v(n-1)^2\Big). \end{aligned}$$

Proof. (i) By Lemma 3.1 with $p = n/2$, $\Phi_{n,n/2}(-t) + a = (n-1)g(t) + \sqrt{g(t)^2 + 1/\gamma}$, which is $\geq (n-1)\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$ because $g(t) \geq \sqrt{\sigma}$ and both terms are monotone in g; equality at $t = \sqrt{\sigma}$ gives the converse (formal: `even_threshold`). (ii) is a ring identity (formal: `even_stationarity_factor`). ☐

When $n = N$ is an even integer, (i) is exactly the statement of the source that (S22) is necessary and sufficient for (S21) at $p = N/2$ and $a > 0$; the source obtains the stationary point $\bar{k}_{N/2} = -\sqrt{\sigma}$ by a symbolic computation it does not display ("one can verify symbolically"), while here nothing is differentiated. The identity (ii) is the stationarity polynomial of $\Phi_{N,N/2}$ in $v = k^2/\sigma$ (Remark 4.2) written as $(v-1)^2$ times a factor that is positive for $v > 0$, $c > 0$, $n \geq 2$; so $\bar{k} = \pm\sqrt{\sigma}$ is the *only* stationary point in the even case, which the source asserts and does not prove. Note that (i) needs no parity: it is a statement about the real-parameter family $\Phi_{n,n/2}$. For $n = 2$ it is the two-player threshold $\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$ of [2, Theorem 4.3].

4. The stationarity cubic

Theorem 4.1 (the cubic)**.** *For real c and w let $h_c(w) = (1-c)w^3 - 3(1+c)w^2 + 4(1+c)$.*

(i) $h_c(w) = 0$ *if and only if* $c(w-1)(w+2)^2 = (2-w)^2(w+1)$.
(ii) *If* $h_c(w) = 0$ *then* $(1+c)(2-w)^2(1+w) = 2c\,w^3$.
(iii) *If* $c > 0$ *and* $0 \le w_1 < w_2 \le 2$ *then* $h_c(w_2) < h_c(w_1)$.
(iv) *If* $c > 0$ *there is exactly one* w *with* $1 < w < 2$ *and* $h_c(w) = 0$.

Proof. (i) and (ii) are rearrangements: $(2-w)^2(w+1) - c(w-1)(w+2)^2 = h_c(w)$ (formal: `hcub_eq_zero_iff`, `hcub_prod`). (iii) From $h_c(w_2) - h_c(w_1) = (w_2 - w_1)\big[(1-c)(w_1^2 + w_1w_2 + w_2^2) - 3(1+c)(w_1 + w_2)\big]$ and, on $[0,2]$, $w^2 \le 2w$ and $w_1w_2 \le w_1 + w_2$, the bracket is negative for every $c > 0$, whether $c \le 1$ or $c > 1$ (formal: `hcub_anti`). (iv) $h_c(1) = 2$ and $h_c(2) = -16c$, independently of the shape of the cubic, so the intermediate value theorem gives a root in $(1,2)$ and (iii) gives its uniqueness (formal: `hcub_existsUnique`; this is the only place where continuity is used). □

We write $\bar w = \bar w(c)$ for the root of (iv). At $c = 1$ the cubic degenerates to $-6w^2 + 8$ and $\bar w = 2/\sqrt{3}$; for $c \ne 1$ its discriminant in w is $1728\,c(1+c)^2 > 0$, so it has three distinct real roots. Over the function field $\mathbb{Q}(c)$ the cubic is irreducible — a root in $\mathbb{Q}(c)$ would invert the degree-three map $w \mapsto (2-w)^2(w+1)/\big((w-1)(w+2)^2\big)$ of Theorem 4.1(i) — so by the *casus irreducibilis* there is no expression for $\bar w$ in real radicals of c; the trigonometric form of Cardano's formula applies, but the usable statement is the polynomial criterion of Section 6. This says nothing about individual parameter values, where the cubic may of course factor: at $c = 5/49$, for instance, $\bar w = 3/2$ exactly. (The discriminant value, the irreducibility and the sample factorisation are exact computations outside the formal development.)

Remark 4.2 (where the cubic comes from; outside the formal development)*.* Clearing the radical in the stationarity equation of $t \mapsto \Phi_{N,p}(-t)$, written through Lemma 3.1, gives in $u = t^2$ the polynomial equation

$$J_{N,p}(u) := \gamma(u+\sigma)^2\big[(1-p)u + \sigma(N-p-1)\big]\big[\sigma(N-p) - pu\big] + u\big[(1-2p)u + \sigma(2N-2p-1)\big]^2 = 0, \tag{5}$$

symbolically in N and p: the cleared equation is exactly $-4J_{N,p}$. Its u^4 coefficient is $\gamma p(p-1)$, so the stationarity equation has degree at most four in k^2 for every (N,p), and among $p \ge 1$ the quartic term vanishes exactly at $p = 1$, where the cubic coefficient is $1 - \gamma\sigma(N-2)$ — so the cubic itself degenerates on the single locus $\gamma\sigma = 1/(N-2)$, which for $N = 3$ is $c = 1$. With $u = \sigma v$, $c = \gamma\sigma$ and $v = 1 + w$ one finds $J_{3,1} = \sigma^3 h_c(w)$, which is how Theorem 4.1 was found; and for $p = N/2$, $J_{N,N/2}/\sigma^3$ is the left side of Theorem 3.3(ii). For $N = 5$, $p = 2$ the polynomial is the genuine quartic $2cw^4 + (5c+9)w^3 - 3(1+c)w^2 - 8(1+c)w + 4(1+c)$ in $w = v - 1$. These identities were verified by exact polynomial arithmetic for this paper; only the two instances named — $p = 1, N = 3$ through Theorem 5.1, and $p = N/2$ as Theorem 3.3(ii) — are in the formal development, which never differentiates and needs the cubic only as the point where Lemma 3.2(i) has a horizontal tangent. Since a quartic is solvable by radicals, the source's "cannot be done in closed-form" is not right if "closed form" means "by radicals" — the source itself solves a cubic that way, closing the proof of its Theorem 1 with "The expressions in (16) are obtained through Cardano's procedures for the solution of cubic expressions" — but it is fair if "closed form" means a short explicit expression, since already at $N = 3$ the answer needs the root of a cubic and, by the remark above, a trigonometric one.

