MARKINGS OF A GRAPH THAT REALISE KEI, SYMMETRIC AND CNS-QUANDLES

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ABSTRACT. A *marking* of a finite (di)graph Γ with vertex set V is a map $R\colon V \to \operatorname{Aut}\Gamma$; it makes V into a right quasigroup, and by a theorem of Ta that right quasigroup is a rack exactly when $R_{R_v(w)} = R_v R_w R_v^{-1}$ for all v, w. Ta tabulated the numbers $\mu_{\mathrm{rack}}(\Gamma)$ and $\mu_{\mathrm{qnd}}(\Gamma)$ of markings realising a rack, respectively a quandle, for complete graphs, stars and cycles, and closed his paper by asking for Cayley (di)graphs of racks carrying extra structure, naming symmetric racks. We do not characterise those Cayley graphs; we cross the extra structure named there with the counting problems of the same list, a question we have not found posed anywhere. For a graph Γ we count the markings whose quandle is a kei (μ_{kei}), admits a good involution (μ_{sym}), or is a connected noninvolutory symmetric quandle (μ_{CNS}), and we count the pairs consisting of a marking and a good involution of it (μ_{pair}). We determine all four counts for K_n with $n \le 7$, for the star $K_{1,n-1}$ with $n \le 7$, for the cycle C_n with $n \le 8$ and for the vertex-transitive graphs on six vertices; on the way we obtain $\mu_{\mathrm{qnd}}(K_7) = 152900$, the quandle half of the last undetermined entry of Ta's complete-graph row. Three structural theorems say where the new columns carry no information: $\mu_{\mathrm{kei}} = \mu_{\mathrm{sym}} = \mu_{\mathrm{qnd}}$ and $\mu_{\mathrm{CNS}} = 0$ for every graph all of whose point stabilisers consist of involutions, in particular for every cycle; $\mu_{\mathrm{CNS}} = 0$ for every graph that is not vertex-transitive, for every star, and for every graph on an odd number of vertices. A fourth says where the μ_{CNS} column lives: if some marking of some graph on n vertices realises a CNS-quandle with good involution ρ, then the *same* marking realises a CNS-quandle on the perfect matching M_ρ of ρ, on its complement, on K_n and on the edgeless graph. At $n = 6$ those four graphs are all of them: exactly 32 of the 32768 labelled graphs on six vertices carry a CNS marking, namely the edgeless graph, the complete graph, the 15 perfect matchings and their 15 complements. So the smallest graph other than a complete or an edgeless one with a nonempty CNS column is $3K_2$, and we write a marking of it out. Every theorem and proposition below has been verified with a proof assistant, apart from statements attributed to the literature and two computations marked in the text as external.

1. INTRODUCTION

A *rack* is a finite set X together with a map $x \mapsto s_x$ from X to the symmetric group $\mathrm{Sym}(X)$ satisfying $s_x s_y = s_{s_x(y)} s_x$; it is a *quandle* if $s_x(x) = x$ for every x, and it is *involutory* — a *kei*, when it is a quandle — if $s_x^2 = \mathrm{id}$ for every x. Racks and quandles come from knot theory [2, 1] and are, at bottom, combinatorial: a rack structure on X is one equation on a map $X \to \mathrm{Sym}(X)$.

Ta [11] studies these structures through graphs, continuing a line he credits to Bardakov [11, §1.1.1]. If Γ is a (di)graph with vertex set V, a *marking* of Γ is a map $R\colon V \to \operatorname{Aut}\Gamma$, written $R_v := R(v)$; it realises the right quasigroup $V_R^\Gamma = (V, R)$ with operation $v \triangleleft w = R_w(v)$, and it is a *q-marking* if $R_v(v) = v$ for all v. By [11, Thm. 5.1], V_R^Γ is a rack (respectively a quandle) if and only if R is a magma homomorphism into $\mathrm{Conj}(\operatorname{Aut}\Gamma)$, which unfolds to the single equation

$$R_{R_v(w)} \;=\; R_v\, R_w\, R_v^{-1} \qquad \text{for all } v, w \in V. \tag{1}$$

Writing $\mu_{\mathrm{rack}}(\Gamma)$ and $\mu_{\mathrm{qnd}}(\Gamma)$ for the number of markings realising a rack, respectively of q-markings realising a quandle, [11] computes the pair $(\mu_{\mathrm{rack}}, \mu_{\mathrm{qnd}})$ for complete graphs, stars and cycles; this is its Table 1, reproduced here as Table 1. Five entries are printed there as "?".

The list of open problems closing [11] asks to "Compute μ_{rack} and μ_{qnd} for more families of (di)graphs" and to "Add more entries to Table 1" [11, Probs. 8.1, 8.2]; its last item asks for something else, and it is the one that motivates this paper. Verbatim, with the reference marks omitted [11, Prob. 8.6]:

TABLE 1. Table 1 of [11]: $(\mu_{\text{rack}}, \mu_{\text{qnd}})$ for the complete graph K_n, the star $K_{1,n-1}$ and the cycle C_n, indexed by the number n of vertices.

n	0	1	2	3	4	5	6	7
K_n	$(1,1)$	$(1,1)$	$(2,1)$	$(13,5)$	$(114,36)$	?	?	?
$K_{1,n-1}$	n/a	$(1,1)$	$(2,1)$	$(4,2)$	$(31,13)$	$(390,114)$	?	?
C_n	n/a	n/a	n/a	$(13,5)$	$(32,8)$	$(41,7)$	$(108,13)$	$(113,9)$

> "Define and characterize Cayley (di)graphs of classes of racks equipped with extra structure (e.g., generalized Legendrian racks, multi-virtual quandles, symmetric racks)."

Problem 8.6 asks for a characterisation of Cayley (di)graphs, which this paper does not give. What we take from it is its list of extra structures, which we cross with the counting of Problems 8.1–8.2. The question answered below is ours, not Ta's, and we have not found it posed in the literature.

Symmetric racks and quandles are Kamada's [3]: a *good involution* of a rack (X, s) is an involution $\rho \in \mathrm{Sym}(X)$ with

$$\rho\, s_x = s_x\, \rho \quad \text{and} \quad s_{\rho(x)} = s_x^{-1} \qquad \text{for all } x \in X, \tag{2}$$

and a *symmetric quandle* is a pair (Q, ρ) consisting of a quandle and a good involution of it [12, Def. 3.8]. Good involutions are what make quandle cocycle invariants work for unoriented and non-orientable objects [4], and Ta's other papers [12, 13] count and classify them. A quandle is *connected* if its inner group $\mathrm{Inn}(Q) = \langle s_x : x \in Q \rangle$ acts transitively, and a *CNS-quandle* is a connected noninvolutory quandle equipped with a good involution [12, §1.1.2].

