COUNTING AMPLITUDE MODULI: $\chi(|H\rangle^{\otimes m}) \geq 4$ AND $\chi(|T\rangle^{\otimes m}) \geq 4$ FOR EVERY $m \geq 31$

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ABSTRACT. The stabilizer rank $\chi(\psi)$ of an n-qubit state ψ is the least number of stabilizer states whose complex span contains ψ. For the two Clifford-inequivalent single-qubit magic-state orbits — the octahedron-edge state $|H\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$ and the octahedron-face state $|T\rangle = \cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle$, $\cos 2\beta = 1/\sqrt{3}$ — we prove that $\chi(|H\rangle^{\otimes m}) \geq 4$ and $\chi(|T\rangle^{\otimes m}) \geq 4$ for every $m \geq 31$. The proof counts moduli. Every amplitude of a stabilizer state is 0 or $\gamma\, i^e$ for one fixed scale γ, so an amplitude of a state in the span of three stabilizer states is determined by a *type* in a $5^3 = 125$-element set; multiplying all three symbols of a type by i multiplies the amplitude by i and leaves its modulus alone, so at most $(125-1)/4 = 31$ distinct nonzero moduli can occur. The $m+1$ Hamming-weight classes of a magic state carry $m+1$ pairwise distinct moduli, and 32 of them do not fit. Both statements quantify over every finite set of stabilizer states: no dictionary of stabilizer states is enumerated, and no floating-point search enters. For comparison, the published lower bounds carrying explicit constants first reach 4 at $m = 40$ for the H orbit and at $m = 148$ for the T orbit; asymptotically they all dominate the count used here, and the improvement is in the constant, not the rate. Every statement below is machine-checked in Lean 4 by kernel reduction alone.

1. INTRODUCTION

Stabilizer rank. A pure n-qubit *stabilizer state* is a joint eigenvector of a maximal abelian subgroup of the Pauli group; there are $2^n \prod_{j=1}^{n}(2^j + 1)$ of them, that is $6, 60, 1080, 36720, 2423520, 315057600$ for $n = 1, \ldots, 6$ [7, Prop. 2]. The *stabilizer rank* of a state ψ,

$$\chi(\psi) \;=\; \min\bigl\{\, k \;:\; \psi \in \operatorname{span}_{\mathbb{C}}\{\sigma_1, \ldots, \sigma_k\},\ \sigma_j \text{ stabilizer states} \,\bigr\},$$

measures how far ψ is from the classically simulable world: a decomposition of length k turns a Clifford circuit fed with ψ into k classically tractable branches. The quantity of interest is the growth of $\chi(\psi^{\otimes m})$ in m for a fixed single-qubit *magic* state ψ, and the exponent $\gamma_\psi = \limsup_m \frac{1}{m}\log_2 \chi(\psi^{\otimes m})$ is what a simulation runtime is exponential in.

Two single-qubit magic states are distinguished by the Clifford action on the Bloch sphere: the octahedron-*edge* state

$$|H\rangle = \cos\tfrac{\pi}{8}|0\rangle + \sin\tfrac{\pi}{8}|1\rangle, \qquad |T\rangle = \cos\beta\,|0\rangle + e^{i\pi/4}\sin\beta\,|1\rangle, \quad \cos 2\beta = \tfrac{1}{\sqrt{3}},$$

the octahedron-*face* state. Naming conventions differ between papers, and the translation matters for reading the table below. The $|H\rangle$ of [2] is the $|H\rangle$ of this paper — printed there as $|H\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$, with χ_n the stabilizer rank of $|H^{\otimes n}\rangle$ — and the second orbit is that paper's $|R\rangle$, the $+1$ eigenvector of

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$(X+Y+Z)/\sqrt{3}$, which is our $|T\rangle$; the $|T\rangle \sim |0\rangle + e^{i\pi/4}|1\rangle$ that [2] names in an appendix is a Clifford-equivalent representative of its own $|H\rangle$ orbit, not of the other one. [4] uses the same $|H\rangle, |R\rangle$ pair, and the $|H\rangle$ of [6] is again ours. In [3], by contrast, the state called $|T\rangle$ is $2^{-1/2}(|0\rangle + e^{i\pi/4}|1\rangle)$, Clifford-equivalent to the $|H\rangle$ of this paper, and their $|F\rangle = \cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle$ is our $|T\rangle$; the T-state of [5] is likewise $2^{-1/2}(|0\rangle + e^{i\pi/4}|1\rangle)$, so their headline bound is an H-orbit bound.

The state of knowledge, and what it rests on. Labib and Russo [1] tabulate what is known for the two qubit orbits at $m \le 6$:

m	1	2	3	4	5	6
$\chi(\lvert H\rangle^{\otimes m})$	2	2	3	4	≤ 6	≤ 6
$\chi(\lvert T\rangle^{\otimes m})$	2	2	3	3	≤ 6	≤ 6

The H row at $2 \le m \le 5$ is attributed to Bravyi, Smith and Smolin [2], who report $\chi_n \le 2, 3, 4, 6, 7$ for $n = 2, \dots, 6$ from “a heuristic algorithm for computing low-rank decompositions of $|H^{\otimes n}\rangle$ into stabilizer states”, print in their Appendix B the decompositions it found at $n = 2, \dots, 5$ together with a seven-state one at $n = 6$, and add “We believe that these upper bounds are tight”; the H cell at $m = 6$ is Qassim, Pashayan and Gosset’s [3], who reach ≤ 6 through $\chi(\mathrm{cat}_6) \le 3$ and thereby disprove the conjectural 7. The T row at $2 \le m \le 6$ is [1]’s own; the two $m = 1$ cells carry no mark, holding as they do for every non-stabilizer single-qubit state. The paper’s text is explicit about what the two directions rest on: the value $\chi(|T\rangle^{\otimes 4}) = 3$ is “established as tight via the $m = 3$ lower-bound certificate by monotonicity”, that certificate appears in no theorem there, and where such certificates are described their soundness is numerical — “the smallest non-witness least-squares residual is ≈ 0.14, nine orders of magnitude above the 10^{-10} tolerance used to flag a true witness” [1, verbatim]. The same paper states its open cells verbatim: “whether genuinely smaller rank witnesses exist at $m \in \{5, 6\}$ is unknown, as is whether the asymptotic exponents γ_H and γ_T in fact coincide”.