5. The threshold

For $\sigma > 0$ and $1 < w < 2$ put

$$g^*(\sigma, w) = \frac{\sqrt{\sigma}\,(2+w)}{2\sqrt{1+w}}, \qquad a^*(\sigma, w) = \frac{4\sqrt{\sigma}}{(2-w)\sqrt{1+w}}. \tag{6}$$

When $w = \bar w(\gamma\sigma)$ we write simply g^* and $a^* = a^*(\sigma, \gamma)$.

Theorem 5.1 (the minimiser and the value)**.** *Let $\sigma > 0$ and $1 < w < 2$.*

(i) $\sqrt{\sigma} < g^*(\sigma, w)$.

(ii) *If moreover* $\gamma > 0$ *and* $h_{\gamma\sigma}(w) = 0$*, then* $\Psi'(g^*(\sigma, w)) = 0$ *and* $\Psi(g^*(\sigma, w)) = a^*(\sigma, w)$*.*

Proof. (i) is $2\sqrt{1+w} < 2+w$ (formal: `sqrt_lt_gstar`). For (ii), the first radical collapses identically, $g^{*2} - \sigma = \sigma w^2/(4(1+w))$; the second collapses *at a root* of the cubic, where Theorem 4.1(i) gives $1/\gamma = \sigma(w-1)(w+2)^2/\big((2-w)^2(w+1)\big)$ and hence

$$\sqrt{g^{*2} - \sigma} = \frac{\sqrt{\sigma}\, w}{2\sqrt{1+w}}, \qquad \sqrt{g^{*2} + 1/\gamma} = \frac{\sqrt{\sigma}\,(2+w)\,w}{(2-w)\,2\sqrt{1+w}}$$

(formal: `sqrt_gstar_sub`, `sqrt_gstar_add`, untagged). Then $\Psi'(g^*) = 2 - \frac{2+w}{w} + \frac{2-w}{w} = 0$ and

$$\Psi(g^*) = \frac{\sqrt{\sigma}}{2\sqrt{1+w}}\Big[2(2+w) - w + \frac{(2+w)w}{2-w}\Big] = \frac{\sqrt{\sigma}}{2\sqrt{1+w}} \cdot \frac{8}{2-w} = a^*(\sigma, w)$$

(formal: `slope_gstar`, `Psi_gstar`). □

Because $t = g^* + \sqrt{g^{*2} - \sigma} = \sqrt{\sigma}\sqrt{1+w}$ (a one-line paper step: $(2+w)+w = 2(1+w)$), the minimiser in the original variable is $\bar{k}_1 = -\sqrt{\sigma(1+\bar{w})}$, with $\bar{k}_1^2 \in (2\sigma, 3\sigma)$ by Theorem 4.1(iv); the source's even-N point $-\sqrt{\sigma}$ is not the odd-N one, which is what its sentence "$\Phi_{\lfloor N/2\rfloor}(\bar{k}_{\lfloor N/2\rfloor}) < \Phi_{\lfloor N/2\rfloor}(-\sqrt{\sigma})$" asserts qualitatively.

Theorem 5.2 (the exact $N = 3$ condition)**.** *Let* $\sigma, \gamma > 0$*, let* $1 < w < 2$ *satisfy* $h_{\gamma\sigma}(w) = 0$*, and let* $a \in \mathbb{R}$*. Then*

$$\big(\exists\, k < 0 :\ \Phi_{3,1}(k) \le 0\big) \quad\iff\quad a^*(\sigma, w) \le a.$$

Proof. ($\Rightarrow$) Let $k = -t < 0$ with $\Phi_{3,1}(-t) \le 0$. By Lemma 3.2(ii), $\Psi(g(t)) \le a$; by Lemma 3.2(i) at $g_0 = g^*$, which is $> \sqrt{\sigma}$ by Theorem 5.1(i), and Theorem 5.1(ii), $\Psi(g(t)) \ge \Psi(g^*) + 0 \cdot (g(t) - g^*) = a^*$. ($\Leftarrow$) At $t = g^* + \sqrt{g^{*2} - \sigma}$, Lemma 3.2(iii) gives $\Phi_{3,1}(-t) = \Psi(g^*) - a = a^* - a \le 0$ (formal: `three_threshold`). □

