

THE BESSEL CONSTANT BEHIND THE SHARP KHINCHIN INEQUALITY FOR STEINHAUS SUMS AT NEGATIVE MOMENTS:

$\sup_{3 \leq t \leq 12} 3J_0(t)^2 |J_1(t)| < \frac{1}{10}$, PROVED FROM THE POWER SERIES

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ABSTRACT. Rapaport, Tkocz and Wu (arXiv:2512.09077) determine the sharp constant A_p in the Khinchin-type inequality $A_p \|S\|_2 \leq \|S\|_p$ for Steinhaus sums $S = \sum_j z_j \xi_j$ in the last open range $-1 < p < 0$. The technical core of their proof, their Lemma 10, bounds $\int_0^\infty |J_0(t)|^3 t^{p-1} dt$ for the Bessel function J_0 by splitting the integral at $t = 1, 3, 12$, and two steps of that bound are asserted rather than proved: the constant $L = \sup_{3 \leq t \leq 12} \left| \frac{d}{dt} |J_0(t)|^3 \right| = \sup_{3 \leq t \leq 12} 3J_0(t)^2 |J_1(t)|$ is stated to be less than 0.1 on the strength of a numerical check, and on $[1, 3]$ the supremum of $|J_0|$ over a subinterval is stated to be attained at an endpoint on the strength of the shape of $|J_0|$. We prove both statements from the power series of J_0 and J_1 in exact rational arithmetic. The obstacle is that on $[3, 12]$ the series cancels catastrophically, so neither interval evaluation of the truncated series nor a grid with a Lipschitz constant obtained from a bound on $|J_0|, |J_1|$ (which is what one is trying to prove) is available. The classical monotonicity of the energy $J_0^2 + J_1^2$, whose derivative is $-2J_1^2/t$, removes the circularity: one exact enclosure at the single point $t = 3$ bounds $|J_0|$ and $|J_1|$ by 0.428 on all of $[3, 12]$, hence $|(3J_0^2 J_1)'| \leq 0.79$ there, and a 289-point dyadic grid of step $1/32$ closes the argument with $\sup_{[3, 12]} 3J_0^2 |J_1| \leq 0.0805 + 0.79/64 < 0.093$. The same certificate shows that 0.08 is not an upper bound and that the bound fails at $t = 1$, so the interval $[3, 12]$ cannot be widened to $[1, 12]$. Every statement is checked by the kernel of the Lean 4 proof assistant over Mathlib, which contains no Bessel function; the numerical part is discharged by kernel evaluation over $\mathbb{Q}$ with the three standard axioms only. The third numerical step of the same proof, a six-node tangent-line table, is not treated here: it requires rigorous enclosures of Γ and its logarithmic derivative, which we price.

1. INTRODUCTION

1.1. The inequality and its Bessel core. Let $\xi_1, \xi_2, \dots$ be independent Steinhaus random variables, that is, random variables uniform on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and for $z_1, \dots, z_n \in \mathbb{C}$ put $S = \sum_{j=1}^n z_j \xi_j$ and $\|S\|_p = (\mathbb{E}|S|^p)^{1/p}$, which is finite for every $p > -1$. For $p > -1$ let A_p be the best constant in $A_p \|S\|_2 \leq \|S\|_p$ over all such sums, that is $A_p = \inf \|S\|_p$ over $\sum_j |z_j|^2 = 1$. The values of A_p for $p \geq 0$ are known (see the account in [13]; the case $0 < p < 1$ is due to König [6], whose proof has a gap that [13] also repairs), and the range $-1 < p < 0$ was the last one open. Rapaport, Tkocz and Wu [13] close it — in their words, together with König–Kwapień [7], Baernstein–Culverhouse [3] and König [6] this “closes the investigation of sharp $L_p - L_2$ Khinchin-type inequalities for the Steinhaus sums”:

Theorem 1 [13]. With notation (3), for $-1 < p < 0$, we have $A_p = \|(\xi_1 + \xi_2)/\sqrt{2}\|_p$.

Numbering of statements and equations of [13] is that of the e-print compiled from the arXiv source (version 2, whose $\text{\LaTeX}$ source is byte-identical to that of version 1); the page numbers we mention are those of that compilation (22 pages; Theorem 1 on p. 5, Lemma 2 on p. 7, Remark 4 on p. 8, Lemmas 9 and 10 on p. 11, the displays (18)–(21) on p. 14, the table on p. 15).