Away from these small cells, three unconditional lower bounds with explicit constants are on record. Peleg, Shpilka and Volk prove, verbatim, that “*Let $F : \mathbb{F}_2^n \to \mathbb{C}$ be a function of stabilizer rank r such that $r \le n/100$. Then, there exist $y, z \in \mathbb{F}_2^n$ such that $|y| \ne |z|$ and $F(y) = F(z)$*” [4, Lemma 3.1]; since both magic states separate weight classes, this gives $\chi > n/100$, and it is how they obtain $\chi(|B\rangle^{\otimes n}) = \Omega(n)$ for $|B\rangle \in \{|H\rangle, |R\rangle\}$ [4, Theorem 3.2]. Lovitz and Steffan prove $\chi(\psi^{\otimes n}) \ge (\lfloor n/k \rfloor + 1)/(4\log_2(\lfloor n/k \rfloor + 1))$ for every non-stabilizer single-qubit ψ [5, proof of Corollary 3.3]: here $|0\rangle + \alpha|1\rangle$ is a representative of the Clifford orbit of ψ with $|\alpha| > 1$ and k is any integer with $|\alpha|^k \ge 2$, so the strongest instance takes $|\alpha|$ largest over the orbit and k least. Mehraban and Tahmasbi prove, for $\phi = 2^{-n/2}\sum_x f(x)|x\rangle$ with $a = \min_x |f(x)|$, $b = \max_x |f(x)|$ and $F(\phi)$ the stabilizer fidelity, that $\chi(\phi) \ge \frac{2}{3}\log_2\big(a^2/(b\sqrt{F(\phi)})\big)$ [11, Prop. A.3], and read this off at the H orbit — where $a = b = 1$ and $F = \cos^{2m}\frac{\pi}{8}$ — as [11, Cor. A.4]:

$$\chi(|H\rangle^{\otimes m}) \;\ge\; \tfrac{2}{3}\log_2\big(1/\cos\tfrac{\pi}{8}\big)\, m \;=\; 0.07614\ldots\, m \;>\; 0.076\, m.$$

None of the three constants is improved elsewhere in the three papers; the other bounds there are asymptotic, or concern the approximate rank. All three are asymptotically far stronger than anything proved here. All three are also, at small m, weak. The first forces $\chi \ge 4$ only from $n = 300$ on. The second only once $\lfloor n/k \rfloor + 1 = 75$ — at 75 the bound is $3.010\ldots$, at 74 still $2.979\ldots$ — i.e.

from $n = 74$ for the H orbit, which contains $|0\rangle + (\sqrt{2}+1)|1\rangle$ and so admits $k = 1$, and from $n = 148$ for the T orbit, where the largest ratio over the orbit is $|\alpha| = \cot\beta = \sqrt{2+\sqrt{3}} = 1.9319\ldots < 2$ and the least admissible k is 2. The third is the strongest of the three, and forces $\chi \geq 4$ from $m = 40$ on ($0.07614\ldots \times 39 = 2.969\ldots$ against $\times 40 = 3.045\ldots$); but it is an H-orbit bound only. Applied to $|T\rangle^{\otimes m}$ — or to any single-qubit-Clifford image of it, all of which have the same amplitude profile — one has $a = 2^{m/2}\sin^m\beta$, $b = 2^{m/2}\cos^m\beta$ and $F \geq \cos^{2m}\beta$, the last because $|0\rangle^{\otimes m}$ is a stabilizer state; hence

$$\frac{a^2}{b\sqrt{F}} \;\leq\; \big(\sqrt{2}\tan^2\beta\big)^m \;=\; \big(\sqrt{2}\,(2-\sqrt{3})\big)^m \;=\; (0.3789\ldots)^m \;<\; 1,$$

so [11, Prop. A.3] returns a negative number on the T orbit for every m and says nothing there.

Results. This paper proves the two statements in the title, by counting how many distinct amplitude *moduli* a low-rank state can display.

- The amplitudes of a state in the complex span of three stabilizer states have at most 31 distinct nonzero moduli; consequently, 32 indices carrying 32 pairwise distinct nonzero moduli force $\chi \geq 4$ (Theorem 10).
- $\chi(|H\rangle^{\otimes m}) \geq 4$ and $\chi(|T\rangle^{\otimes m}) \geq 4$ for every $m \geq 31$ (Theorem 15).

The mechanism is elementary and is the point. Every nonzero amplitude of a stabilizer state is $\gamma\, i^e$ for a single scale γ and $e \in \mathbb{Z}_4$, so for $\psi = Au_1 + Bu_2 + Cu_3$ the value ψ_x depends only on the *type* of x: the triple recording, for each j, whether u_j vanishes at x and if not which fourth root of unity it carries. There are $5^3 = 125$ types. Multiplying all three symbols of a type by i multiplies ψ_x by i, hence preserves $\|\psi_x\|$; the resulting $\mathbb{Z}_4$-action on the 125 types has exactly one fixed point (the all-zero type, which forces $\psi_x = 0$) and 31 free orbits. Hence 31 moduli, and hence the threshold: $|H\rangle^{\otimes m}$ and $|T\rangle^{\otimes m}$ have $m+1$ Hamming-weight classes carrying the $m+1$ pairwise distinct moduli $\cos^{m-w}\theta\,\sin^w\theta$, $w = 0,\ldots,m$, with $\theta = \frac{\pi}{8}$ and $\theta = \beta$ respectively; so $m + 1 \geq 32$ is already a contradiction.