The left side is the source's condition (S21) at $p = \hat{p} = 1$ for $a > 0$, where $\mathrm{sign}(-a) = -1$ and (S21) reads $\Phi_1(\bar{k}_1) \le 0$ at the stationary point $\bar{k}_1$: the proof shows that the minimum of $\Phi_{3,1}$ over the negative axis is attained at $\bar{k}_1 = -\sqrt{\sigma(1+\bar{w})}$, the point where the tangent of Theorem 5.1(ii) is horizontal, so "$\Phi_{3,1}(\bar{k}_1) \le 0$" and "$\Phi_{3,1} \le 0$ somewhere on the negative axis" are the same statement. (That $\bar{k}_1$ is the *only* stationary point on the negative axis is not formalised; it is not needed for the equivalence, since a minimum is attained there.) The theorem carries no hypothesis on the sign of a; for $a \le 0$ both sides are false, since $a^* > 0$. Combined with the source's Theorem 2, which needs its Assumption 1: at $N = 3$ the index $\hat{p}$ of (S20) equals 1 — so that the source's bound M takes its maximal value $2\binom{3}{1} = 6 = 2^3 - 2$, and with the symmetric equilibrium of its Theorem 1 the total is $7 = 2^3 - 1$ — precisely when $a \ge a^*(\sigma, \gamma)$. The source states that bound as "admits up to", so this settles its *criterion* rather than the count itself; the count is checked independently, at rational points, in Remark 7.2.

Theorem 5.3 (the gap)**.** *Let* $\sigma, \gamma > 0$ *and let* $1 < w < 2$ *satisfy* $h_{\gamma\sigma}(w) = 0$*. Then*

$$a^*(\sigma, w) \ <\ 2\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}.$$

Proof. Evaluate the tangent identity of Lemma 3.2(i) at $g = \sqrt{\sigma}$, $g_0 = g^*$, where the slope vanishes: writing $A = \sqrt{g^{*2} - \sigma}$ and $B = \sqrt{g^{*2} + 1/\gamma}$,

$$a^* = \Psi(g^*) = 2\sqrt{\sigma} - \Big(A + \frac{g^*(\sqrt{\sigma} - g^*)}{A}\Big) + \Big(B + \frac{g^*(\sqrt{\sigma} - g^*)}{B}\Big) = 2\sqrt{\sigma} - \frac{\sqrt{\sigma}(g^* - \sqrt{\sigma})}{A} + \frac{g^*\sqrt{\sigma} + 1/\gamma}{B},$$

and the last term is at most $\sqrt{\sigma + 1/\gamma}$ by Cauchy–Schwarz while the middle term is strictly positive by Theorem 5.1(i) (formal: `athr_lt_paper`). □

So at $N = 3$ the source's (S22) is sufficient and not necessary, and the failure is quantified: for every $\sigma, \gamma > 0$ the interval $\big[a^*, 2\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}\big)$ is nonempty, and on it the maximal-multiplicity condition holds while (S22) fails. The source states the non-necessity and gives no gap. Numerically (exact bisection of the cubic, outside the formal development): $\bar{w}(1) = 2/\sqrt{3} = 1.1547\ldots$ and $a^*(1, 1) =$

$108^{1/4} = 3.22370979547\ldots$ (the cubic degenerates at $c = 1$) against $2+\sqrt{2} = 3.41421\ldots$; and $\bar{w}(4/5) = 1.18080978\ldots$, $a^*(1, 4/5) = 3.30648334772\ldots$ against $7/2$. In units of $\sqrt{\sigma}$ the threshold depends on (σ, γ) only through $c = \gamma\sigma$: $a^*/\sqrt{\sigma} = 4/\big((2 - \bar{w}(c))\sqrt{1 + \bar{w}(c)}\big)$.

6. The polynomial form

Theorem 6.1 (the criterion with the root eliminated)**.** *Let $\sigma, \gamma, a > 0$ and let $1 < w < 2$ satisfy $h_{\gamma\sigma}(w) = 0$. Put $X = \gamma a^2$ and $c = \gamma\sigma$. Then*

$$\big(\exists\, k < 0 : \ \Phi_{3,1}(k) \le 0\big) \quad \iff \quad 5c + 1 \le X \ \ \text{and} \ \ 27(1 + c)^2 X \le (X + 2 - 2c)^3.$$

Proof. By Theorem 5.2 the left side is $a^* \le a$, i.e. $X^* \le X$ with $X^* = \gamma a^{*2} = 16c/\big((2 - w)^2(1 + w)\big)$. Theorem 4.1(ii) turns this into $X^* = 8(1 + c)/w^3$, and two more consequences of $h_c(w) = 0$,

$$X^* + 2 - 2c = \frac{6(1 + c)}{w} \quad \text{and} \quad 8(1 + c) - (5c + 1)w^3 = 3(1 + c)(2 - w)w^2 > 0,$$

give $F(X^*) = 0$ for $F(X) = (X + 2 - 2c)^3 - 27(1 + c)^2 X$ (because $6^3 = 27 \cdot 8$) and $X^* > 5c + 1$. Finally F is strictly increasing on $[5c + 1, \infty)$: for $5c + 1 \le X < Y$, $F(Y) - F(X) = (Y - X)\big[(Y + 2 - 2c)^2 + (Y + 2 - 2c)(X + 2 - 2c) + (X + 2 - 2c)^2 - 27(1 + c)^2\big]$ and each of the three quadratic terms is $\ge 9(1 + c)^2$ there. Hence $X \ge X^*$ iff $X \ge 5c + 1$ and $F(X) \ge 0$ (formal: `three_threshold_poly`; the monotonicity is `cube_mono` and `cube_mono_strict`, untagged; the identity $\gamma a^{*2} w^3 = 8(1 + c)$, i.e. $a^{*2} = 8(1 + \gamma\sigma)/(\gamma\bar{w}^3)$, is established inside that proof and is not a separate declaration). $\square$

The criterion is a single polynomial inequality plus a linear side condition in the game's own parameters, so it is decidable by exact rational arithmetic at every rational (a, σ, γ); that is how Proposition 8.1 is proved.