Crossing the two lines of work gives four counts on Ta's own objects. For a (di)graph Γ on n vertices put

$$\begin{aligned}
\mu_{\text{kei}}(\Gamma) &= \#\{R : \ V_R^\Gamma \text{ is a kei}\},\\
\mu_{\text{sym}}(\Gamma) &= \#\{R : \ V_R^\Gamma \text{ admits a good involution}\},\\
\mu_{\text{CNS}}(\Gamma) &= \#\{R : \ V_R^\Gamma \text{ is a CNS-quandle}\},\\
\mu_{\text{pair}}(\Gamma) &= \#\{(R, \rho) : \ \rho \text{ is a good involution of } V_R^\Gamma\},
\end{aligned}$$

all four ranging over q-markings R of Γ realising a quandle. The first three are three new columns beside Ta's; the fourth counts pairs, which is how a symmetric quandle is defined and how Ta's own sequences count [7]. Since a kei has the identity as a good involution and a good involution need not be unique,

$$\mu_{\text{kei}}(\Gamma) \ \le \ \mu_{\text{sym}}(\Gamma) \ \le \ \mu_{\text{qnd}}(\Gamma), \qquad \mu_{\text{sym}}(\Gamma) \ \le \ \mu_{\text{pair}}(\Gamma),$$

and all three inequalities are strict for some Γ (Section 7).

Results. *The columns* (Section 5). We compute all four counts for the complete graph on at most seven vertices (Table 2), for the star on at most seven vertices (Table 4), for the cycle on at most eight vertices (Table 5) and for every vertex-transitive graph on six vertices (Table 3). Each is an exact cardinality of the set of markings that the definitions cut out, not a sample. The first coordinate of every computed tuple is Ta's own μ_{qnd}, so each of these theorems re-derives a published or previously computed value before quoting a new one; in particular we obtain

$$\mu_{\text{qnd}}(K_7) = 152900,$$

the quandle half of the last entry of Table 1 still printed as "?". (The other four "?" cells were determined in the companion manuscript [8]; their μ_{qnd} halves are re-derived here.) The first place where the symmetric column separates from the kei column is $n = 5$: of the 404 labelled quandle structures on five points, 232 are kei and 272 admit a good involution, so a noninvolutory quandle of order 5 carries a good involution — necessarily a disconnected one, since $\mu_{\text{CNS}}(K_5) = 0$.

Where the new columns collapse (Section 3). Call Γ *point-involutory* if every automorphism of Γ fixing a vertex is an involution. Then every quandle marking of Γ is a kei, so $\mu_{\text{kei}}(\Gamma) = \mu_{\text{sym}}(\Gamma) =$

$\mu_{\mathrm{qnd}}(\Gamma)$ and $\mu_{\mathrm{CNS}}(\Gamma) = 0$ (Theorem 3.1). Cycles are point-involutory at every $n \geq 3$ (Theorem 3.3); combined with Ta's closed form $\mu_{\mathrm{qnd}}(C_n) = \sigma(n) + 1$ [11, Prop. 5.7] this determines the whole cycle row of the kei and symmetric columns at every n. If a marking realises a *connected* rack then Γ is vertex-transitive (Theorem 3.5), so μ_{CNS} vanishes on every graph that is not; on a star with $n \geq 3$ no marking at all realises a connected rack. Finally $\mu_{\mathrm{CNS}}(\Gamma) = 0$ for every graph on an odd number of vertices (Theorem 3.7). Two of Ta's three families are therefore exactly the wrong place to look for the phenomenon the μ_{CNS} column measures, and the third is the complete graph, where a CNS marking is nothing but a labelled CNS-quandle structure.

Where it does live (Section 4). A good involution commutes with every translation, and the translations of a marking are automorphisms of the marked graph. Hence the graph M_ρ whose edges are the pairs $\{x, \rho(x)\}$ is preserved by every translation: the same table is a marking of M_ρ. On a CNS-quandle ρ is fixed-point-free, so M_ρ is a perfect matching, and since the marking condition is vacuous for K_n and for the edgeless graph and survives complementation, one CNS-quandle of order n produces four graphs with $\mu_{\mathrm{CNS}} > 0$ at once (Theorem 4.1):

$$\overline{K_n}, \qquad M_\rho \cong \tfrac{n}{2} K_2, \qquad \overline{M_\rho}, \qquad K_n.$$

The order-6 column, exhaustively (Section 6). Every clause in the definition of a CNS marking except adjacency-preservation is a property of the table alone, and adjacency-preservation is automatic for K_n. Hence, for *every* graph Γ on n vertices,

$$\mu_{\mathrm{CNS}}(\Gamma) = \#\{T \text{ a CNS marking of } K_n : \text{ every translation of } T \text{ lies in } \operatorname{Aut}\Gamma\},$$

a filter of one fixed list rather than a fresh search per graph (Theorem 6.1). At $n = 6$ that list has 30 entries, and scanning all 2^{15} labelled graphs on six vertices against it gives the complete answer (Theorem 6.2): exactly 32 of them have $\mu_{\mathrm{CNS}} > 0$, namely the edgeless graph, the complete graph, the 15 perfect matchings and their 15 complements, $1+15+15+1$. So the four graphs that Theorem 4.1 forces are, at order 6, all the graphs there are; the smallest graph other than a complete or edgeless one with a nonempty CNS column is the perfect matching $3K_2$, together with its complement the octahedron $K_{2,2,2}$. A marking realising a CNS-quandle on $3K_2$ is written out in Theorem 6.4, and its good involution *is* the matching it marks.

Relation to other work. The propagating search that makes the counts of Section 5 feasible is the one introduced in the companion manuscript [8], which determines four of the five undetermined entries of Table 1. The classification of CNS-quandles by order is the subject of a second companion [9]: a CNS-quandle of order $k \leq 47$ exists exactly for $k \in \{6, 12, 18, 20, 24, 30, 36, 40, 42\}$. Combined with Theorem 6.1 this says at once that $\mu_{\mathrm{CNS}}(\Gamma) = 0$ for every graph whose number of vertices is at most 47 and not one of those nine numbers. A third companion [10] counts good involutions of linear quandles; the objects here are disjoint from the objects there, because by [13, Thm. 4.2] a linear quandle with a good involution is a kei, whereas the quandles counted by the gap $\mu_{\mathrm{sym}} - \mu_{\mathrm{kei}}$ are noninvolutory. What we use from outside is short: Ta's Theorem 5.1, Kamada's definition (2), the elementary fixed-point-freeness fact of [9] in Theorem 3.7, and Ta's closed form for $\mu_{\mathrm{qnd}}(C_n)$ in Corollary 3.4. The classifications of [12] and [9] enter only as cross-checks and in remarks.