Writing p for $-p$, so that $0 < p < 1$ throughout, the proof in [13] starts from the Fourier-analytic formula (their equation (6))

$$\mathbb{E} \left| \sum_{j=1}^n z_j \xi_j \right|^{-p} = \kappa_p \int_0^\infty \prod_{j=1}^n J_0(t|z_j|) t^{p-1} dt, \quad \kappa_p = 2^{1-p} \frac{\Gamma(1-p/2)}{\Gamma(p/2)},$$

where J_0 is the Bessel function of the first kind of order zero, and applies Hölder's inequality to the product. With

$$\Psi_p(s) = s^{p/2} \int_0^\infty |J_0(t)|^s t^{p-1} dt \quad (\text{their (9)}),$$

everything reduces to their Lemma 2, $\Psi_p(s) \leq \Psi_p(2)$ for $s \geq 2$, and, with $F_p(s) = \int_0^\infty |J_0(t)|^s t^{p-1} dt$ and $G_p(s) = s^{-p/2} \Psi_p(2)$ (their (15)), Lemma 2 is obtained from three lemmas, of which the third is

Lemma 10 [13]. For every $0 < p < 1$, we have $F_p(3) \leq e^{-p/4} G_p(2)$.

The authors call it “perhaps the most technical part of this paper”; its proof follows the scheme of Chasapis, Singh and Tkocz [4], while the inductive part of the paper follows Nazarov and Podkorytov [12]. The proof of Lemma 10 splits $F_p(3) = A_1 + A_2 + A_3 + A_4$ at the nodes $t_1 = 1$, $t_2 = 3$, $t_3 = 12$, bounds the four pieces by their displays (18)–(21) with subintervals of length $1/m$, $m = 100$, on $(1, 3)$ and on $(3, 12)$, and finishes by comparing the resulting function of p with $e^{-p/4} G_p(2)$ through tangent lines at six nodes.

1.2. Three steps the source verifies numerically. Three steps of that proof are asserted rather than proved. We quote them from the compiled e-print.

First step (the bound (20) on $A_3 = \int_3^{12} |J_0|^3 t^{p-1} dt$, by midpoint Riemann sums with a derivative-controlled error): the error term contains the constant

$$L = \sup_{t \in [t_2, t_3]} \left| \frac{d}{dt} |J_0(t)|^3 \right| < 0.1.$$

Since $\left| \frac{d}{dt} |J_0(t)|^3 \right| = 3|J_0(t)|^2 |J_1(t)|$, this is directly verified numerically (there are concrete precise approximations to J_0 and J_1 , see e.g. 9.4.3 in [1]).

Their reference [1] is the Abramowitz–Stegun handbook [1]. Nothing further is said: no code, no interval arithmetic, no error analysis.

Second step (the bound (19) on $A_2 = \int_1^3 |J_0|^3 t^{p-1} dt$, by a piecewise constant majorant): the authors write that they bound $|J_0|^3$

using $\sup_{[u,v]} |J_0| \leq \max\{|J_0(a)|, |J_0(b)|\}$ [sic] for every $t_1 = 1 \leq a < b \leq 3 = t_2$ (since $|J_0(t)|$ is first decreasing to its first zero occurring at about $t = 2.40..$ and then increasing to its first local maximum occurring at about $t = 3.83..$),

the parenthetical being the whole justification.

Third step (the endgame): with $L(p) = p(B_1(p) + \dots + B_4(p))$ the sum of the four bounds and $R(p) = e^{-p/4} 2^{p/2} \Gamma(1 + p/2) D(p)$, $D(p) = 2^{p/2} \Gamma(1 - p) / \Gamma(1 - p/2)^3$, the authors “use the tangent $\ell_0(p) = R(0) + R'(0)p$ to lower bound $R(p)$, and calculate that $\ell_0(p) - L(p) > 10^{-5}$ ” on $0 < p \leq 0.02$, then on the segments between the nodes $u_1 = 0.02$, $u_2 = 0.06$, $u_3 = 0.15$, $u_4 = 0.3$, $u_5 = 0.7$, $u_6 = 1$ “verify that the differences $d_-(j) = \ell_j(u_j) - L(u_j)$ and $d_+(j) = \ell_j(u_{j+1}) - L(u_{j+1})$ are positive”, the values being collected in their Table 1.

This paper proves the first two steps. The third is discussed, and priced, in Section 6.

1.3. Results. Throughout, J_0 and J_1 are defined by their power series

$$(1) \quad J_0(t) = \sum_{k \geq 0} \frac{(-1)^k (t/2)^{2k}}{(k!)^2}, \quad J_1(t) = \sum_{k \geq 0} \frac{(-1)^k (t/2)^{2k+1}}{k! (k+1)!}, \quad t \in \mathbb{R},$$

the first being, with the factors rearranged, the series printed below equation (6) of [13]. The main result is the first step, in the form to which [13] itself reduces it.

Theorem 1.1 (Theorem 4.1 below). *For every real t with $3 \leq t \leq 12$, $3J_0(t)^2 |J_1(t)| < \frac{1}{10}$.*

The proof produces more. The same certificate gives $3J_0(t)^2 |J_1(t)| \leq 0.093$ for $3 \leq t \leq 12$, while at $t = 103/32$ the value exceeds 0.08; so the constant L of [13] lies in the interval $(0.08, 0.093]$ (Proposition 4.2). At $t = 1$ the value exceeds 0.1; so the bound $L < 0.1$ is genuinely a statement about $[3, 12]$ and would fail on $[1, 12]$ (Proposition 4.3). For the second step we prove $J_1 > 0$ on $[1, 3]$, hence J_0 is decreasing there, hence

$$|J_0(t)| \leq \max\{|J_0(a)|, |J_0(b)|\} \quad \text{for } 1 \leq a \leq b \leq 3 \text{ and } t \in [a, b]$$

(Propositions 5.1 and 5.2). The positivity of J_1 on $[1, 3]$ is classical (its first positive zero is $j_{1,1} = 3.8317\dots$, [1, Section 9.5], OEIS A115369), and the endpoint statement is two lines from it; we record them because [13] proves neither and because the library we verify in contains no Bessel function to state them about.