Remark 6.2 (the criterion as a discriminant; outside the formal development). The same polynomial arises directly from the source's first-order condition (S6), without Φ. At a fixed closed loop $A = a + \sum_i k_i$ every gain of the symmetric game solves the one quadratic $\gamma A k^2 + (1 - \gamma A^2)k + \gamma\sigma A = 0$ (Proposition 9.1(i)), so a profile has at most two distinct gains and two distinct gains have product σ; for $N = 3$ the profile $(\sigma/y, y, y)$ satisfies (S6) for all three players exactly when

$$E(y) := y - \gamma\,(y + a)\,(2y^2 + ay + \sigma) = 0,$$

a cubic in y, and its discriminant is $\operatorname{disc}_y E = \gamma\big[(X + 2 - 2c)^3 - 27(1 + c)^2 X\big]$, the polynomial of Theorem 6.1. So $F(X) \ge 0$ says that E has three real roots; the side condition $X \ge 5c + 1$ selects the branch on which they are admissible (for small X one has $F > 0$ and three real roots, none of them stable). For $N = 2$ the same computation gives the quadratic $y - \gamma a(y^2 + ay + \sigma)$ with discriminant $(1 - \gamma a^2)^2 - 4\gamma^2 a^2 \sigma$, whose positive zeros are $\sqrt{\sigma + 1/\gamma} \pm \sqrt{\sigma}$; the larger of the two is the threshold $\sqrt{\sigma} + \sqrt{\sigma + 1/\gamma}$ of [2, Theorem 4.3]. Both identities were verified by exact polynomial arithmetic for this paper; neither is formalised.

7. From a root to an equilibrium

The source's Φ_p is an auxiliary function; the objects the game is about are gain vectors satisfying (S6) for every player, together with stability. The following closes that gap for $N = 3$; all of it is the source's own content (its Propositions 1 and 2 and equation (15a)) specialised, and it is recorded because the formal development rests on it.

Proposition 7.1 (the hyperbolic profile)**.** *Let $\sigma, \gamma > 0$.*

- (i) *If $k \ne 0$ and $\Phi_{3,1}(k) = 0$, then $f(k, 2\sigma/k) = 0$ and $f(\sigma/k, k + \sigma/k) = 0$: the profile $(k, \sigma/k, \sigma/k)$ satisfies* (S6) *for the player at k (whose $S_{-i} = 2\sigma/k$) and for each player at σ/k (whose $S_{-i} = k + \sigma/k$).*
- (ii) *For $t > 0$, $a + h^-(-t) + (-t) = \sqrt{g(t)^2 + 1/\gamma} - g(t)$.*
- (iii) *If $\gamma \le 1$, $\sigma > (1 - \gamma)^2/(4\gamma^2)$, $k < 0$ and $\Phi_{3,1}(k) = 0$, then $\big|a + (k + 2\sigma/k)\big| < 1$.*

Proof. (i) By Proposition 2.1(iv), $\Phi_{3,1}(k) = 0$ gives $h^-(k) = 2\sigma/k$, and (S9) vanishes at $S = h^-(k)$; by Proposition 2.1(iii), $\Phi_{3,2}(\sigma/k) = \Phi_{3,1}(k) = 0$, so $h^-(\sigma/k) = k + \sigma/k$ and (S9) vanishes for the partner (formal: `foc_of_root`). (ii) is (S9)'s h^- in the aggregate coordinate (formal: `acl_eq`). (iii) The closed loop of the profile is $a + k + 2\sigma/k = a + h^-(k) + k = \sqrt{g^2 + 1/\gamma} - g$ with $g = g(-k) \geq \sqrt{\sigma}$, which lies in $(0, 1)$ because $1/\gamma < 2g + 1$ under (S11) (formal: `stable_of_root`). This is the direction "$\Leftarrow$" of the source's Proposition 2 made explicit. □

So each root $k < 0$ of $\Phi_{3,1}$ yields, under (S11), the three stable equilibria obtained by permuting $(k, \sigma/k, \sigma/k)$, i.e. $\binom{3}{1} = 3$ of the six; two roots give six. What is *not* formalised, and is the content of the source's Assumption 1 at $N = 3$, is that $\Phi_{3,1}$ has exactly two roots on the negative axis when $a > a^*$ (unimodality of Ψ, which follows from Lemma 3.2(i) and the monotonicity of $g/\sqrt{g^2 - \sigma}$ and $g/\sqrt{g^2 + 1/\gamma}$, plus an intermediate-value argument) and that every non-symmetric stable FNE is a permutation of such a profile.

Remark 7.2 (exact counts at rational points; outside the formal development). The enumeration just described is complete, since a profile has at most two distinct gains (Remark 6.2): the stable FNE of the symmetric three-player game are the stable symmetric profiles (z, z, z), roots of a cubic in z, together with the three permutations of each stable $(\sigma/y, y, y)$ with $y^2 \neq \sigma$, $E(y) = 0$. Isolating the real roots of these cubics by Sturm sequences and deciding $|a_{\mathrm{cl}}| < 1$ exactly on the isolating intervals, using only (S6), we find at $(\sigma, \gamma) = (1, 4/5)$ one symmetric and six hyperbolic stable FNE at $a \in \{357/100,\ 133/40,\ 3.3064834\}$ and one symmetric and no hyperbolic at $a \in \{651/200,\ 3.3064833\}$; at $(1, 1)$ seven at $a = 33/10$ and one at $a = 16/5$; and for $N = 2$ at $(1, 4/5)$ one stable FNE at $a = 999/400$ and three at $a = 1001/400$, the source's even-N threshold $5/2$ reproduced. The switch between 3.3064833 and 3.3064834 agrees with $a^*(1, 4/5) = 3.30648334772\ldots$, and the count $6 = 2^3 - 2$ on the whole range $a \geq a^*$ tested is the source's Theorem 2 at those points without its Assumption 1. This is a computation, not a theorem.