Organisation. Section 2 fixes notation and records the two characterisations that make the counts decidable. Section 3 proves the three vanishing theorems, Section 4 the matching theorem, Section 5 presents the tables and describes the search, Section 6 the order-6 classification and the explicit witness. Section 7 records the controls that show no hypothesis is vacuous and no column is a relabelling of another; Section 8 says what is not settled here, and Section 9 describes the machine verification.

2. Preliminaries

Throughout, Γ is a (di)graph on the vertex set $V = \{0, 1, \ldots, n-1\}$, given by its adjacency function $A\colon V \times V \to \{\mathrm{true}, \mathrm{false}\}$; $\operatorname{Aut}\Gamma$ is the group of permutations π of V with $A(\pi y, \pi z) = A(y, z)$ for all

y, z. The complete graph K_n, the edgeless graph $\overline{K_n}$, the star $K_{1,n-1}$ with centre 0 and the cycle C_n are as usual; $\overline{\Gamma}$ denotes the loopless complement of Γ.

A marking R of Γ is stored as its table of rows: row x is the value list of the permutation R_x, and we write $s_x := R_x$ for the *translations* of the realised right quasigroup. By [11, Thm. 5.1], R realises a rack precisely when (1) holds, equivalently when

$$s_v\big(s_w(z)\big) = s_{s_v(w)}\big(s_v(z)\big) \qquad \text{for all } v, w, z \in V, \tag{3}$$

and a quandle precisely when in addition $s_v(v) = v$ for all v.

Definition 2.1. Let Γ be a (di)graph on n vertices.

(1) $\mu_{\mathrm{kei}}(\Gamma)$ is the number of q-markings R of Γ such that V_R^Γ is a quandle with $s_x^2 = \mathrm{id}$ for all x;
(2) $\mu_{\mathrm{sym}}(\Gamma)$ is the number of q-markings R of Γ such that V_R^Γ is a quandle admitting a good involution;
(3) $\mu_{\mathrm{CNS}}(\Gamma)$ is the number of q-markings R of Γ such that V_R^Γ is a quandle which is connected, not involutory, and admits a good involution.

Definition 2.2. $\mu_{\mathrm{pair}}(\Gamma)$ is the number of pairs (R, ρ) in which R is a q-marking of Γ realising a quandle and ρ is a good involution of V_R^Γ.

Each of the four is defined as the cardinality of a set of markings, and each is finite because a marking is a map $V \to \operatorname{Aut}\Gamma$. A blind sweep over all $|\operatorname{Aut}\Gamma|^n$ of them is out of reach beyond the smallest cases — it is about $8.2 \cdot 10^{25}$ maps for K_7 — so each count is computed instead from an enumeration of the markings satisfying (3), and two of the defining clauses have to be replaced by decidable equivalents before that can be done. Both replacements are proved in both directions.

Lemma 2.3. *Let T be a table of n permutations of V. Then T is a marking of K_n, and it is a marking of $\overline{K_n}$: for these two graphs the adjacency clause holds for every table of permutations, and the two clauses are equivalent. Consequently the q-markings of K_n realising a quandle are exactly the quandle structures on the labelled set V, and*

$$\mu_{\mathrm{kei}}(\overline{K_n}) = \mu_{\mathrm{kei}}(K_n), \qquad \mu_{\mathrm{sym}}(\overline{K_n}) = \mu_{\mathrm{sym}}(K_n), \qquad \mu_{\mathrm{CNS}}(\overline{K_n}) = \mu_{\mathrm{CNS}}(K_n),$$

these three numbers being the numbers of labelled kei, symmetric and CNS-quandle structures on n points.

Proof. A permutation π satisfies $\pi y = \pi z \iff y = z$, which is the adjacency clause for $\overline{K_n}$ and, negated, for K_n. The remaining clauses in the definition of a marking realising a quandle — that the rows are permutations, (3), idempotence — do not mention the graph, and neither do the kei, symmetric and CNS conditions. $\square$

Remark 2.4. All four counts are *labelled* counts. If $Q_1, \ldots, Q_m$ are the isomorphism classes of quandles of order n with a given isomorphism-invariant property, then the number of labelled structures on V with that property is $\sum_i n!/|\operatorname{Aut} Q_i|$, because the labelled structures isomorphic to Q_i form one orbit of $\mathrm{Sym}(V)$ with stabiliser $\operatorname{Aut} Q_i$. Together with Lemma 2.3 this says that the complete-graph column of each of our counts is, in principle, determined by a classification of the relevant quandles together with their automorphism orders; what is new is that nobody had computed it. For a graph other than K_n and $\overline{K_n}$ no such reduction is available, since a marking is then constrained by $\operatorname{Aut}\Gamma$. Read backwards, the same formula turns a computed count into structural information about a classified quandle; see the discussion after Table 2. Note also that the sequences of [7] attached to symmetric quandles count isomorphism classes of *pairs* (Q, ρ) by order, which is a different indexing from every column here.

Proposition 2.5. *Let R be a marking of Γ realising a rack. Then V_R^Γ is connected if and only if the only subset $S \subseteq V$ with $0 \in S$ and $s_x(S) \subseteq S$ for every $x \in V$ is $S = V$ itself.*

Proof. Connectivity of a rack is transitivity of Inn, equivalently: every vertex is reachable from 0 by a finite composite of translations. If that holds and S is closed under all translations with $0 \in S$, an induction on the length of the composite puts every vertex in S. Conversely the set of vertices reachable from 0 is itself closed and contains 0, so by hypothesis it is V. □

Proposition 2.5 turns connectivity into a quantification over the 2^n subsets of V, which is what makes it decidable without computing an orbit closure. For good involutions we use the second clause of (2) as a constraint rather than a test.

Lemma 2.6. *Let R be a marking of Γ realising a rack and ρ a good involution of V_R^Γ. Then for every $x \in V$ the translation at $\rho(x)$ is the inverse permutation of the translation at x: $s_{\rho(x)} = s_x^{-1}$ as permutations of V. Consequently, if ρ is enumerated one value at a time, the value $\rho(x)$ is confined to the set of vertices y with $s_y = s_x^{-1}$.*

Proof. The second clause of (2) is exactly $s_{\rho(x)} = s_x^{-1}$; the point is that it is a condition on the single value $\rho(x)$ and can therefore be imposed while ρ is still partial. □

Proposition 2.7. *Fix a marking R realising a rack. Enumerating the candidate involutions ρ prefix by prefix under the necessary conditions of Lemma 2.6 and of the first clause of* (2)*, and testing each completed candidate against the full definition* (2)*, produces exactly the good involutions of V_R^Γ: no good involution is lost by the pruning, and nothing that fails* (2) *survives the test.*

Proof. Soundness is the final test. For completeness, let ρ be a good involution; each of the conditions imposed on prefixes is a consequence of (2) restricted to the arguments already assigned, so every prefix of ρ satisfies them and ρ survives to the last level. □

The distinction in Proposition 2.7 — pruning conditions used only for completeness, soundness supplied by a full test at the end — is not pedantry; see Remark 9.1.