The inequality of Theorem 4.1 is not new: it is asserted in [13], and its truth is not in doubt, since in floating point $\sup_{[3,12]} 3J_0^2|J_1| \approx 0.0804$. What is new is a proof, and one whose every step, numerical steps included, is checked by a proof kernel (Section 7). A referee of [13] can now discharge the first two numerical steps of Lemma 10 by citation instead of by trust.

1.4. Why a proof needed building. A rigorous bound on a Bessel function over an interval is the kind of thing one expects to find in a library. We did not find one. At the time of writing (September 2026) the Mathlib library of Lean 4 [11] (revision of July 2026) contains no Bessel function of the first kind of any order (its three files mentioning “Bessel” concern Bessel’s inequality and the Bessel potential). The operator set recognised by the CoqInterval tactic [9] (version 4.11.4, February 2026: π , negation, absolute value, reciprocal, square, square root, cos, sin, tan, arctan, exp, log, integer and real powers, and the four arithmetic operations) contains no Bessel function; its `integral` tactic accepts integrands with constant leaves only, so it could enclose J_0 at a fixed point through $J_0(t) = \frac{1}{\pi} \int_0^\pi \cos(t \sin u) du$, but not as a function of t over an interval. The Isabelle Archive of Formal Proofs has, since July 2026, an entry on Bessel functions of the first kind [2], which defines J_a by its series and proves the derivative formulas and the contiguous relations, but states no numerical bound, no zero and no sign of J_0 or J_1 on any interval. The one artifact connecting Bessel functions to a proof assistant that evaluates them is the Lean–Mathematica interface of [8], whose Bessel tactic delegates to Mathematica and produces no kernel proof. The Arb library [5] can enclose J_0 and J_1 rigorously in ball arithmetic, but it is a C library, not a proof, and it states no bound of the kind needed.

Nor is the bound a routine consequence of the series (1) once one has it. On $[3, 12]$ the series cancels badly: at $t = 12$ the terms of the J_0 series reach about 4200 in magnitude while the sum $J_0(12) = 0.0477\dots$ is below 0.05, so interval evaluation of the truncated series over a subinterval of $[3, 12]$ returns widths in the hundreds, and recentring a degree-46 polynomial whose coefficients have denominators of up to sixty digits at each of several hundred midpoints is a proof obligation of its own. The natural alternative, exact evaluation on a grid plus a Lipschitz constant, needs an a-priori bound on $|J_0|$ and $|J_1|$ over $[3, 12]$, which is what one is trying to prove.

The classical energy identity breaks the circle. From $J'_0 = -J_1$ and $J'_1 = J_0 - J_1/t$,

$$\frac{d}{dt}(J_0(t)^2 + J_1(t)^2) = -\frac{2J_1(t)^2}{t} \leq 0 \quad (t > 0),$$

so $J_0^2 + J_1^2$ is non-increasing on $(0, \infty)$, and one exact enclosure at the single point $t = 3$ bounds $|J_0|$ and $|J_1|$ on all of $[3, 12]$. Exact evaluation of a truncated alternating series at a rational point has no cancellation problem at all (the partial sum is an exact rational; only the tail needs a bound), and the energy supplies the missing Lipschitz constant for free. That is the whole method; Sections 2–5 carry it out.

1.5. What is not claimed. The third step of Lemma 10 (the tangent-line table) is not treated; Section 6 explains why and what it would cost. Remark 4 of [13], “Numerical simulations strongly suggest that $s \mapsto \Psi_p(s)$ is in fact strictly decreasing in an open neighbourhood of $s = 2$ ”, is an open question that we do not address. The formal development states the first step in the source’s own equivalent form $3J_0^2|J_1|$; the elementary identity $|\frac{d}{dt}|J_0(t)|^3| = 3J_0(t)^2|J_1(t)|$, which needs the differentiability of $y \mapsto |y|^3$ at $y = 0$ because J_0 vanishes three times inside $[3, 12]$, is not part of it. Finally, the numerical value of the supremum, $0.0804\dots$, is not certified: the certified statement is the enclosure $[0.08, 0.093]$.