Neither ingredient is new by itself. The framing — amplitudes of a low-rank state are subset sums of few numbers — is [5, §1.1], where a rank-r decomposition is converted into a "length-$4r$-subset-sum representation of the coordinates"; and counting the distinct *values* of a rank-χ state by C^χ, for a constant C left unspecified, is already the first step of the $\Omega(n^{1/2})$ lower bound of [2, App. C]. What the present work supplies is the sharp small-k constant: not C^k but 5^k, and then not 5^k but $(5^k - 1)/4$ for the moduli. The raw type count 5^k gives $\chi > 3$ only from $m = 125$; the orbit count 31 gives it from $m = 31$. Measured against the published explicit thresholds recalled above, we have located no unconditional exact argument proving $\chi(|T\rangle^{\otimes m}) \geq 4$ for even one $m \leq 147$, nor $\chi(|H\rangle^{\otimes m}) \geq 4$ for even one $m \leq 39$; Theorem 15 proves both for every $m \geq 31$.

What is not claimed. Three limitations are worth stating at once.

First, the improvement is in the constant only. The bound proved here is a $\log_5$ rate (Proposition 14); [4] and [11] are linear and [5] is near-linear, and all three overtake it. For the H orbit the crossover falls between the $k = 3$ and $k = 4$ instances of the count, against [5] and [11] alike: at $k = 3$ the count wins ($m = 31$ against 74 and 40), at $k = 4$ it loses ($m = 156$ against 108 and 53). For the T

orbit, where [11] is silent, the crossover against [5] sits one step later, between $k = 4$ ($m = 156$ against 216) and $k = 5$ ($m = 781$ against 286).

Second, for the H orbit alone one may reach $\chi(|H\rangle^{\otimes m}) \geq 4$ for all $m \geq 4$ by chaining monotonicity of χ under tensoring with the value $\chi(|H\rangle^{\otimes 4}) = 4$ of the table above. That route is not available in exact form: the lower half of that cell is the floating-point exhaustive triple search whose soundness [1] rests on the 0.14 residual gap quoted above. The T orbit has no analogue at all, since $\chi(|T\rangle^{\otimes 4}) = 3$. Monotonicity of χ under tensoring is not used anywhere below and is not part of the formal development.

Third, nothing here decides the two cells [1] calls open. We bracket them exactly, $3 \leq \chi(|T\rangle^{\otimes 5}), \chi(|T\rangle^{\otimes 6}) \leq 6$ and $3 \leq \chi(|H\rangle^{\otimes 5}), \chi(|H\rangle^{\otimes 6}) \leq 6$ (§5), and record in Remark 24 the restricted search that failed to improve the upper halves.

2. Preliminaries

Exact arithmetic. Fix a prime q and write $\mathbb{Z}[i, \sqrt{q}]$ for the free $\mathbb{Z}$-module on $1, i, \sqrt{q}, i\sqrt{q}$ with the evident multiplication.

Proposition 1. *For q prime, $\mathbb{Z}[i, \sqrt{q}]$ is a commutative ring and the evaluation map $\mathrm{ev}_q\colon \mathbb{Z}[i, \sqrt{q}] \to \mathbb{C}$, $(a, b, c, d) \mapsto a + bi + c\sqrt{q} + di\sqrt{q}$, is an injective ring homomorphism.*

Injectivity follows from the irrationality of $\sqrt{q}$. Its use is uniform: every finite computation below is performed on 4-tuples of integers, and Proposition 1 transports both the identities ($= 0$) and the non-identities ($\neq 0$) to $\mathbb{C}$, so that an exact integer computation rules out *complex* coefficients and not merely real or algebraic ones.

Stabilizer states, in canonical form. An amplitude vector on n qubits is a function $\psi\colon \{0, 1, \dots\} \to \mathbb{C}$ indexed by bitmasks, with $\psi_x = 0$ for $x \geq 2^n$. Stabilizer states are handled through the qubit canonical form of Dehaene and De Moor [8, Thm. 5(ii)]: the data $d = (x_0, W, \ell, Q)$ with $x_0 \in \mathbb{F}_2^n$, W an $n \times k$ matrix over $\mathbb{F}_2$ of full column rank, ℓ a linear and Q a quadratic form on $\mathbb{F}_2^k$, determine

$$\sigma_d \;\propto\; \sum_{y \in \mathbb{F}_2^k} i^{\,\ell(y) + 2Q(y)}\, |x_0 \oplus Wy\rangle. \tag{1}$$

(The qudit analogue for odd prime p, with $\omega_p^{Q(y)}$ in place of $i^{\ell(y)+2Q(y)}$, is [1, eq. (1)]; that paper states it for odd p only, so the $p = 2$ form used here is the qubit one just cited.) We work projectively: v is a *stabilizer ray* on n qubits when $v = c\,\sigma_d$ for some $c \neq 0$ and some well-formed d, the normalisation $2^{-k/2}$ being irrelevant to rank. Write

$$\chi(\psi) \;=\; \inf\bigl\{\, |S| \;:\; S \text{ a finite set of stabilizer rays},\; \psi \in \operatorname{span}_{\mathbb{C}} S \,\bigr\}.$$

Remark 2. That (1) parametrises exactly the Pauli stabilizer states is cited, not proved, in the development below. One direction — every n-qubit stabilizer state has this form — is [8, Thm. 5(ii)], which exhibits the amplitude vector as supported on an affine coset and equal there, up to a global scale, to i raised to a linear plus a quadratic $\mathbb{F}_2$-form. That the correspondence runs both ways is stated in [4, §2], citing [8] and [9]. What was independently checked here — in exact computation outside the formal development — is that the parametrisation reproduces the counts 6, 60, 1080 at $n = 1, 2, 3$ and, by an echelon enumeration, 36 720 at

$n = 4$, agreeing with [7, Prop. 2]. All the lower bounds below are therefore lower bounds on the number of states of the form (1) needed, and the reader who wants them as statements about Pauli stabilizer states should read them through that identification.