3. Where the new columns carry no information

3.1. **A point-stabiliser criterion.** Call Γ *point-involutory* if every $\pi \in \operatorname{Aut}\Gamma$ with a fixed vertex satisfies $\pi^2 = \mathrm{id}$.

Theorem 3.1. *Let Γ be point-involutory. Then every q-marking of Γ realising a quandle realises a kei; consequently*

$$\mu_{\mathrm{kei}}(\Gamma) = \mu_{\mathrm{sym}}(\Gamma) = \mu_{\mathrm{qnd}}(\Gamma) \qquad \text{and} \qquad \mu_{\mathrm{CNS}}(\Gamma) = 0.$$

Moreover, for an arbitrary Γ, every kei marking is a symmetric marking, so $\mu_{\mathrm{kei}}(\Gamma) \le \mu_{\mathrm{sym}}(\Gamma) \le \mu_{\mathrm{qnd}}(\Gamma)$.

Proof. Let R be a q-marking realising a quandle. Each row s_x is an automorphism of Γ and fixes x, so $s_x^2 = \mathrm{id}$ by hypothesis and V_R^Γ is a kei; this gives the first equality. For the second, the identity permutation is a good involution of any kei: it commutes with everything, and $s_{\mathrm{id}(x)} = s_x = s_x^{-1}$ because $s_x^2 = \mathrm{id}$. Hence every kei marking is symmetric, which is the last assertion, and under the hypothesis it forces $\mu_{\mathrm{sym}} = \mu_{\mathrm{qnd}}$ as well. Finally a CNS-quandle is by definition not involutory, so no marking of Γ realises one. □

3.2. **Cycles.**

Lemma 3.2. *Let $n \ge 3$ and let f be an automorphism of C_n fixing two adjacent vertices a and $a+1$. Then $f = \mathrm{id}$.*

Proof. Induct along the cycle. If f fixes $a+k$ and $a+k+1$, then $f(a+k+2)$ is adjacent to $f(a+k+1) = a+k+1$, hence is $a+k+2$ or $a+k$; the second is excluded by injectivity, since $f(a+k) = a+k$ and $a+k+2 \ne a+k$ for $n \ge 3$. Indices are read modulo n. □

Theorem 3.3. *For every $n \ge 3$ the cycle C_n is point-involutory. Hence*

$$\mu_{\mathrm{kei}}(C_n) = \mu_{\mathrm{sym}}(C_n) = \mu_{\mathrm{qnd}}(C_n) \qquad \text{and} \qquad \mu_{\mathrm{CNS}}(C_n) = 0$$

for every $n \ge 3$.

Proof. Let $\pi \in \operatorname{Aut} C_n$ fix a vertex x. The two neighbours of x are the only vertices adjacent to it, so π either fixes both or interchanges them; in both cases π^2 fixes x and its successor, so $\pi^2 = \mathrm{id}$ by Lemma 3.2. Now apply Theorem 3.1. □

Corollary 3.4. *With Ta's closed form $\mu_{\mathrm{qnd}}(C_n) = \sigma(n) + 1$, where σ is the sum-of-divisors function* [11, Prop. 5.7]*, we get $\mu_{\mathrm{kei}}(C_n) = \mu_{\mathrm{sym}}(C_n) = \sigma(n) + 1$ for every $n \geq 3$.*

The involutivity behind Theorem 3.3 is close to the surface of [11]: in the proof of Proposition 5.7 there one reads that, for a marking R of C_n realising a quandle, "each right-multiplication map R_v is either the identity map or a reflection. This is because every rotation in D_n has no fixed points" [11, App. A.3], and a reflection is an involution. What that source does not draw is the consequence for the kei, symmetric and CNS columns, which is Corollary 3.4; and Theorem 3.1 is graph-independent, so it also covers graphs whose automorphism group is not dihedral.

Corollary 3.4 is the reason the cycle row of Table 5 is a consistency check rather than data: three of its four columns are pinned by a theorem, and only the pair column μ_{pair} is not.

3.3. **Connectedness forces vertex-transitivity.**

Theorem 3.5. *Let $n \geq 1$ and let R be a marking of Γ realising a* connected *rack. Then Γ is vertex-transitive. Consequently $\mu_{\mathrm{CNS}}(\Gamma) = 0$ for every graph on $n \geq 1$ vertices that is not vertex-transitive.*

Proof. The translations s_x lie in $\operatorname{Aut}\Gamma$, so $\operatorname{Inn}(V_R^\Gamma) \leq \operatorname{Aut}\Gamma$. Connectedness says $\operatorname{Inn}(V_R^\Gamma)$ is transitive on V, hence so is $\operatorname{Aut}\Gamma$. A CNS-quandle is connected by definition, so a graph with $\mu_{\mathrm{CNS}} > 0$ is vertex-transitive. □

Theorem 3.6. *Let $n \geq 3$. No marking of the star $K_{1,n-1}$ realises a connected rack, and therefore $\mu_{\mathrm{CNS}}(K_{1,n-1}) = 0$.*

Proof. Every automorphism of $K_{1,n-1}$ with $n \geq 3$ fixes the centre, since the centre is the unique vertex of degree $n - 1 > 1$. So every translation fixes the centre, and an induction on the length of a composite of translations shows that the only vertex reachable from the centre is the centre itself. As $n \geq 3$ there is another vertex, so no marking realises a connected rack. □

3.4. **Odd order.**

Theorem 3.7. *$\mu_{\mathrm{CNS}}(\Gamma) = 0$ for every graph Γ on an odd number of vertices.*

Proof. A good involution ρ of a connected noninvolutory quandle Q is fixed-point-free [9]. The argument is elementary: ρ commutes with every translation, so it lies in the centralizer of $\operatorname{Inn}(Q)$, which acts transitively; the centralizer of a transitive permutation group is semiregular, so an element of it with a fixed point is the identity, and $\rho = \mathrm{id}$ would give $s_x = s_{\rho(x)} = s_x^{-1}$ for every x, making Q involutory. A fixed-point-free involution partitions its underlying set into two-element orbits, so that set has even cardinality. A marking of Γ realising a CNS-quandle would supply such an involution of V, which is impossible when $|V|$ is odd. □

Theorem 3.7 subsumes, in one line and for every graph at once, the computed zeros $\mu_{\mathrm{CNS}}(K_3) = \mu_{\mathrm{CNS}}(K_5) = \mu_{\mathrm{CNS}}(K_7) = 0$, $\mu_{\mathrm{CNS}}(C_3) = \mu_{\mathrm{CNS}}(C_5) = \mu_{\mathrm{CNS}}(C_7) = 0$ and the odd stars; those cells become checks that the exhaustive search agrees with a theorem. It is not an equivalence: K_4 has even order and $\mu_{\mathrm{CNS}}(K_4) = 0$ (Section 7).