2. THE BESSEL FUNCTIONS FROM THEIR SERIES

2.1. Definitions and the classical facts. For $t \in \mathbb{R}$ and $k \geq 0$ write

$$b_0(t, k) = \frac{(-1)^k (t/2)^{2k}}{(k!)^2}, \quad b_1(t, k) = \frac{(-1)^k (t/2)^{2k+1}}{k! (k+1)!},$$

so that $J_0(t) = \sum_k b_0(t, k)$ and $J_1(t) = \sum_k b_1(t, k)$ as in (1). Both series converge absolutely for every real t , by comparison with $\sum_k x^k/k!$ at $x = (t/2)^2$. Let

$$S_0(t) = \sum_{k=0}^{23} b_0(t, k), \quad S_1(t) = \sum_{k=0}^{23} b_1(t, k), \quad \varepsilon = \frac{2 \cdot 6^{48}}{(24!)^2} = 1.1665 \dots \times 10^{-10}$$

be the 24-term truncations and the tail constant. For a rational q the numbers $S_0(q)$, $S_1(q)$, ε are rationals computable exactly.

Proposition 2.1 (classical). (i) *For every real t the series defining $J_0(t)$ and $J_1(t)$ converge.*
(ii) *For $|t| < 13$, $J'_0(t) = -J_1(t)$, and for $0 < |t| < 13$, $J'_1(t) = J_0(t) - J_1(t)/t$.*
(iii) *The energy $E(t) = J_0(t)^2 + J_1(t)^2$ is non-increasing on $[1, 12]$.*
(iv) *For $|t| \leq 12$, $|J_0(t) - S_0(t)| \leq \varepsilon$ and $|J_1(t) - S_1(t)| \leq \varepsilon$.*

All four are textbook facts [14]; the restrictions to $|t| < 13$ and $[1, 12]$ are those of the formal development, which needs no more. We indicate the proofs because the constants of (iv) and the shape of (ii) are used below.

Proof sketch. (i) $|b_0(t, k)| \leq x^k/k!$ and $|b_1(t, k)| \leq |t/2| x^k/k!$ with $x = (t/2)^2$, using $(k!)^2 \geq k!$ and $(k+1)! \geq 1$.

(ii) Term by term. The termwise derivatives are $d_0(t, k) = (-1)^k k (t/2)^{2k-1}/(k!)^2$ and $d_1(t, k) = (-1)^k \frac{2k+1}{2} (t/2)^{2k}/(k!(k+1)!)$, and for $|y| \leq 13$ both are bounded by the single summable majorant $169^k/k!$, using $k \leq k!$ for d_0 and $(2k+1)/2 \leq (k+1)!$ for d_1 . Differentiation under the sum on the open interval $(-13, 13)$ is therefore legitimate, and the exact reindexings $d_0(t, j+1) = -b_1(t, j)$, $d_0(t, 0) = 0$ and $d_1(t, k) = b_0(t, k) - b_1(t, k)/t$ give the two identities.

(iii) By (ii), $E'(t) = 2J_0(-J_1) + 2J_1(J_0 - J_1/t) = -2J_1(t)^2/t \leq 0$ for $0 < t < 13$.

(iv) The exact recurrences

$$b_0(t, m+1) = -b_0(t, m) \frac{(t/2)^2}{(m+1)^2}, \quad b_1(t, m+1) = -b_1(t, m) \frac{(t/2)^2}{(m+1)(m+2)}$$

give, for $|t| \leq 12$ and $m \geq 24$, ratios of consecutive terms at most $36/625 < 1/2$, hence $|b_0(t, k+24)| \leq |b_0(t, 24)| 2^{-k}$ and the tail past index 24 is at most $2|b_0(t, 24)| \leq 2 \cdot 6^{48}/(24!)^2 = \varepsilon$; likewise for b_1 , using

$$\frac{6^{49}}{24! 25!} = \frac{6}{25} \cdot \frac{6^{48}}{(24!)^2} \leq \frac{6^{48}}{(24!)^2}. \quad \square$$

2.2. Enclosures at rational points. For rational q with $|q| \leq 12$, part (iv) gives the two-sided enclosures

$$(2) \quad |S_0(q)| - \varepsilon \leq |J_0(q)| \leq |S_0(q)| + \varepsilon, \quad |S_1(q)| - \varepsilon \leq |J_1(q)| \leq |S_1(q)| + \varepsilon, \quad J_1(q) \geq S_1(q) - \varepsilon,$$

whose outer bounds are exact rationals. Every numerical fact used in this paper is a finite conjunction of inequalities between such rationals, evaluated exactly. We now list them; for size, the numerators of the reduced rationals $S_0(t_i)$, $S_1(t_i)$ over the grid below reach about 130 digits.

Definition 2.2 (the finite computation). Let $t_i = 3 + i/32$ for $0 \leq i \leq 288$ (so $t_0 = 3$, $t_{288} = 12$) and $u_i = 1 + i/32$ for $0 \leq i \leq 64$ (so $u_0 = 1$, $u_{64} = 3$). The certificate consists of the following exact rational inequalities.