8. Controls

Proposition 8.1 (rational points on both sides)**.** (i) *At $\sigma = \gamma = 1$: there is $k < 0$ with $\Phi_{3,1}(k) \leq 0$ for $a = 33/10$, and $2\sqrt{1} + \sqrt{1 + 1/1} \leq 33/10$ is false.*

(ii) *At $\sigma = \gamma = 1$ and $a = 16/5$ there is no $k < 0$ with $\Phi_{3,1}(k) \leq 0$.*

(iii) *For the root $w \in (1, 2)$ of h_1: $16/5 < a^*(1, w) \leq 33/10$.*

(iv) *At $\sigma = 1$, $\gamma = 4/5$: there is $k < 0$ with $\Phi_{3,1}(k) \leq 0$ for $a = 17/5$, there is none for $a = 33/10$, and $2\sqrt{1} + \sqrt{1 + 1/(4/5)} = 7/2$.*

(v) *$(1 - 1)^2/(4 \cdot 1^2) < 1$ and $(1 - 4/5)^2/(4 \cdot (4/5)^2) < 1$: the standing assumption* (S11) *holds at both parameter points.*

Proof. (i), (ii) and (iv) instantiate Theorem 6.1 and reduce to rational arithmetic — e.g. at $(33/10, 1, 1)$, $X = 1089/100 \geq 6$ and $(1089/100)^3 \geq 108 \cdot 1089/100$; at $(16/5, 1, 1)$, $(256/25)^3 < 108 \cdot 256/25$ — together with $(33/10 - 2)^2 = 169/100 < 2$ and $1 + 5/4 = (3/2)^2$ (formal: `control_undiscounted_holds`, `control_undiscounted_fails`, `control_discounted`). (iii) follows from (i), (ii) and Theorem 5.2 (formal: `athr_bracket_one`); (v) is `norm_num` (formal: `control_assumption_11`). □

These are the negative controls of the paper. Against a too-large claim: the condition does not hold for every $a > 0$ (it fails at $16/5$). Against a too-small claim: it does not require the source's (S22) (it holds at $33/10 < 2 + \sqrt{2}$ when $\gamma = 1$, and at $17/5 < 7/2$ when $\gamma = 4/5$). Both halves of Theorem 5.2 are inhabited, the threshold is bracketed, $16/5 < a^*(1, 1) \leq 33/10$ and $33/10 < a^*(1, 4/5) \leq 17/5$, and by (v) and Proposition 7.1(iii) every root of $\Phi_{3,1}$ at these points is a stable equilibrium.

9. A correction to the source

The line displayed immediately above equation (13) of [1] reads, verbatim, "After few steps one obtains that the squared expression is a quadratic in $S^\star$, i.e., $\gamma k_i^2(S^\star + a) + k_i\big(1 - (S^\star + a)^2\big) + \gamma\sigma_i(S^\star + a) = 0$". Writing $A = S^\star + a$ for the closed loop, the first-order condition (S6) gives the middle coefficient as $1-\gamma A^2$, not $1-A^2$. The line is the same word for word in the arXiv version 1 and in the published version (page 921 of [1]), so the slip stands in the version of record.

Proposition 9.1 (the missing γ). *Let $Q_{\text{printed}}(\sigma,\gamma,A,k)=\gamma k^2A+k(1-A^2)+\gamma\sigma A$ be the displayed quadratic and $Q(\sigma,\gamma,A,k)=\gamma k^2A+k(1-\gamma A^2)+\gamma\sigma A$.*

(i) $f(k,S_{-i})=Q(\sigma,\gamma,\,a+S_{-i}+k,\,k)$ *for all real* a,σ,γ,k,S_{-i}.
(ii) $Q_{\text{printed}}(1,\frac{1}{2},2,1)\neq f(1,0)$ *at* $a=1$, $\sigma=1$, $\gamma=\frac{1}{2}$*: the values are* -1 *and* 1.
(iii) $Q_{\text{printed}}(\sigma,1,A,k)=Q(\sigma,1,A,k)$ *for all* σ,A,k.
(iv) *If* $\gamma\neq 0$, $A\neq 0$, $e^2=1$ *and* $R^2=(\gamma A^2-1)^2-4\gamma^2\sigma A^2$*, then* $Q\big(\sigma,\gamma,A,\,((\gamma A^2-1)+eR)/(2\gamma A)\big)=0$*; in particular this holds with* $R=\sqrt{(\gamma A^2-1)^2-4\gamma^2\sigma A^2}$ *whenever the radicand is nonnegative.*

Proof. (i) and (iii) are ring identities; (ii) is an evaluation; (iv) is a ring identity modulo R^2 and e^2 (formal: `foc_as_quadratic`, `printed_form_ne_foc`, `printed_form_eq_foc_of_undiscounted`, `quadratic_root`, `quadratic_root_sqrt`). In general $Q_{\text{printed}}-Q=k(\gamma-1)A^2$. □

So the displayed line is not the first-order condition when $\gamma<1$ (ii), coincides with it when $\gamma=1$ (iii) — which is why the slip is invisible in the undiscounted literature the source generalises — and, by (iv), the source's equation (13), $k^\star_{i,\pm}(S^\star)=\big(\gamma(S^\star+a)^2-1\pm\sqrt{\Delta^{(i)}}\big)/(2\gamma(S^\star+a))$ with $\Delta^{(i)}=(\gamma(S^\star+a)^2-1)^2-4\gamma^2\sigma_i(S^\star+a)^2$, is correct as printed: its roots solve the corrected quadratic. Nothing downstream in the source, and nothing in this paper, depends on the displayed line.