4. Every CNS marking marks a perfect matching

For a permutation ρ of V let M_ρ be the graph on V whose edges are the pairs $\{x, \rho(x)\}$ with $x \neq \rho(x)$.

Theorem 4.1. *Let R be a marking of a graph Γ on n vertices and let ρ be a good involution of the realised rack. Then:*

(1) *R is also a marking of M_ρ;*

(2) *R is also a marking of $\overline{\Gamma}$, of K_n and of $\overline{K_n}$;*
(3) *if moreover R realises a CNS-quandle, then ρ is fixed-point-free, so n is even and M_ρ is a perfect matching $\frac{n}{2}K_2$; and R realises a CNS-quandle on each of*

$$\overline{K_n}, \qquad M_\rho, \qquad \overline{M_\rho}, \qquad K_n,$$

so all four of these graphs have $\mu_{\mathrm{CNS}} > 0$.

Proof. (i) Let $x, y \in V$. Since ρ commutes with s_x we have $s_x(\rho(y)) = \rho(s_x(y))$, and s_x is injective; hence $s_x(z) = \rho(s_x(y))$ holds exactly when $z = \rho(y)$, and s_x preserves the adjacency of M_ρ in both directions. So every row of R lies in $\operatorname{Aut} M_\rho$, which is all that is required of a marking of M_ρ: the two remaining clauses — that the rows are permutations, and (3) — do not mention the graph.

(ii) The adjacency clause is vacuous for K_n and $\overline{K_n}$, because a permutation preserves equality; and it is invariant under passing to the loopless complement, because a permutation preserves adjacency if and only if it preserves non-adjacency between distinct vertices.

(iii) Fixed-point-freeness is the fact quoted in the proof of Theorem 3.7; a fixed-point-free involution has all orbits of size two, so M_ρ is a perfect matching and n is even. Being a CNS-quandle is a property of the table — connectedness, noninvolutivity and the existence of a good involution never mention the graph — so the marking, transported by (i) and (ii) to each of the four graphs, realises a CNS-quandle there as well. $\square$

Theorem 4.1 is the positive half of the question "which graphs carry a CNS marking?", at every order at once, and Section 6 shows that at order 6 it is also the negative half. Part (ii) also explains an agreement visible throughout Table 3: complementary pairs of graphs have the same markings, hence the same five counts. For μ_{rack} and μ_{qnd} this is a special case of Ta's corollary that the two counts depend only on the $\operatorname{Aut}\Gamma$-set structure of V [11, Cor. 5.4].

5. The tables

Every theorem in this section states a tuple whose first coordinate is Ta's own μ_{qnd}; where Table 1 prints that entry, the theorem re-derives the published number from the definitions before quoting a new one. We call this the gate, and it passes at every entry of Table 1 that the computations below reach.

Theorem 5.1. *For $0 \le n \le 5$ the quadruple $(\mu_{\mathrm{qnd}}, \mu_{\mathrm{kei}}, \mu_{\mathrm{sym}}, \mu_{\mathrm{CNS}})(K_n)$ is*

$$(1,1,1,0),\ (1,1,1,0),\ (1,1,1,0),\ (5,5,5,0),\ (36,26,26,0),\ (404,232,272,0).$$

Theorem 5.2. $(\mu_{\mathrm{qnd}}, \mu_{\mathrm{kei}}, \mu_{\mathrm{sym}}, \mu_{\mathrm{CNS}})(K_6) = (6658, 3472, 3922, 30)$*, and the last three of these four numbers are also μ_{kei}, μ_{sym} and μ_{CNS} of the edgeless graph $\overline{K_6}$.*

Theorem 5.3.

$$(\mu_{\mathrm{qnd}}, \mu_{\mathrm{kei}}, \mu_{\mathrm{sym}}, \mu_{\mathrm{pair}}, \mu_{\mathrm{CNS}})(K_7) = (152900,\ 69694,\ 77632,\ 523966,\ 0),$$

and μ_{kei}, μ_{sym}, μ_{CNS} take the values 69694, 77632, 0 *on $\overline{K_7}$ as well.*

The transport to the edgeless graph in Theorems 5.2 and 5.3 is Lemma 2.3, not a second computation.

Two entries of Table 2 deserve comment. The gap $\mu_{\mathrm{sym}}(K_6) - \mu_{\mathrm{kei}}(K_6) = 3922 - 3472 = 450$ is the number of labelled noninvolutory symmetric quandle structures on six points. And $\mu_{\mathrm{CNS}}(K_6) = 30$ is the number of labelled CNS-quandle structures on six points, which cross-checks against a published classification: by [12, Prop. 1.5] the symmetric Galkin quandle of order 6 is, up to isomorphism, the only CNS-quandle of order at most 8, so by Remark 2.4 the 30 labelled copies force its automorphism group to have order $6!/30 = 24$. We have not found that order stated in the literature; it is derived here from the published classification and our count, and is not independently verified. The value $\mu_{\mathrm{CNS}}(K_7) = 0$ agrees with the same proposition, which forbids a CNS-quandle of order 7, and with Theorem 3.7, which forbids one without any computation; the search knows about neither.

TABLE 2. The five counts for the complete graph K_n. The μ_{qnd} row is Ta's; its entries at $n = 5, 6$ were determined in [8] and its entry at $n = 7$ here.

n	0	1	2	3	4	5	6	7
μ_{qnd}	1	1	1	5	36	404	6658	152900
μ_{kei}	1	1	1	5	26	232	3472	69694
μ_{sym}	1	1	1	5	26	272	3922	77632
μ_{pair}	1	1	2	11	74	1002	18202	523966
μ_{CNS}	0	0	0	0	0	0	30	0

Theorem 5.4. *For $3 \le n \le 7$ the quadruple $(\mu_{\mathrm{qnd}}, \mu_{\mathrm{kei}}, \mu_{\mathrm{sym}}, \mu_{\mathrm{CNS}})(K_{1,n-1})$ is*

$$(2,2,2,0),\ (13,11,11,0),\ (114,74,82,0),\ (1708,962,1072,0),\ (36538,17572,19306,0).$$

Proposition 5.5. *For $3 \le n \le 8$ the quadruple $(\mu_{\mathrm{qnd}}, \mu_{\mathrm{kei}}, \mu_{\mathrm{sym}}, \mu_{\mathrm{CNS}})(C_n)$ is*

$$(5,5,5,0),\ (8,8,8,0),\ (7,7,7,0),\ (13,13,13,0),\ (9,9,9,0),\ (16,16,16,0),$$

in particular $\mu_{\mathrm{qnd}}(C_8) = 16 = \sigma(8) + 1$, one step past Table 1.