- (C1) $3(|S_0(t_i)| + \varepsilon)^2 (|S_1(t_i)| + \varepsilon) \leq \frac{805}{10000}$ for each of the 289 indices i .
- (C2) $S_1(u_i) - \varepsilon \geq \frac{339}{1000}$ for each of the 65 indices i .
- (C3) $(|S_0(3)| + \varepsilon)^2 + (|S_1(3)| + \varepsilon)^2 \leq \frac{183}{1000}$.
- (C4) $(|S_0(1)| + \varepsilon)^2 + (|S_1(1)| + \varepsilon)^2 \leq \frac{78}{100}$.

(C5) For each of the three pairs $(q, c) = (103/32, 8/100)$, $(1, 1/10)$ and $(3, 6/100)$: $\varepsilon \leq |S_0(q)|$, $\varepsilon \leq |S_1(q)|$, and $c < 3(|S_0(q)| - \varepsilon)^2(|S_1(q)| - \varepsilon)$.

All of (C1)–(C5) hold; this is the exhaustive computation on which the results rest, and it is discharged in the manner described in Section 7. For orientation (these values are not part of any statement): the largest left-hand side in (C1) is $0.080428\dots$, at $i = 7$, $t_7 = 103/32$; the smallest left-hand side in (C2) is $0.33905\dots$, at $u_{64} = 3$; the left-hand sides of (C3) and (C4) are $0.18258\dots$ and $0.77917\dots$.

3. THE A-PRIORI BOUND AND THE LIPSCHITZ CONSTANT ON $[3, 12]$

Lemma 3.1. *For $3 \leq t \leq 12$, $|J_0(t)| \leq \frac{428}{1000}$ and $|J_1(t)| \leq \frac{428}{1000}$. For $1 \leq t \leq 12$, $|J_0(t)| \leq \frac{884}{1000}$ and $|J_1(t)| \leq \frac{884}{1000}$.*

Proof. By (2) and (C3), $E(3) = J_0(3)^2 + J_1(3)^2 \leq 0.183$; by Proposition 2.1(iii), $E(t) \leq E(3) \leq 0.183$ for $3 \leq t \leq 12$, and $0.428^2 = 0.183184 \geq 0.183$. The second half is the same with (C4) at $t = 1$: $E(t) \leq E(1) \leq 0.78 \leq 0.884^2 = 0.781456$. $\square$

Let $g(t) = 3J_0(t)^2 J_1(t)$, so that $|g(t)| = 3J_0(t)^2 |J_1(t)|$ is the quantity of Theorem 4.1.

Lemma 3.2. *For $0 < t < 13$, g is differentiable at t with*

$$g'(t) = -6J_0(t)J_1(t)^2 + 3J_0(t)^3 - \frac{3J_0(t)^2 J_1(t)}{t},$$

and for $3 \leq t \leq 12$, $|g'(t)| \leq \frac{79}{100}$.

Proof. The formula is Proposition 2.1(ii) and the product rule. With $\rho = 0.428$ and $t \geq 3$, Lemma 3.1 gives $|g'(t)| \leq 6\rho^3 + 3\rho^3 + 3\rho^3/3 = 10\rho^3 = 0.78402752 \leq 0.79$. $\square$

4. THE SUPREMUM BOUND

Theorem 4.1. *For every real t with $3 \leq t \leq 12$,*

$$3J_0(t)^2 |J_1(t)| < \frac{1}{10}.$$

Consequently the constant $L = \sup_{t \in [3, 12]} \left| \frac{d}{dt} |J_0(t)|^3 \right|$ of [13, proof of Lemma 10] satisfies $L < 0.1$.

Proof. Let $3 \leq t \leq 12$. Since the grid $t_i = 3 + i/32$ has step $1/32$ and covers $[3, 12]$, there is an index $0 \leq i \leq 288$ with $|t - t_i| \leq 1/64$ (take $i = \lfloor 32(t - 3) \rfloor$ or $i + 1$, whichever is nearer). By (2) and (C1),

$$|g(t_i)| = 3|J_0(t_i)|^2 |J_1(t_i)| \leq 3(|S_0(t_i)| + \varepsilon)^2(|S_1(t_i)| + \varepsilon) \leq 0.0805.$$

By Lemma 3.2 and the mean value inequality on the convex set $[3, 12]$, $|g(t) - g(t_i)| \leq 0.79|t - t_i| \leq 0.79/64$. Hence

$$3J_0(t)^2 |J_1(t)| = |g(t)| \leq 0.0805 + \frac{0.79}{64} = 0.09284375 < \frac{1}{10}.$$

For the consequence: on $[3, 12]$ the function $|J_0|^3$ is differentiable with $\left| \frac{d}{dt} |J_0|^3 \right| = 3J_0^2 |J_1|$ (as [13] notes; at a zero of J_0 both sides vanish), so L is the supremum over $[3, 12]$ of a quantity we have just shown to be at most 0.09284375 . This last identity is elementary and, as said in Section 1.5, is the one step here that is not part of the formal development. $\square$

The proof used 0.0805 and 0.79 only through the sum $0.0805 + 0.79/64$, so it also gives a sharper constant.