10. Remarks and open questions

Remark 10.1 ($N=5$). At $N=5$, $\hat p=2$, the stationarity polynomial is the quartic of Remark 4.2, so a closed form by radicals exists. Two things do not carry over from $N=3$: the root selection — the localisation $\bar w\in(1,2)$ came from $h_c(1)=2$, $h_c(2)=-16c$, which is special to the cubic — and the collapse of both radicals at the root, which at $N=3$ telescopes to $8/(2-w)$. Neither is attempted here.

Remark 10.2 (general odd N). Equation (5) has degree at most four in k^2 for every (N,p), so for every odd N the condition (S21) at $\hat p=\lfloor N/2\rfloor$ is solvable by radicals; a statement quantified over N and p, and the count (S20) itself, are not formalised here — the development states Φ with real N and p and proves the two instances it needs.

Remark 10.3 (exactly two roots). Theorem 5.2 settles the source's condition (S21); the step from "there is a root of $\Phi_{3,1}$" to "there are exactly two, hence exactly six hyperbolic equilibria" needs the unimodality of Ψ and the converse enumeration of Section 7, which is the source's Assumption 1 territory. Remark 7.2 confirms the count at rational points by exact computation, not by proof.

11. Verification

Proposition 2.1, Lemmas 3.1 and 3.2, Theorems 3.3, 4.1, 5.1, 5.2, 5.3, 6.1 and Propositions 7.1, 8.1, 9.1 rest on declarations formally verified in Lean 4 against *Mathlib* [15, 16] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. Where a printed statement says more than its Lean counterpart the text names the extra step: the identification of the left side of Theorem 5.2 with the source's (S21) (a reading of the source, Section 5), the consequence "only stationary point" drawn from Theorem 3.3(ii), the numerical values of $\bar w$ and a^*, and the three labelled remarks. The development is seven modules, 1206 lines, 14 definitions, 38 lemmas and 19 theorems, of which 33 declarations are the ones cited above: `Basic` (the source's objects, Proposition 2.1, Lemma 3.1), `Bound` (Ψ, Lemma 3.2, Theorem 3.3(i)), `Cubic` (Theorem 4.1, Theorem 3.3(ii)), `Threshold` (Theorems 5.1–6.1), `Fne` (Proposition 7.1), `Erratum` (Proposition 9.1) and `Controls` (Proposition 8.1). Every proof is checked by the Lean kernel; there is no compiled evaluation (`native_decide`), no `sorry`, no `decide`, no certificate file and no axiom declared by hand; the rational identities of Proposition 8.1 are discharged by `norm_num`, and the only analytic ingredients are `Real.sqrt` and the intermediate value theorem in Theorem 4.1(iv). The axiom print (`#print axioms`) of each of the 33 declarations of Appendix A, re-run for this paper against the compiled modules, is `propext`, `Classical.choice`, `Quot.sound`. As recorded at the time of development, a clean check of the seven modules took 84 s wall-clock (116 s of processor time) with a peak resident set of 7.1 GB, of which about 6.9–7.0 GB is the cost of importing *Mathlib* itself; the exact computations of Remarks 4.2, 6.2 and 7.2 ran in under two seconds. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). All variables are real; √ is `Real.sqrt`, `|·|` the absolute value, and `∃!` unique existence. In the module `Threshold` the variables `σ γ w : ℝ` are section variables, and in `Fne` the variables `a σ γ k : ℝ`; they are implicit arguments of the declarations shown from those modules.

Definitions.

```
def foc (a σ γ ki Smi : ℝ) : ℝ :=
  -γ * ki ^ 2 * (a + Smi) + γ * σ * (a + Smi) - ki * (γ * (a + Smi) ^ 2 - γ * σ - 1)

noncomputable def rad (σ γ k : ℝ) : ℝ := √(γ ^ 2 * (k ^ 2 + σ) ^ 2 + 4 * γ * k ^ 2)

noncomputable def hMinus (a σ γ k : ℝ) : ℝ :=
  (σ - k * (2 * a + k)) / (2 * k) - rad σ γ k / (2 * γ * k)

noncomputable def hPlus (a σ γ k : ℝ) : ℝ :=
  (σ - k * (2 * a + k)) / (2 * k) + rad σ γ k / (2 * γ * k)

noncomputable def PhiPaper (n p a σ γ k : ℝ) : ℝ :=
  (-2 * n * γ * σ - 2 * a * γ * k + γ * (σ + k ^ 2)) / (2 * γ * k)
    + (p * (2 * γ * σ - 2 * γ * k ^ 2) - rad σ γ k) / (2 * γ * k)

noncomputable def Phi (n p a σ γ k : ℝ) : ℝ :=
  (γ * σ * (1 + 2 * p - 2 * n) + γ * k ^ 2 * (1 - 2 * p) - 2 * a * γ * k - rad σ γ k)
    / (2 * γ * k)

noncomputable def gg (σ k : ℝ) : ℝ := (k ^ 2 + σ) / (2 * k)

noncomputable def Psi (σ γ g : ℝ) : ℝ := 2 * g - √(g ^ 2 - σ) + √(g ^ 2 + 1 / γ)

noncomputable def slope (σ γ g : ℝ) : ℝ :=
  2 - g / √(g ^ 2 - σ) + g / √(g ^ 2 + 1 / γ)

def hcub (c w : ℝ) : ℝ := (1 - c) * w ^ 3 - 3 * (1 + c) * w ^ 2 + 4 * (1 + c)

noncomputable def gstar (σ w : ℝ) : ℝ := √σ * (2 + w) / (2 * √(1 + w))

noncomputable def athr (σ w : ℝ) : ℝ := 4 * √σ / ((2 - w) * √(1 + w))

def printedQuadratic (σ γ Scl k : ℝ) : ℝ := γ * k ^ 2 * Scl + k * (1 - Scl ^ 2) + γ * σ * Scl

def correctQuadratic (σ γ Scl k : ℝ) : ℝ := γ * k ^ 2 * Scl + k * (1 - γ * Scl ^ 2) + γ * σ * Scl
```