Proposition 5.5 is not new data: every entry is forced by Theorem 3.3 and Corollary 3.4. It is recorded because it is the check that the theorem and the exhaustive search agree, and because $\sigma(8) + 1 = 16$ tests Ta's closed form one step beyond the printed table.

Theorem 5.6. *The pair counts of the complete graph and of the star are*

$$\mu_{\mathrm{pair}}(K_n) = 1, 1, 2, 11, 74, 1002, 18202 \quad (0 \le n \le 6),$$

$$\mu_{\mathrm{pair}}(K_{1,n-1}) = 6, 35, 348, 5362, 131278 \quad (3 \le n \le 7),$$

each obtained as the fourth coordinate of a quintuple whose other four coordinates are those of Theorem 5.1, Theorem 5.2 and Theorem 5.4.

Theorem 5.7. $\mu_{\mathrm{pair}}(C_n) = 11, 30, 57, 152, 373, 1120$ *for* $3 \le n \le 8$.

The pair column is the only one of the four whose cycle row is not pinned by a theorem: at $n = 8$ the quintuple reads $(16, 16, 16, 1120, 0)$. We do *not* claim that $\mu_{\mathrm{pair}}(C_n)$ fails to be a function of $\sigma(n)$; the six values of $\sigma(n)$ occurring here are pairwise distinct, so this data cannot separate that.

Theorem 5.8. *For the vertex-transitive graphs on six vertices other than K_6 and $\overline{K_6}$, the quadruple $(\mu_{\mathrm{qnd}}, \mu_{\mathrm{kei}}, \mu_{\mathrm{sym}}, \mu_{\mathrm{CNS}})$ is*

$$\begin{aligned} (344, 294, 308, 2) &\quad \textit{for } 3K_2 \textit{ and } K_{2,2,2}, \\ (787, 571, 595, 0) &\quad \textit{for } 2K_3 \textit{ and } K_{3,3}, \\ (13, 13, 13, 0) &\quad \textit{for } C_6 \textit{ and the prism } C_3 \square K_2. \end{aligned}$$

Theorem 5.9. *The pair counts of the same graphs are*

$$\mu_{\mathrm{pair}}(3K_2) = \mu_{\mathrm{pair}}(K_{2,2,2}) = 1860, \qquad \mu_{\mathrm{pair}}(2K_3) = \mu_{\mathrm{pair}}(K_{3,3}) = 3121, \qquad \mu_{\mathrm{pair}}(C_3 \square K_2) = 152.$$

Each is the fourth coordinate of a quintuple whose other four coordinates are those of Theorem 5.8.

With $\mu_{\mathrm{pair}}(C_6) = 152$ from Theorem 5.7 and $\mu_{\mathrm{pair}}(K_6) = 18202$ from Theorem 5.6, this completes Table 3.

TABLE 3. The five counts for the eight vertex-transitive graphs on six vertices. The four complementary pairs agree column by column, as Theorem 4.1(ii) requires. For the three lower pairs both members are covered by separate computations in the first four columns, and in the μ_{pair} column as well except that 152 is computed for C_6 in Theorem 5.7 and for the prism in Theorem 5.9. In the top row the computations are for K_6; the μ_{kei}, μ_{sym} and μ_{CNS} entries of $\overline{K_6}$ are Theorem 5.2, and the other two follow from the marking equivalence of Lemma 2.3, which we do not restate as separate identities.

Γ	μ_{qnd}	μ_{kei}	μ_{sym}	μ_{pair}	μ_{CNS}
$\overline{K_6}$, K_6	6658	3472	3922	18202	30
$3K_2$, $K_{2,2,2}$	344	294	308	1860	2
$2K_3$, $K_{3,3}$	787	571	595	3121	0
C_6, $C_3 \square K_2$	13	13	13	152	0

TABLE 4. The star $K_{1,n-1}$, indexed by the number n of vertices. The μ_{CNS} row is a theorem (Theorem 3.6), valid at every $n \geq 3$.

n	3	4	5	6	7
$\mu_{\text{qnd}}(K_{1,n-1})$	2	13	114	1708	36538
$\mu_{\text{kei}}(K_{1,n-1})$	2	11	74	962	17572
$\mu_{\text{sym}}(K_{1,n-1})$	2	11	82	1072	19306
$\mu_{\text{pair}}(K_{1,n-1})$	6	35	348	5362	131278
$\mu_{\text{CNS}}(K_{1,n-1})$	0	0	0	0	0

TABLE 5. The cycle C_n. The first three rows are $\sigma(n) + 1$ by Corollary 3.4.

n	3	4	5	6	7	8
$\mu_{\text{qnd}} = \mu_{\text{kei}} = \mu_{\text{sym}}$	5	8	7	13	9	16
μ_{pair}	11	30	57	152	373	1120
μ_{CNS}	0	0	0	0	0	0

5.1. **Reading the tables.** Four things in Tables 2–5 seem worth pointing out.

Where the symmetric column starts to say something. On the complete graph the kei and symmetric columns agree for $n \leq 4$ and first separate at $n = 5$: 232 against 272. So there are exactly 40 labelled noninvolutory quandle structures on five points that admit a good involution, and 450 on six points. The 40 are all disconnected, since a connected one would be a CNS-quandle and $\mu_{\text{CNS}}(K_5) = 0$; of the 450 exactly 30 are connected, namely the CNS structures counted by Theorem 5.2. The star row separates at the same place, 74 against 82. This gap is the whole reason the symmetric column is not a renaming of the kei column, and it is what Proposition 7.1(ii) records.

Where the CNS column is nonzero. In every table above it vanishes except at K_6 and $\overline{K_6}$, where it is 30, and at $3K_2$ and $K_{2,2,2}$, where it is 2. Every one of the zeros is also a consequence of a theorem — odd order for $n = 1, 3, 5, 7$ (Theorem 3.7), the star row (Theorem 3.6), the cycle row (Theorem 3.3), and the prism, whose markings are those of its complement C_6 by Theorem 4.1(ii) — except those at K_0, K_2, K_4 and at $2K_3$ and $K_{3,3}$, which are computations. So the exhaustive searches and the structural theorems check each other cell by cell, in both directions.

The pair column is not proportional to the symmetric column. The average number of good involutions carried by a symmetric marking is $18202/3922 \approx 4.6$ for K_6, $1860/308 \approx 6.0$ for $3K_2$ and $152/13 \approx 11.7$ for C_6: three graphs of the same order with three different values. The pair column is therefore an invariant of its own, not a rescaling; Proposition 7.2 makes that precise at the smallest cells.

Complementary pairs. Every complementary pair in Table 3 agrees column by column. For μ_{rack} and μ_{qnd} that is predicted by Ta's corollary on $\operatorname{Aut}\Gamma$-set structure; for the four columns here it is Theorem 4.1(ii). For three of the four pairs both members are computed by separate searches, so there the agreement is also an independent check on the computation.