Proposition 3. *Let d be well formed on n qubits. Then each amplitude of σ_d is 0 or a power of i, and all nonzero amplitudes of a stabilizer ray have equal modulus. Moreover:*

1. *(upper bound) an explicit list $\sigma_1, \ldots, \sigma_k$ of stabilizer rays and coefficients $c_1, \ldots, c_k$ with $\psi_x = \sum_j c_j(\sigma_j)_x$ for all x gives $\chi(\psi) \leq k$;*
2. *(non-membership) a functional φ with $\sum_{x<2^n} \varphi_x(\sigma_j)_x = 0$ for all j and $\sum_{x<2^n} \varphi_x\psi_x \neq 0$ shows $\psi \notin \operatorname{span}_{\mathbb{C}}\{\sigma_1, \ldots, \sigma_k\}$, and the certificate may be computed in any subring of $\mathbb{C}$;*
3. *(lower bound) if ψ has at least one stabilizer decomposition — which Proposition 4 below supplies once and for all — and no set of at most k stabilizer rays spans ψ, then $\chi(\psi) > k$.*

Item (iii) is the shape every lower bound below takes, and it is worth emphasising what it quantifies over: an arbitrary finite set of stabilizer rays, not a list of them. No enumeration of stabilizer states occurs anywhere in this paper.

Proposition 4. *Every amplitude vector vanishing outside the 2^n basis indices lies in the span of the 2^n computational basis states, which are stabilizer states. Hence the set of cardinalities of stabilizer decompositions of ψ is nonempty and $\chi(\psi)$ is a genuine infimum.*

This is the hypothesis every application of Proposition 3(iii) needs, and discharging it once from the computational basis — rather than per cell from an explicit witness — is what makes statements quantified over all m possible.

The two orbits, exactly. Both magic states become integral after one rescaling.

Proposition 5. *With $c_\beta = \cos\beta$, $s_\beta = \sin\beta$ and $\cos 2\beta = 1/\sqrt{3}$ one has $c_\beta^2 + s_\beta^2 = 1$ and $\sqrt{2}\, s_\beta = c_\beta(\sqrt{3}-1)$, so that*

$$|T\rangle = \tfrac{c_\beta}{2}\big(2|0\rangle + (1+i)(\sqrt{3}-1)|1\rangle\big), \qquad |H\rangle = \cos\tfrac{\pi}{8}\big(|0\rangle + (\sqrt{2}-1)|1\rangle\big),$$

the second from $\tan\frac{\pi}{8} = \sqrt{2}-1$. Consequently $|T\rangle^{\otimes m} = (c_\beta/2)^m \operatorname{ev}_3(t^{(m)})$ and $|H\rangle^{\otimes m} = \cos^m\frac{\pi}{8}\operatorname{ev}_2(h^{(m)})$ coordinatewise, for explicit integral vectors $t^{(m)} \in \mathbb{Z}[i,\sqrt{3}]^{2^m}$ and $h^{(m)} \in \mathbb{Z}[i,\sqrt{2}]^{2^m}$.

The change of variables matters more than it looks. Individually $\cos\beta$ is a nested radical and the odd tensor powers have no entries in $\mathbb{Q}(i,\sqrt{3})$; the ratio $e^{i\pi/4}\tan\beta = (1+i)(\sqrt{3}-1)/2$ does lie in $\mathbb{Q}(i,\sqrt{3})$, and since stabilizer rank is invariant under rescaling by a nonzero scalar the whole problem moves into a rank-4 $\mathbb{Z}$-algebra where every check is a finite integer computation.

Finally, the modulus profile, which is the only property of the magic states the main theorems use. Writing $|x|$ for the Hamming weight of the low m bits of x:

(2)
$$\big\|(|H\rangle^{\otimes m})_x\big\| = \cos^{m-|x|}\tfrac{\pi}{8}\,\sin^{|x|}\tfrac{\pi}{8}, \qquad \big\|(|T\rangle^{\otimes m})_x\big\| = c_\beta^{m-|x|} s_\beta^{|x|} \qquad (x < 2^m).$$

3. How many values, and how many moduli

Throughout this section ψ is an amplitude vector on n qubits satisfying the hypothesis of Proposition 4.

Values.

Lemma 6. *Let ψ lie in the span of at most k stabilizer rays. Then ψ takes at most 5^k distinct values. Consequently, if some index set carries more than 5^k pairwise distinct values of ψ, then $\chi(\psi) > k$.*

Proof sketch. Write $\psi = \sum_{j\le k} c_j v_j$ with $v_j = \gamma_j \sigma_{d_j}$. By Proposition 3 each $(v_j)_x$ lies in $\{0, \gamma_j, \gamma_j i, -\gamma_j, -\gamma_j i\}$, a set of five elements when $\gamma_j \neq 0$; so the map sending x to the k-tuple of these symbols has at most 5^k values and determines ψ_x. No computation is involved. □

Lemma 6 is not new either. It is exactly the observation of [2, App. C] — "any stabilizer state has $C = O(1)$ distinct amplitudes in the computational basis. Thus any linear combination of χ stabilizer states has at most C^χ distinct amplitudes" — with C pinned to 5; there it is used, with C left unspecified, inside the proof of $\chi_n = \Omega(n^{1/2})$, and an unspecified C forces $\chi > k$ at no particular m. Our 5 is exact at $k = 1$: a single three-qubit stabilizer ray attaining all five values $0, 1, i, -1, -i$ is exhibited in §6. Compared with the 2^{4k} implicit in the subset-sum framing of [5, §1.1], the saving is that at most one fourth root of unity per ray is selected at a given index.

Moduli at rank two. A common phase does not change a modulus, and that already improves the count.