Proposition 4.2. *For every real t with $3 \leq t \leq 12$, $3J_0(t)^2 |J_1(t)| \leq \frac{93}{1000}$; and there is $t \in [3, 12]$, namely $t = 103/32$, with $3J_0(t)^2 |J_1(t)| > \frac{8}{100}$. Hence*

$$0.08 < \sup_{3 \leq t \leq 12} 3J_0(t)^2 |J_1(t)| \leq 0.093.$$

Proof. The upper bound is the computation in the proof of Theorem 4.1, $0.09284375 \leq 0.093$. For the lower bound, (C5) with $(q, c) = (103/32, 8/100)$ and the lower enclosures in (2) (which are non-negative by the first two clauses of (C5)) give $3J_0(q)^2 |J_1(q)| \geq 3(|S_0(q)| - \varepsilon)^2(|S_1(q)| - \varepsilon) > 0.08$. $\square$

We stress that the value is routine and only the certification is ours: in double precision the supremum is 0.08043..., attained near $t \approx 3.214$, a number any Bessel library returns at once. The enclosure is what a sharpening of Lemma 10 of [13] would cite: replacing 0.1 by 0.093 in the error term $L(t_3 - t_2)t_2^{p-1}/(2m)$ of their (20) shrinks it by seven per cent. A finer grid would certify (0.0804, 0.0805) at a cost linear in the number of points, the grid step entering the final bound only through 0.79δ .

The remaining two evaluations in (C5) are controls: they show that the constant and the interval in Theorem 4.1 are both doing work.

Proposition 4.3. (i) *There is $t \in [3, 12]$ with $3J_0(t)^2|J_1(t)| > \frac{8}{100}$; so 0.08 is not an admissible value of the constant.*

(ii) *$3J_0(1)^2|J_1(1)| > \frac{1}{10}$; so the bound of Theorem 4.1 fails on $[1, 12]$, and the choice $t_2 = 3$ in [13] cannot be replaced by $t_2 = 1$.*

(iii) *$3J_0(3)^2|J_1(3)| > \frac{6}{100}$; so the quantity bounded is not identically zero on $[3, 12]$.*

Proof. Exactly as the lower bound of Proposition 4.2, from the three cases of (C5). $\square$

In double precision (not certified) $3J_0(1)^2|J_1(1)| = 0.773$, $3J_0(3)^2|J_1(3)| = 0.0688$, the bound < 0.1 would still hold on $[2, 12]$ (where the supremum is about 0.0867), and it first fails around $t \approx 1.97$.

5. THE ENDPOINT-MAXIMUM STEP ON $[1, 3]$

Proposition 5.1 (classical). *$J_1(t) > 0$ for $1 \leq t \leq 3$. Consequently J_0 is non-increasing on $[1, 3]$.*

Proof. Same shape as Theorem 4.1, on the grid $u_i = 1 + i/32$. For $1 \leq t \leq 3$ pick $0 \leq i \leq 64$ with $|t - u_i| \leq 1/64$. By (2) and (C2), $J_1(u_i) \geq 0.339$. By Proposition 2.1(ii) and Lemma 3.1, for $1 \leq z \leq 3$, $|J_1'(z)| = |J_0(z) - J_1(z)/z| \leq 0.884 + 0.884 = 1.768$, so by the mean value inequality $J_1(t) \geq 0.339 - 1.768/64 = 0.311375 > 0$. Then $J_0' = -J_1 < 0$ on $[1, 3]$ and J_0 is non-increasing there. $\square$

The formal statement records only the positivity, not the bound 0.311; the minimum of J_1 on $[1, 3]$ is attained at $t = 3$, where $J_1(3) = 0.33905\dots$

Proposition 5.2. *Let $1 \leq a \leq b \leq 3$ and $a \leq t \leq b$. Then $|J_0(t)| \leq \max\{|J_0(a)|, |J_0(b)|\}$. In particular $\sup_{[a,b]} |J_0| = \max\{|J_0(a)|, |J_0(b)|\}$ for $1 \leq a < b \leq 3$, which is the statement used for the bound (19) on A_2 in [13, proof of Lemma 10].*

Proof. By Proposition 5.1, $J_0(b) \leq J_0(t) \leq J_0(a)$. If $J_0(t) \geq 0$ then $|J_0(t)| = J_0(t) \leq J_0(a) \leq |J_0(a)|$; if $J_0(t) \leq 0$ then $|J_0(t)| = -J_0(t) \leq -J_0(b) \leq |J_0(b)|$. $\square$

This is the two-line argument that the parenthetical of [13] quoted in Section 1.2 gestures at; what it needs is only that J_0 is monotone on $[1, 3]$, not the positions of the first zero ($j_{0,1} = 2.4048\dots$) or of the first local maximum of $|J_0|$ beyond it ($j_{1,1} = 3.8317\dots$).