5.2. **What the computations are.** Every entry above rests on an exhaustive enumeration, and it is worth saying exactly which one. The markings realising a rack are enumerated by the propagating search of [8]: equation (1) is a forcing rule, so once the rows R_v and R_w are known the row $R_{R_v(w)}$ is determined, and the sweep over all $|\operatorname{Aut}\Gamma|^n$ maps becomes a search tree. At K_7 that tree has 1 330 371 nodes and 838 227 600 candidate trials, against about $8.2 \cdot 10^{25}$ maps for the blind sweep. Each enumerated table is then filtered: μ_{kei} by the test $s_x^2 = \mathrm{id}$; μ_{sym} by the enumeration of good involutions of Proposition 2.7; μ_{CNS} by that enumeration together with the connectivity criterion of Proposition 2.5, evaluated over the 2^n subsets of V, and the negation of the kei test. For μ_{pair} we take a deliberately different route: the good involutions of a table are found by filtering *all* involutions of $\mathrm{Sym}(V)$ — 232 of the 5040 elements of $\mathrm{Sym}(V)$ when $n = 7$ — against the definition (2) directly, with no pruning to justify. Since μ_{sym} counts the tables with at least one good involution and μ_{pair} sums their numbers, the two routes share only the test (2), and the agreement of the μ_{sym} coordinate computed both ways is a check on the pruned search of Proposition 2.7.

The largest single computation is Theorem 5.3, which took 1400 seconds and 6.7 gigabytes. Every other cell above is minutes or less. All counts were also reproduced by two independent implementations in C: the μ_{qnd} coordinates by two programs that share no code, and all five counts by one of them, written independently of the verified development. Those programs also return $\mu_{\mathrm{rack}}(K_7) = 1164056$, which we record as computed but not machine-checked, since the enumeration used here reads off μ_{qnd} rather than μ_{rack}.

6. The order-6 column, exhaustively

The theorems of Section 5 give the CNS column for eight graphs on six vertices, and Theorem 3.5 says that only a vertex-transitive graph can contribute to that column at all. Up to isomorphism the vertex-transitive graphs on six vertices are exactly the eight of Table 3 — there are eight of them by [7, A006799] — so the order-6 column is decided by those computations *modulo* that enumeration. The enumeration is quoted rather than proved, and the following reduction makes it unnecessary.

Theorem 6.1. *Let Γ be any graph on n vertices. Then*

$$\mu_{\mathrm{CNS}}(\Gamma) \;=\; \#\{T : \; T \text{ a CNS marking of } K_n \text{ and every row of } T \text{ lies in } \operatorname{Aut}\Gamma\}.$$

Proof. A CNS marking of Γ consists of a table T whose rows are permutations, together with five conditions: the rows preserve the adjacency of Γ; (3); idempotence; non-involutivity; connectedness; and the existence of a good involution. All but the first are conditions on T alone. For $\Gamma = K_n$ the first is automatic, because a permutation preserves equality. So a CNS marking of Γ is exactly a CNS marking of K_n whose rows happen to lie in $\operatorname{Aut}\Gamma$, and the two sets being equal, so are their cardinalities. □

Theorem 6.1 replaces one marking search per graph by one filter of a fixed list: the labelled CNS-quandle structures on n points, of which there are $\mu_{\mathrm{CNS}}(K_n)$. At $n = 6$ that list has 30 entries by Theorem 5.2, so the whole column on six vertices costs 32768×30 membership tests.

Theorem 6.2. *Exactly* 32 *of the* 32768 *labelled graphs on the vertex set* $\{0, 1, \ldots, 5\}$ *satisfy* $\mu_{\mathrm{CNS}} > 0$*, namely*

the edgeless graph, the 15 *perfect matchings, their* 15 *complements, the complete graph,*

that is, $1 + 15 + 15 + 1$. *The scan misses no labelled graph: every symmetric irreflexive adjacency function on* $\{0, \ldots, 5\}$ *is the one encoded by some* $m < 2^{15}$ *in the* 15*-bit incidence encoding used, and every such* m *encodes one.*

So at order 6 the four graphs produced by Theorem 4.1 from a single CNS-quandle are all the graphs there are: the mechanism is not merely sufficient but exhaustive. Up to isomorphism the four classes are $\overline{K_6}$, $3K_2$, $K_{2,2,2}$ and K_6, matching Table 3, and the vanishing of the CNS column on the four other vertex-transitive classes is now a consequence of the scan rather than an input to it.

Remark 6.3. Theorem 6.2 was also obtained outside the verified development by an entirely different route, as a cross-check. The inner group of each of the 30 labelled CNS-quandle structures on six points has exactly two orbits on the 15 unordered pairs of vertices, of sizes 3 and 12; a graph is preserved by all the translations of one such structure exactly when its edge set is a union of that structure's orbits, which gives four graphs per structure, and over the 30 structures those unions are 32 graphs in all — the same 32. That computation is recorded here as a check and is not among the machine-checked statements.

Theorem 6.4. *Let $V = \{0, 1, 2, 3, 4, 5\}$, let Γ be the perfect matching with edges $\{0, 1\}, \{2, 3\}, \{4, 5\}$, and let R be the marking whose translations are the 4-cycles*

$$s_0 = (2\,4\,3\,5),\quad s_1 = (2\,5\,3\,4),\quad s_2 = (0\,5\,1\,4),\quad s_3 = (0\,4\,1\,5),\quad s_4 = (0\,2\,1\,3),\quad s_5 = (0\,3\,1\,2).$$

Then R is a q-marking of Γ and of its complement $K_{2,2,2}$, it realises a connected noninvolutory quandle, and

$$\rho = (0\,1)(2\,3)(4\,5)$$

is a good involution of it. Moreover $M_\rho = \Gamma$: the graph of the good involution is the matching that R marks.

Theorem 6.4 exhibits Theorem 4.1 in the smallest case where it says something new, and the coincidence $M_\rho = \Gamma$ is the mechanism of that theorem rather than an accident of order 6. By [12, Prop. 1.5] the quandle realised here is isomorphic to the symmetric Galkin quandle of order 6, the unique CNS-quandle of order at most 8; Theorem 5.8 says that $3K_2$ carries exactly two CNS markings, of which this is one.

7. Controls

The following are recorded to show that no column is a relabelling of another and no hypothesis above is vacuous.