Proposition 7. *A state in the span of two stabilizer rays has at most six distinct nonzero moduli, namely $\|a\|$, $\|b\|$, $\|a \pm b\|$ and $\|a \pm ib\|$ for the two ray scales a, b. Hence seven indices carrying seven pairwise distinct nonzero moduli force $\chi(\psi) \ge 3$. No support hypothesis and no finite computation are used.*

Proposition 8. *Suppose in addition that ψ has full support, that its moduli are constant on the classes of some function* wt *and separate them, and that every class has fewer than 2^{n-1} elements. Then both rays have full support, only the four moduli $\|a \pm b\|, \|a \pm ib\|$ occur, and five separated classes force $\chi(\psi) \ge 3$.*

The support step is the reason for the class-size hypothesis: on the part of the support of one ray that the other misses, $\|\psi_x\|$ is constant, so that part fits inside one modulus class; a canonical state's support has 2^k elements, and once more than 2^{n-1} of them are forced the support is everything.

Moduli at rank three. At $k = 3$ the same phase symmetry is best organised as a group action. Code a type as $c = t_1 + 5t_2 + 25t_3$ with $t_j \in \{0, 1, 2, 3, 4\}$, the symbol 0 meaning "the j-th ray vanishes here" and $t_j = e + 1$ meaning "it carries i^e"; let ρ multiply every nonzero symbol of a type by i.

Lemma 9. *ρ is a $\mathbb{Z}_4$-action on the 125 codes with exactly one fixed point, the all-zero code; the remaining 124 codes fall into 31 free orbits, so $125 = 1 + 4 \cdot 31$. If $F(c)$ denotes the amplitude a code determines from three ray scales A, B, C, then $F(\rho c) = i\,F(c)$, and hence $\|F\|$ is constant on orbits, while the fixed point has $F = 0$.*

The orbit count is the one exhaustive computation in the proof of the main theorem: the canonical representative $c \mapsto \min\{c, \rho c, \rho^2 c, \rho^3 c\}$ is evaluated at all 125 codes, the image is checked to have 32 elements, and the all-zero code is checked to be the unique fixed point. Three finite checks over 125 items; nothing larger.

Theorem 10. *A state in the complex span of three stabilizer rays has at most* 31 *distinct nonzero moduli. Consequently, if $x_1, \ldots, x_{32}$ are indices with $\psi_{x_r} \neq 0$ and $\|\psi_{x_r}\|$ pairwise distinct, then $\chi(\psi) \geq 4$.*

Proof sketch. By Proposition 3(iii) it suffices to rule out every finite set S of stabilizer rays with $|S| \leq 3$ whose span contains ψ. Pad S to exactly three rays (a computational basis state is available for padding by Proposition 4), write $\psi = Au_1 + Bu_2 + Cu_3$ with u_j the amplitude vectors of well-formed canonical data, and let $c(x) < 125$ be the type code of x. Proposition 3 gives $\psi_x = F(c(x))$ with F as in Lemma 9. If $c(x) = 0$ then $\psi_x = 0$, so each x_r has $c(x_r) \neq 0$ and its canonical orbit representative lies in a set of size 31; by Lemma 9 two indices with the same representative have equal moduli, so $r \mapsto$ representative of $c(x_r)$ is injective on 32 indices into a 31-element set — a contradiction. The only exhaustive computation is the 125-code orbit count of Lemma 9. ☐

Remark 11. The pattern $(5^k - 1)/4$ is visible — $1, 6, 31$ at $k = 1, 2, 3$ — but is established here only at $k \leq 3$, and at $k = 2$ and $k = 3$ by different means (an explicit six-element list of moduli, and the orbit computation). A uniform statement would need a normalised type construction over an abstract index set; it is elementary but is not proved below, and it would not help at the scale that matters: $k = 4$ would give $\chi \geq 5$ only from $m = 156$, past the point where [5] and [11] are stronger on the H orbit ($m = 108$ and $m = 53$). On the T orbit, where [11] is silent, $k = 4$ would still win (156 against 216), and $k = 5$ would not (781 against 286).

4. The magic states

The bridge from §3 to the magic states is the profile (2).

Lemma 12. *Let $0 < s < c$ and $\mu_w = c^{m-w} s^w$ for $0 \leq w \leq m$. Then $w \mapsto \mu_w$ is strictly decreasing, hence injective; the indices $2^w - 1$, $w = 0, \ldots, m$, are representatives of the $m + 1$ Hamming-weight classes and are pairwise distinct. Consequently a state ψ vanishing outside $x < 2^m$ and satisfying $\|\psi_x\| = \mu_{|x|}$ has $m + 1$ indices with pairwise distinct nonzero moduli.*

Combining Lemma 12 with the results of §3 gives three ready-made bounds for such a state: $\chi \geq 3$ as soon as $m \geq 6$ (seven classes against six moduli, Proposition 7); $\chi \geq 3$ for $m = 4, 5$ as well, using Proposition 8 together with the class sizes $1, 4, 6, 4, 1$ and $1, 5, 10, 10, 5, 1$, all below 2^{m-1}; and $\chi > k$ whenever $5^k < m + 1$ from Lemma 6.

Theorem 13. *$\chi(|H\rangle^{\otimes m}) \geq 3$ and $\chi(|T\rangle^{\otimes m}) \geq 3$ for every $m \geq 4$.*

These values are known to any reader by monotonicity of χ under tensoring from $\chi(|H\rangle^{\otimes 3}) = \chi(|T\rangle^{\otimes 3}) = 3$, and nothing new is claimed for them as mathematics. What is worth recording is the shape of the argument: it is direct, uses no monotonicity, enumerates nothing, and for $m \geq 6$ contains no finite computation at all — the finite input at $m = 4, 5$ is the two class-size checks above. We know of no other machine-checked stabilizer-rank lower bound covering infinitely many m.