The source's identities and the aggregate coordinate (Proposition 2.1, Lemma 3.1).

```
lemma foc_factor {a σ γ k S : ℝ} (hγ : 0 < γ) (hk : k ≠ 0)
    (hrad : rad σ γ k ^ 2 = γ ^ 2 * (k ^ 2 + σ) ^ 2 + 4 * γ * k ^ 2) :
    foc a σ γ k S = -γ * k * (S - hMinus a σ γ k) * (S - hPlus a σ γ k)

lemma Phi_step {n p a σ γ k : ℝ} (hγ : 0 < γ) (hk : k ≠ 0) :
    Phi n (p + 1) a σ γ k - Phi n p a σ γ k = σ / k - k
```

```
lemma Phi_hyper {n p a σ γ k : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (hk : k ≠ 0) :
    Phi n p a σ γ (σ / k) = Phi n (n - p) a σ γ k

lemma Phi_eq_hMinus {n p a σ γ k : ℝ} (hγ : 0 < γ) (hk : k ≠ 0) :
    Phi n p a σ γ k = hMinus a σ γ k + k - (p * k + (n - p) * (σ / k))

lemma Phi_neg {n p a σ γ t : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (ht : 0 < t) :
    Phi n p a σ γ (-t)
      = (2 * p - 1) * gg σ t + (n - 2 * p) * (σ / t) + √(gg σ t ^ 2 + 1 / γ) - a
```

Tangent lines and the even case (Lemma 3.2, Theorem 3.3).

```
lemma Psi_tangent {σ γ g g₀ : ℝ} (hσ : 0 < σ) (hγ : 0 < γ)
    (hg : √σ ≤ g) (hg₀ : √σ < g₀) :
    Psi σ γ g₀ + slope σ γ g₀ * (g - g₀) ≤ Psi σ γ g

lemma Psi_le_Phi {a σ γ t : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (ht : 0 < t) :
    Psi σ γ (gg σ t) ≤ Phi 3 1 a σ γ (-t) + a

lemma Phi_eq_Psi {a σ γ g₀ : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (hg₀ : √σ ≤ g₀) :
    Phi 3 1 a σ γ (-(g₀ + √(g₀ ^ 2 - σ))) = Psi σ γ g₀ - a

theorem even_threshold {n a σ γ : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (hn : 1 ≤ n) :
    (∃ k : ℝ, k < 0 ∧ Phi n (n / 2) a σ γ k ≤ 0) ↔ (n - 1) * √σ + √(σ + 1 / γ) ≤ a

lemma even_stationarity_factor (n c v : ℝ) :
    c * (v + 1) ^ 2 * ((1 - n / 2) * v + (n - n / 2 - 1)) * ((n - n / 2) - (n / 2) * v)
        + v * ((1 - 2 * (n / 2)) * v + (2 * n - 2 * (n / 2) - 1)) ^ 2
      = (v - 1) ^ 2 * (c * (n / 2) * (n / 2 - 1) * (v + 1) ^ 2 + v * (n - 1) ^ 2)
```

The cubic (Theorem 4.1).

```
lemma hcub_eq_zero_iff (c w : ℝ) :
    hcub c w = 0 ↔ c * (w - 1) * (w + 2) ^ 2 = (2 - w) ^ 2 * (w + 1)

lemma hcub_prod (c w : ℝ) (h : hcub c w = 0) :
    (1 + c) * ((2 - w) ^ 2 * (1 + w)) = 2 * c * w ^ 3

lemma hcub_anti {c w₁ w₂ : ℝ} (hc : 0 < c) (h0 : 0 ≤ w₁) (h : w₁ < w₂) (h2 : w₂ ≤ 2) :
    hcub c w₂ < hcub c w₁

theorem hcub_existsUnique {c : ℝ} (hc : 0 < c) :
    ∃! w : ℝ, 1 < w ∧ w < 2 ∧ hcub c w = 0
```