Proposition 7.1.
(1) *$\mu_{\mathrm{sym}}(K_5) \neq \mu_{\mathrm{qnd}}(K_5)$: not every quandle marking is symmetric — 132 of the 404 quandle structures on five points admit no good involution.*
(2) *$\mu_{\mathrm{kei}}(K_5) \neq \mu_{\mathrm{sym}}(K_5)$: the symmetric column is not a renaming of the kei column.*
(3) *$\mu_{\mathrm{CNS}}(K_6) \notin \{0, 29, 31\}$.*
(4) *K_4 is not point-involutory: the 3-cycle $(1\,2\,3)$ is an automorphism of K_4 fixing 0 and is not an involution. So Theorem 3.1 is not a statement about all graphs; indeed $\mu_{\mathrm{kei}}(K_4) = 26 < 36 = \mu_{\mathrm{qnd}}(K_4)$.*
(5) *The star $K_{1,2}$ is not vertex-transitive, so the hypothesis of Theorem 3.5 is satisfiable; and C_6 is vertex-transitive with $\mu_{\mathrm{CNS}}(C_6) = 0$, so vertex-transitivity is necessary and not sufficient.*
(6) *The tetrahedral quandle is a connected noninvolutory quandle marking of K_4 with no good involution at all. Hence the good-involution clause in the definition of a CNS marking is not implied by the other three. In particular 4 is even and $\mu_{\mathrm{CNS}}(K_4) = 0$, so Theorem 3.7 is not an equivalence.*

Proposition 7.2.
(1) *$\mu_{\mathrm{pair}}(K_4) = 74$ is neither $\mu_{\mathrm{sym}}(K_4) = 26$ nor $\mu_{\mathrm{qnd}}(K_4) = 36$; the pair count is a genuinely different column, and it is not even ordered the same way as μ_{qnd}, since $\mu_{\mathrm{pair}}(K_5) = 1002 > 404 = \mu_{\mathrm{qnd}}(K_5)$.*
(2) *$\mu_{\mathrm{pair}}(K_2) = 2 > 1 = \mu_{\mathrm{sym}}(K_2)$: the unique quandle structure on two points has two good involutions, so a good involution is not unique and the two columns separate at the smallest possible place.*

(3) $\mu_{\text{pair}}(C_5) = 57 \neq 7 = \mu_{\text{sym}}(C_5)$*: the pair column separates on the cycle row, where Theorem 3.3 makes the other three columns coincide.*
(4) $\mu_{\text{pair}}(K_6) \neq 18203$ *and* $\mu_{\text{pair}}(2K_3) = 3121 \neq 595 = \mu_{\text{sym}}(2K_3)$.

8. What is not settled

The complete graph on eight vertices. The measured trial counts of the propagating search are 5 124 240 at K_6 and 838 227 600 at K_7; extrapolating the two measured points puts K_8 at order 10^{11} trials, which is hours in C and tens of hours in the verified setting. We did not run it.

Order 12. By [9] there are three CNS-quandles of order 12 up to isomorphism, so Theorem 4.1 already gives $\mu_{\text{CNS}} > 0$ for the perfect matching $6K_2$ and its complement, and Theorem 6.1 reduces the whole order-12 column to a filter of the labelled CNS structures of that order. What is missing is the count itself: $\mu_{\text{CNS}}(K_{12})$, i.e. the number of labelled CNS-quandle structures on twelve points, which is not reachable by the marking search. Whether the analogue of Theorem 6.2 holds at order 12 — that is, whether the four graphs of Theorem 4.1 are again all of them — is open, and we regard it as the natural next question.

The pair column of the cycle. It is the one column of the cycle row that no theorem pins; values past $n = 8$ are cheap but we have not computed them, and no closed form is proposed.

Digraphs. Every definition above is stated for (di)graphs and every proof uses only that automorphisms preserve the adjacency function, but all computations here are for undirected graphs.

9. Machine verification

Every theorem and proposition in this paper has been verified in Lean 4 against *Mathlib* [5, 6]; the development contains no `sorry`. The exceptions are the statements attributed to the literature, the arithmetic and the transports read off the tables in the surrounding discussion — including the μ_{qnd} and μ_{pair} entries of $\overline{K_6}$ in Table 3, which follow from Lemma 2.3 but are not stated as identities of their own — and the two computations marked in the text as external (the value of $\mu_{\text{rack}}(K_7)$ in Section 5.2 and the orbit computation of Remark 6.3). The repository is private: a repository reference will be added at submission.

The counts of Definitions 2.1 and 2.2 are defined there as cardinalities of the sets of markings that the definitions cut out — never as the output of a program — and a proved identity connects each cardinality with the length of an explicitly filtered list, which is what the evaluation then computes. The two replacements of Section 2 are proved in both directions (Propositions 2.5 and 2.7), so the enumerated list really is the set defined. The structural results — Theorems 3.1, 3.3, 3.5, 3.6, 3.7, 4.1 and 6.1, the encoding statement in Theorem 6.2, and the explicit witness of Theorem 6.4 including its connectivity certificate and its good involution — are checked by the kernel alone. The numerical statements, that is all of Section 5, the two scans behind Theorem 6.2 and the numbered controls, rest in addition on exhaustive computation: each is obtained by evaluating a compiled program on a closed term, so for those the compiler is part of the trusted base, while the definitions and the identities connecting them to the enumerations are not. The finite searches run are exactly the ones described in Section 5.2: for each graph, the propagating enumeration of all markings satisfying (3), then the filters; and, for Theorem 6.2, a scan of all 2^{15} labelled graphs on six vertices against the 30 CNS tables. Because the largest evaluations are far beyond an interpreter, they were run through an import-free mirror of the enumeration code that is compiled to a shared library; the mirror is proved equal to the original definitions, most of it by reflexivity, so nothing about the compiled route is assumed.

Remark 9.1. One methodological observation is worth recording, because it is the reason for the soundness/completeness split of Proposition 2.7. An early prototype of the good-involution search, written in C, pruned with an incremental involutivity test that checked $\rho(\rho(k)) = k$ only when $\rho(k) \le k$; that test accepts maps that are not injective. The prototype agreed with the verified values at every $n \le 5$ and returned $\mu_{\text{sym}}(K_6) = 4002$ and $\mu_{\text{sym}}(K_{1,5}) = 1112$ instead of the correct 3922 and 1072. The wrong program was the faster one and matched on every small case. In the verified development the same pruning cannot be unsound, because the pruning conditions are used only to prove completeness and every completed candidate is tested against the full definition (2).

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[12] L. Ta, *Good involutions of conjugation subquandles*, arXiv:2505.08090v5 (2025). Definition 3.8 (good involution; symmetric rack and quandle), Definition 2.24 (connected) and Proposition 1.5 are numbered as in v5; in v3 and v4 the good-involution definition is Definition 3.7.

[13] L. Ta, *Good involutions of twisted conjugation subquandles and Alexander quandles*, arXiv:2508.16772v1 (2025); Comm. Algebra, published online 30 April 2026, DOI 10.1080/00927872.2026.2643415. Theorem 4.2 is numbered as in v1.