Proposition 14. *$\chi(|H\rangle^{\otimes m}) > k$ and $\chi(|T\rangle^{\otimes m}) > k$ for every k with $5^k < m+1$.*

This ladder is uniform in both m and k and consumes no data whatever, but its rate is $\log_5$, far below the $\Omega(n)$ of [4] and the $\Omega(n/\log n)$ of [5]. At $k = 3$ it gives $\chi \geq 4$ from $m = 125$. The modulus refinement of Theorem 10 replaces that by:

Theorem 15. *$\chi(|H\rangle^{\otimes m}) \geq 4$ and $\chi(|T\rangle^{\otimes m}) \geq 4$ for every $m \geq 31$.*

Proof sketch. By (2) and Lemma 12 the $m+1 \geq 32$ indices $2^w - 1$ carry pairwise distinct nonzero moduli, since $\sin\frac{\pi}{8} < \cos\frac{\pi}{8}$ and $s_\beta < c_\beta$; apply Theorem 10. The hypothesis of Proposition 4 holds because the state vanishes outside $x < 2^m$. The only exhaustive computation anywhere in the proof is the 125-code orbit count. □

Comparison. At $m = 31$ the three published bounds with explicit constants give, as recalled in the introduction, nothing: $31/100 < 1$; $32/(4\log_2 32) = 32/20 = 1.6$ for the H orbit, less for the T orbit; and $0.07614\ldots \times 31 = 2.360\ldots$ for the H orbit, nothing at all for the T orbit. They first give $\chi \geq 4$ at $m = 40$ (H) for [11], at $m = 74$ (H) and $m = 148$ (T) for [5], and at $m = 300$ for [4]. So Theorem 15 settles $\chi(|T\rangle^{\otimes m}) \geq 4$ at 117 values of m — namely $31 \leq m \leq 147$ — and $\chi(|H\rangle^{\otimes m}) \geq 4$ at 9 values, $31 \leq m \leq 39$, that we have not been able to locate in the literature as consequences of any exact unconditional argument. The T-orbit window is the substantial one; the H-orbit window is nine values wide, and would close entirely against any linear bound with a constant above $\frac{3}{31} = 0.0967$. The searches behind that negative are as follows. The arXiv title search `ti:"stabilizer rank"` returns nine records, one of them a spurious phrase match; the newest genuine one is [1], and the abstract searches `abs:"stabilizer rank"` (48 records) and `abs:"stabilizer extent"` (14) turned up nothing further. We read in full [1, 2, 3, 4, 5, 6, 11] and [7], and checked the two remaining title matches — [10, 12] — for an explicit constant: [10] extends the $\Omega(n)$ of [4] to qudits of any prime dimension, with the constant left unspecified ("let $C > 0$ be a large enough constant"), and [12] bounds stabilizer *fidelity* in terms of rank and states its rank consequence as $\Omega(n/\log n)$; so neither forces $\chi \geq 4$ at any particular m. Not searched, so not a clean negative: MathSciNet, zbMATH, Google Scholar, and theses. As of 30 August 2026 no paper indexed by Semantic Scholar, OpenAlex or INSPIRE cites [1], so its citation graph is no help either. The negative is a "not found", not a proof of absence. The extent-based route is unavailable and is worth naming as a trap: the inequality $\xi(\psi) \geq 1/F(\psi)$ of [6] bounds the stabilizer *extent* ξ, which controls the ℓ_1 norm of the coefficient vector and not the number of terms, so it yields no lower bound on χ. That paper's only lower bound on a rank is restricted to linear combinations of the product states $|\tilde{x}_1\rangle \otimes \cdots \otimes |\tilde{x}_n\rangle$ with $|\tilde{0}\rangle = |0\rangle$ and $|\tilde{1}\rangle = |+\rangle$, and it says of the general problem that "we have no techniques that provide lower bounds on the stabilizer rank that scale exponentially with the number of copies".

Asymptotically the comparison runs the other way, and we state it plainly: Theorem 15 and Proposition 14 are $\log_5$-rate statements and are dominated by both published bounds for large m. The contribution is the constant.

5. The cells $m \leq 6$

The results of §4 sit on top of a layer of exact certificates for the small cells, which we summarise because it is what makes the two-sided brackets below exact rather than numerical. None of the *values* in this section is new, and neither are the three H-orbit witnesses at $m \leq 5$: as §5 records below, each of them reproduces — state for state and coefficient for coefficient — a decomposition already printed in [2, App. B]. What is supplied here is the exact certificate: each identity is checked in $\mathbb{Z}[i, \sqrt{q}]$ over all 2^m basis indices by kernel reduction, and each state is checked to satisfy the concrete stabilizer predicate rather than assumed to be a stabilizer state.

Proposition 16. $\chi(|T\rangle) = \chi(|T\rangle^{\otimes 2}) = 2$. *The lower half is the modulus obstruction of Proposition 3: all nonzero amplitudes of a stabilizer ray share a modulus, while for $m = 1, 2$ the two amplitudes $\cos^m\beta$ and $\cos^{m-1}\beta \sin\beta$ of $|T\rangle^{\otimes m}$ differ.*

Proposition 17. *Let ψ be a three-qubit state whose moduli are a function of the Hamming weight and separate the four weight classes, and whose values at the four class representatives $0, 1, 3, 7$ are $\alpha \neq 0$ times exact elements $T_0, \dots, T_3$ of $\mathbb{Z}[i, \sqrt{q}]$. If no combination $\sum_w T_w\, i^{m_w}$ with $m \in \{0, 1, 2, 3\}^4$ vanishes, then $\chi(\psi) \geq 3$.*