The threshold (Theorems 5.1, 5.2, 5.3, 6.1).

```
lemma sqrt_lt_gstar (hσ : 0 < σ) (hw1 : 1 < w) (hw2 : w < 2) : √σ < gstar σ w

theorem slope_gstar (hσ : 0 < σ) (hγ : 0 < γ) (hw1 : 1 < w) (hw2 : w < 2)
    (hroot : hcub (γ * σ) w = 0) : slope σ γ (gstar σ w) = 0

theorem Psi_gstar (hσ : 0 < σ) (hγ : 0 < γ) (hw1 : 1 < w) (hw2 : w < 2)
    (hroot : hcub (γ * σ) w = 0) : Psi σ γ (gstar σ w) = athr σ w

theorem three_threshold {a : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (hw1 : 1 < w) (hw2 : w < 2)
    (hroot : hcub (γ * σ) w = 0) :
    (∃ k : ℝ, k < 0 ∧ Phi 3 1 a σ γ k ≤ 0) ↔ athr σ w ≤ a

theorem athr_lt_paper (hσ : 0 < σ) (hγ : 0 < γ) (hw1 : 1 < w) (hw2 : w < 2)
```

```
    (hroot : hcub (γ * σ) w = 0) : athr σ w < 2 * √σ + √(σ + 1 / γ)

theorem three_threshold_poly {a : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (ha : 0 < a)
    (hw1 : 1 < w) (hw2 : w < 2) (hroot : hcub (γ * σ) w = 0) :
    (∃ k : ℝ, k < 0 ∧ Phi 3 1 a σ γ k ≤ 0)
      ↔ (5 * (γ * σ) + 1 ≤ γ * a ^ 2 ∧
          27 * (1 + γ * σ) ^ 2 * (γ * a ^ 2) ≤ (γ * a ^ 2 + 2 - 2 * (γ * σ)) ^ 3)
```

The hyperbolic profile (Proposition 7.1).

```
theorem foc_of_root (hσ : 0 < σ) (hγ : 0 < γ) (hk : k ≠ 0) (hroot : Phi 3 1 a σ γ k = 0) :
    foc a σ γ k (2 * (σ / k)) = 0 ∧ foc a σ γ (σ / k) (k + σ / k) = 0

lemma acl_eq {t : ℝ} (hσ : 0 < σ) (hγ : 0 < γ) (ht : 0 < t) :
    a + hMinus a σ γ (-t) + (-t) = √(gg σ t ^ 2 + 1 / γ) - gg σ t

theorem stable_of_root (hσ : 0 < σ) (hγ : 0 < γ) (hγ1 : γ ≤ 1)
    (hstab : (1 - γ) ^ 2 / (4 * γ ^ 2) < σ) (hk : k < 0) (hroot : Phi 3 1 a σ γ k = 0) :
    |a + (k + 2 * (σ / k))| < 1
```

Controls (Proposition 8.1).

```
theorem control_undiscounted_holds :
    (∃ k : ℝ, k < 0 ∧ Phi 3 1 (33/10) 1 1 k ≤ 0) ∧ ¬ (2 * √1 + √(1 + 1 / 1) ≤ (33/10 : ℝ))

theorem control_undiscounted_fails :
    ¬ ∃ k : ℝ, k < 0 ∧ Phi 3 1 (16/5) 1 1 k ≤ 0

theorem athr_bracket_one {w : ℝ} (hw1 : 1 < w) (hw2 : w < 2) (hroot : hcub ((1:ℝ) * 1) w = 0) :
    16/5 < athr 1 w ∧ athr 1 w ≤ 33/10

theorem control_discounted :
    (∃ k : ℝ, k < 0 ∧ Phi 3 1 (17/5) 1 (4/5) k ≤ 0)
      ∧ (¬ ∃ k : ℝ, k < 0 ∧ Phi 3 1 (33/10) 1 (4/5) k ≤ 0)
      ∧ 2 * √1 + √(1 + 1 / (4/5 : ℝ)) = 7/2

theorem control_assumption_11 :
    ((1 - (1:ℝ)) ^ 2 / (4 * (1:ℝ) ^ 2) < 1) ∧ ((1 - (4/5:ℝ)) ^ 2 / (4 * (4/5:ℝ) ^ 2) < 1)
```

The correction (Proposition 9.1).

```
theorem foc_as_quadratic (a σ γ k Smi : ℝ) :
    foc a σ γ k Smi = correctQuadratic σ γ (a + Smi + k) k

theorem printed_form_ne_foc :
    printedQuadratic 1 (1/2) (1 + 0 + 1) 1 ≠ foc 1 1 (1/2) 1 0

theorem printed_form_eq_foc_of_undiscounted (σ Scl k : ℝ) :
    printedQuadratic σ 1 Scl k = correctQuadratic σ 1 Scl k

theorem quadratic_root {σ γ Scl e R : ℝ} (hγ : γ ≠ 0) (hS : Scl ≠ 0) (he : e ^ 2 = 1)
    (hR : R ^ 2 = (γ * Scl ^ 2 - 1) ^ 2 - 4 * γ ^ 2 * σ * Scl ^ 2) :
    correctQuadratic σ γ Scl (((γ * Scl ^ 2 - 1) + e * R) / (2 * γ * Scl)) = 0

theorem quadratic_root_sqrt {σ γ Scl e : ℝ} (hγ : γ ≠ 0) (hS : Scl ≠ 0) (he : e ^ 2 = 1)
    (hΔ : 0 ≤ (γ * Scl ^ 2 - 1) ^ 2 - 4 * γ ^ 2 * σ * Scl ^ 2) :
    correctQuadratic σ γ Scl
        (((γ * Scl ^ 2 - 1)
          + e * √((γ * Scl ^ 2 - 1) ^ 2 - 4 * γ ^ 2 * σ * Scl ^ 2)) / (2 * γ * Scl)) = 0
```

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