6. THE THIRD STEP, AND SOME OBSERVATIONS

6.1. The tangent-line table. The endgame of Lemma 10 compares $L(p) = p(B_1(p) + \dots + B_4(p))$, a sum over $201 + 900 = 1,101$ values of $|J_0|^3$ at the points $1 + k/100$ ($0 \leq k \leq 200$) and $3 + (k + \frac{1}{2})/100$ ($0 \leq k \leq 899$) times elementary functions of p , with tangent lines of $R(p) = e^{-p/4}2^{p/2}\Gamma(1 + p/2)D(p)$, whose slope involves the digamma function $\psi = \Gamma'/\Gamma$ and Euler's constant γ (indeed $(\log R)'(0) = -\frac{1}{4} + \log 2 + \psi(1)$). The Bessel side of this is affordable by the method of this paper: the measured cost of one exact-rational evaluation of the truncations at a point is about 0.1 s of kernel time, so all of them would take about two minutes. The obstacle is Γ . Mathlib has the real Gamma function but no numerical enclosure of it, has a complex digamma function with no numerical evaluation, and its only explicit numerical bounds on Euler's constant are $\frac{1}{2} < \gamma < \frac{2}{3}$ (the two-sided bounds $H_n - \log(n+1) < \gamma < H_n - \log n$ it also has converge too slowly to yield ten decimals); and the margins are thin: the smallest printed entry of Table 1 is $d_-(1) = 0.00017$, and $R(p) - L(p)$, which vanishes at $p = 0$ where both sides equal 1, is in double precision only about 5.7×10^{-7} at $p = 10^{-4}$ and 5.5×10^{-6} at $p = 10^{-3}$. What is needed is a layer of rational enclosures of Γ on $(0, 2)$, of ψ there, and of γ to about ten

decimals, with proved remainders (for instance Stirling’s series with an explicit remainder for Γ , the series $\psi(x) = -\gamma + \sum_{n \geq 0} (\frac{1}{n+1} - \frac{1}{n+x})$ with a tail bound for ψ , and Euler–Maclaurin for γ). Such a layer is separate work, in progress at the time of writing and not used here; once it exists, the third step, and the two-point tangent check of Lemma 9 of [13] (where the margin at $p = 0$ is 0.400016... against the printed “greater than 0.4”), are each a few point evaluations.

6.2. Observations on the printed numbers. Before any proof was attempted, the numbers printed in [13] were recomputed from the definitions in double precision (J_0 from its series, Γ and ψ from standard library routines), and then again, independently, in 25-digit arithmetic with the `mpmath` library, which agrees with every digit shown. These are reproductions, not certificates, and none of them affects a conclusion of [13]; we record them because they locate the tight spots for the work just described.

- Observation.* (a) The d_- row of Table 1 reproduces: computed 0.0001695, 0.000857, 0.00447, 0.01995, 0.2858 against printed 0.00017, 0.0008, 0.004, 0.02, 0.28. Two entries of the d_+ row do not: computed $d_+(3) = 0.0062$ where 0.06 is printed, and computed $d_+(1) = 0.000158$ where 0.00017 is printed (the other three, 0.000604, 0.00315, 0.7012, reproduce). Both are positive either way, which is all the proof uses.
- (b) The statement that $\ell_0(p) - L(p) > 10^{-5}$ on $0 < p \leq 0.02$ computes to 5.7×10^{-7} at $p = 10^{-4}$ and 2.1×10^{-7} at $p = 0.02$, exceeding 10^{-5} only in the middle of the range (2.9×10^{-5} at $p = 0.01$); it tends to 0 as $p \rightarrow 0$ since $\ell_0(0) = L(0) = 1$. The difference is positive throughout, which is what the argument needs.
- (c) The cross-reference to the table in the text of the proof reads “Table 2.4” while the table’s caption reads “Table 1”.

7. VERIFICATION

Every statement above marked as a theorem, proposition or lemma is checked, in the form stated, by the kernel of the Lean 4 proof assistant [10] (version 4.32.0) over the Mathlib library [11] (revision 81a5d25 of July 2026). The development is three modules of about 1,090 lines: the first defines J_0, J_1 as sums of their series and proves Proposition 2.1; the second states the finite computation of Definition 2.2 over the rationals and discharges it; the third proves Lemmas 3.1 and 3.2 and the results of Sections 4 and 5. The exact statements of the definitions and of the thirteen declarations that carry the results of this paper are reproduced below, verbatim from the source files apart from the omitted documentation strings and attributes; Table 1 gives the correspondence. (The summability of Proposition 2.1(i) is the pair of declarations `summable_b0`, `summable_b1`, on which the rest depends.)

statement	declaration(s)
Proposition 2.1(ii)	<code>hasDerivAt_J0</code> , <code>hasDerivAt_J1</code>
Proposition 2.1(iii)	<code>energy_antitone0n</code>
Proposition 2.1(iv)	<code>abs_J0_sub_S0_le</code> , <code>abs_J1_sub_S1_le</code>
Theorem 4.1	<code>three_J0_sq_abs_J1_lt</code>
Proposition 4.2	<code>three_J0_sq_abs_J1_le_sharp</code> , <code>sup_not_lt_eight_hundredths</code>
Proposition 4.3(i)	<code>sup_not_lt_eight_hundredths</code>
Proposition 4.3(ii)	<code>sup_fails_on_one_twelve</code>
Proposition 4.3(iii)	<code>not_identically_zero</code>
Proposition 5.1	<code>J1_pos_on_one_three</code> , <code>J0_antitone0n_one_three</code>
Proposition 5.2	<code>abs_J0_le_max_endpoints</code>