Proof sketch. Suppose $\psi = av_1 + bv_2$. The class sizes are $1, 3, 3, 1$, so the part of one support the other misses meets at most one class and has at most 3 elements; since a canonical support has $2^k \leq 8$ elements, both supports are full. At the four class representatives $0, 1, 3, 7$ write $(v_1)_x = i^{p(x)}$, $(v_2)_x = i^{r(x)}$ and $s = r - p \bmod 4$; equal s-values would force equal moduli, so s is a bijection onto $\mathbb{Z}_4$, whence $\sum_w i^{s(w)} = \sum_w i^{2s(w)} = 0$. Multiplying the w-th equation by $i^{s(w)-p(w)}$ and summing eliminates both a and b and leaves $\sum_w T_w i^{m_w} = 0$ for some $m \in \{0, 1, 2, 3\}^4$, contradicting the hypothesis. For each of the two orbits that hypothesis is a closed finite statement about four fixed elements of $\mathbb{Z}[i, \sqrt{q}]$, and it is checked over all $4^4 = 256$ patterns. This is the whole computation: 256 exact evaluations, no stabilizer states enumerated. □

Theorem 18. $\chi(|T\rangle^{\otimes 3}) = \chi(|H\rangle^{\otimes 3}) = 3$, *both halves exactly certified.*

The four weight-class values entering Proposition 17 are $\big(8,\ 4(1+i)(\sqrt{3}-1),\ 2((1+i)(\sqrt{3}-1))^2,\ ((1+i)(\sqrt{3}-1))^3\big)$ for the T orbit and $(1, \sqrt{2}-1, 3-2\sqrt{2}, 5\sqrt{2}-7)$ for the H orbit. The upper half at $m = 3$ for the H orbit is an explicit witness — $|111\rangle$, the uniform even-parity state, and the triangle graph state $\sum_x (-1)^{x_1x_2+x_1x_3+x_2x_3}|x\rangle$, with the integral coefficients $-8+6\sqrt{2}$, $2-\sqrt{2}$, $-1+\sqrt{2}$ against the rescaled representative $\cos^{-3}\frac{\pi}{8}|H\rangle^{\otimes 3}$. It was found here by the exact search described under Theorem 22, but it is not new: it is, term for term, the decomposition $|H^{\otimes 3}\rangle = (-8+6\sqrt{2})|B_{3,3}\rangle + (2-\sqrt{2})|E_3\rangle + (-1+\sqrt{2})|K_3\rangle$ printed in [2, App. B] in the same normalisation $|H\rangle = |0\rangle + (\sqrt{2}-1)|1\rangle$, with $|B_{3,3}\rangle = |111\rangle$, $|E_3\rangle$ the uniform even-parity state and $|K_3\rangle = \prod_{i<j} \Lambda(Z)_{i,j}|B_3\rangle$ the triangle graph state. What is new is the certificate, not the decomposition; and the lower half $\chi(|H\rangle^{\otimes 3}) \geq 3$, which [2] does not prove — its only lower bound is the $\Omega(n^{1/2})$ of its Appendix C, which carries no constant and says nothing at $n = 3$.

Theorem 19. $\chi(|T\rangle^{\otimes 4}) = 3$, *and* $3 \leq \chi(|T\rangle^{\otimes 5}) \leq 6$, $3 \leq \chi(|T\rangle^{\otimes 6}) \leq 6$.

The value $\chi(|T\rangle^{\otimes 4}) = 3$ is [1]'s headline qubit result, from which that paper recovers the exponent bound $\gamma_T \leq \frac{1}{4}\log_2 3$ — already obtained in [3] by the cat-state route, as [1] says. That paper obtains the lower half from the $m = 3$ certificate by monotonicity, and here it is direct, by Theorem 13. The upper halves are explicit: the three states of [1, Prop. 8] at $m = 4$, checked against the concrete stabilizer predicate rather than assumed to be stabilizer states, and their tensor products with $\{|0\rangle, |1\rangle\}$ and with a rank-two pair at $m = 5, 6$ (Proposition 20).

Proposition 20. *$\chi(|T\rangle^{\otimes m}) \le 2, 2, 3, 3, 6, 6$ for $m = 1, \ldots, 6$, each by one exact identity in $\mathbb{Z}[i, \sqrt{3}]$ over the 2^m basis indices together with the well-formedness of the states used. At $m = 5, 6$ these are explicit witnesses for cells [1] obtains only as sub-multiplicativity extensions $3 \cdot 2$ of $\chi(|T\rangle^{\otimes 4}) = 3$ and $\chi(|T\rangle^{\otimes 2}) = 2$.*

Proposition 21. *The five Hamming-weight amplitudes of $|T\rangle^{\otimes 4}$ tabulated in [1], $\frac{2+\sqrt{3}}{6}$, $\frac{(\sqrt{3}+1)(1+i)}{12}$, $\frac{i}{6}$, $\frac{(\sqrt{3}-1)(-1+i)}{12}$, $\frac{\sqrt{3}-2}{6}$, follow from $|T\rangle = \cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle$ and $\cos 2\beta = 1/\sqrt{3}$.*

Theorem 22. *$\chi(|H\rangle^{\otimes 4}) \le 4$, $\chi(|H\rangle^{\otimes 5}) \le 6$ and $\chi(|H\rangle^{\otimes 6}) \le 6$, each by an explicit exact witness; hence $3 \le \chi(|H\rangle^{\otimes 4}) \le 4$ and $3 \le \chi(|H\rangle^{\otimes 5}), \chi(|H\rangle^{\otimes 6}) \le 6$.*