TABLE 1. Statements of this paper and the declarations that check them.

```

noncomputable def b0 (t : ℝ) (k : ℕ) : ℝ := (-1) ^ k * (t / 2) ^ (2 * k) / (k ! : ℝ) ^ 2
noncomputable def b1 (t : ℝ) (k : ℕ) : ℝ :=
  (-1) ^ k * (t / 2) ^ (2 * k + 1) / ((k ! : ℝ) * ((k + 1)! : ℝ))

```

```

noncomputable def J0 (t : ℝ) : ℝ := ∑' k, b0 t k
noncomputable def J1 (t : ℝ) : ℝ := ∑' k, b1 t k
noncomputable def energy (t : ℝ) : ℝ := J0 t ^ 2 + J1 t ^ 2
noncomputable def S0 (t : ℝ) : ℝ := ∑ k ∈ Finset.range 24, b0 t k
noncomputable def S1 (t : ℝ) : ℝ := ∑ k ∈ Finset.range 24, b1 t k
noncomputable def tailEps : ℝ := 2 * 6 ^ 48 / (Nat.factorial 24 : ℝ) ^ 2

lemma hasDerivAt_J0 {t : ℝ} (ht : t ∈ Set.Ioo (-13 : ℝ) 13) : HasDerivAt J0 (-J1 t) t
lemma hasDerivAt_J1 {t : ℝ} (ht : t ∈ Set.Ioo (-13 : ℝ) 13) (ht0 : t ≠ 0) :
  HasDerivAt J1 (J0 t - J1 t / t) t
lemma energy_antitone0n : Antitone0n energy (Set.Icc 1 12)
lemma abs_J0_sub_S0_le {t : ℝ} (ht : |t| ≤ 12) : |J0 t - S0 t| ≤ tailEps
lemma abs_J1_sub_S1_le {t : ℝ} (ht : |t| ≤ 12) : |J1 t - S1 t| ≤ tailEps

theorem three_J0_sq_abs_J1_lt {t : ℝ} (h3 : 3 ≤ t) (h12 : t ≤ 12) :
  3 * J0 t ^ 2 * |J1 t| < 1 / 10
theorem three_J0_sq_abs_J1_le_sharp {t : ℝ} (h3 : 3 ≤ t) (h12 : t ≤ 12) :
  3 * J0 t ^ 2 * |J1 t| ≤ 93 / 1000
theorem sup_not_lt_eight_hundredths :
  ∃ t : ℝ, 3 ≤ t ∧ t ≤ 12 ∧ 8 / 100 < 3 * J0 t ^ 2 * |J1 t|
theorem sup_fails_on_one_twelve :
  ∃ t : ℝ, 1 ≤ t ∧ t ≤ 12 ∧ 1 / 10 < 3 * J0 t ^ 2 * |J1 t|
theorem not_identically_zero : 6 / 100 < 3 * J0 3 ^ 2 * |J1 3|

theorem J1_pos_on_one_three {t : ℝ} (h1 : 1 ≤ t) (h3 : t ≤ 3) : 0 < J1 t
theorem J0_antitone0n_one_three : Antitone0n J0 (Set.Icc 1 3)
theorem abs_J0_le_max_endpoints {a b t : ℝ} (h1 : 1 ≤ a) (hab : a ≤ b) (hb3 : b ≤ 3)
  (hat : a ≤ t) (htb : t ≤ b) : |J0 t| ≤ max |J0 a| |J0 b|

```

The finite computation (C1)–(C5) is stated as five Boolean-valued functions over $\mathbb{Q}$ (the 24-term truncations of the series at a rational point, the tail constant, and the inequalities of Definition 2.2, with the two grids as lists of 289 and 65 rationals) and each is proved equal to **true** by *kernel* evaluation (**decide +kernel**): the kernel of Lean unfolds the definitions and evaluates the rational arithmetic itself, so no compiled code, no external solver, no certificate file and no floating-point or interval arithmetic is trusted anywhere. The development contains no **sorry** and declares no axiom of its own. A print of the axiom set of each of the thirteen declarations listed above, re-run against the compiled development while this draft was written, returns **propext**, **Classical.choice** and **Quot.sound** and nothing else — in particular no **sorryAx** and no compiled-evaluation axiom, there being no use of **native_decide**. Cost: the module holding the finite computation elaborates in 59 s of CPU time with a peak of 8.2 GB of memory (about 1.7 GB above the baseline of importing **Mathlib**), that is about 0.1 s of kernel time per exact-rational evaluation of the truncated series (the reduced numerators run to about 130 digits), the points near $t = 12$ being the expensive ones; the module holding the results of Sections 3–5 elaborates in 29 s. The verification is the method; the mathematics is that of Sections 2–5. Repository reference to be added at submission.

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