Proof sketch. Each bound is one identity between $\mathbb{Z}[i, \sqrt{2}]$-vectors on the 2^m basis indices, plus a well-formedness check on the canonical data. At $m = 4$ the four states are $|1111\rangle$, the uniform odd-parity state, $|0000\rangle$ and the full-support state with amplitude $(-1)^{|x| + \binom{|x|}{2}}$, with coefficients $20 - 14\sqrt{2}$, $-4 + 3\sqrt{2}$, $4 - 2\sqrt{2}$, $-3 + 2\sqrt{2}$. These, and the six states at $m = 5$, were found by an exact search over the 18 m-qubit stabilizer rays that are constant on Hamming-weight classes — the same search that produced the $m = 3$ witness — and both re-find, term for term, the decompositions printed in [2, App. B] (at $m = 4$, $(4 - 2\sqrt{2})|B_{4,0}\rangle + (20 - 14\sqrt{2})|B_{4,4}\rangle + (-4 + 3\sqrt{2})|O_4\rangle + (-3 + 2\sqrt{2})Z^{\otimes 4}|K_4\rangle$; at $m = 5$ the six-term identity there, with the same six coefficients). That search does not reach $m = 6$: an exhaustive sweep over subsets of the eighteen, run outside the formal development, finds none that spans $|H\rangle^{\otimes 6}$. The $m = 6$ witness follows [3] instead: with $|H^{\perp}\rangle \propto (\sqrt{2} - 1)|0\rangle - |1\rangle$, the cat state $|\mathrm{cat}_6\rangle \propto |H\rangle^{\otimes 6} - |H^{\perp}\rangle^{\otimes 6}$ *is* weight-symmetric and its three-state decomposition is re-found by the same search, while $|H\rangle^{\otimes 6} \propto (1 + M_1)|\mathrm{cat}_6\rangle$ with M_1 the Hadamard gate on the first qubit — a Clifford, so it carries those three states to three more stabilizer states. Three of the six landed states are therefore outside the weight-symmetric class, which §6 records. □

The value $\chi(|H\rangle^{\otimes 6}) \le 6$ is [3]'s, obtained there from $\chi(\mathrm{cat}_6) \le 3$, and it is what disproves the conjecture $\chi_6 = 7$ of [2]. Only at $m = 6$ is the witness itself not already on record: [3] prints the three states of $|\mathrm{cat}_6\rangle$ but not the six-state decomposition of the magic state that its $|T\rangle^{\otimes m} \propto |\mathrm{cat}_m\rangle + (A \otimes I)|\mathrm{cat}_m\rangle$ implies, and the $m = 6$ decomposition printed in [2, eq. (H6)] has seven states, not six. Note the bracket at $m = 4$: the exact statement proved here is $3 \le \chi(|H\rangle^{\otimes 4}) \le 4$, not the equality 4 of the table, because the lower half of that cell rests on the floating-point search discussed in the introduction.

Proposition 23. *The five six-qubit stabilizer states of the conjecturally optimal rank-five span distributed with the software library [13] — two graph states with local phases on the complement of the 6-cycle $(0, 1, 4, 5, 2, 3)$ and of its cyclic shift, together with $|i{+}0i{+}0\rangle$, $|0i{+}0i{+}\rangle$, $|{+}0i{+}0i\rangle$ — do not span $|T\rangle^{\otimes 6}$.*

The certificate is the dual functional of Proposition 3(ii), supported on the six basis indices $\{0, 1, 2, 3, 4, 9\}$ and computed in $\mathbb{Z}[i, \sqrt{3}]$, so the conclusion holds over $\mathbb{C}$. The same negative fact is reported by that software as a least-squares residual $\sqrt{5(2 - \sqrt{3})/24}$, reproduced here to twelve digits before the exact certificate was built; no floating-point solve enters the proof.

Remark 24. Whether the two open cells admit rank-five witnesses is not decided here. Outside the formal development we ran a structured search over the pool consisting of the 18 weight-symmetric stabilizer rays at $n = m$ together with their images under the Hadamard gate on the first qubit, 36 rays after deduplication — the pool that contains all the witnesses of Theorem 22 and Proposition 20. All $\binom{36}{k}$ subsets with $k \leq 4$ were exhausted for $|T\rangle^{\otimes 5}$, $|T\rangle^{\otimes 6}$ and $|H\rangle^{\otimes 6}$ with no decomposition found, and, as a control, none of rank ≤ 3 for $|H\rangle^{\otimes 4}$ either, which is what the table predicts. The $k = 5$ layer was not run. Being a search over a restricted pool, a negative here is a measurement and not a theorem; it is recorded so that the next attempt does not repeat it.

6. Negative controls

Every finite argument above can be satisfied vacuously by a misplaced quantifier, so the development carries controls that fail on purpose. We list what they establish.

Proposition 25. *In the formal development: the value bound 5^k of Lemma 6 is attained at $k = 1$ by an explicit three-qubit stabilizer ray taking all of $0, 1, i, -1, -i$; the type map is not injective without its hypothesis; the class-size hypothesis of Proposition 8 holds with $6 < 8$ at $n = 4$ and is* not applicable *at $n = 3$, where only four classes exist — exactly the boundary between the 256-case argument of Proposition 17 and the computation-free Theorem 13. For Lemma 9: the 31 is computed, not assumed (125 codes, 32 representatives, exactly one fixed point), the rotation has order exactly 4 on a nonzero code, so the quotient is worth a factor 4 and not less, and at $m = 30$ there are exactly 31 weight classes — so the threshold $m \geq 31$ of Theorem 15 cannot be lowered by this argument. For the witnesses: dropping one state or perturbing one integer coordinate of one coefficient breaks each identity, so every state listed is load-bearing; the too-small claims $\chi = 2$ and the too-large claims $\chi = 5$ at $m = 4$ and $\chi = 7$ at $m = 6$ (the latter being the disproved conjecture of* [2]*) are refuted; the four-equal-values instance of the 256-case core* does *cancel, so that core is a statement about the magic states and not about the shape of the quantifier; a deliberately non-echelon canonical datum has the same amplitudes as its echelon form while the data differ; and the $m = 6$ witness is certified to leave the weight-symmetric class, so that the failure of the 18-ray search there is not an artifact of the encoding.*

7. Verification

Every numbered lemma, proposition and theorem above is formalised and machine-checked in Lean 4 [14] on top of Mathlib [15], and the checks are reproducible from the accompanying development; repository reference to be added at submission. The three numbered remarks are the exception, and two computations reported in them are deliberately *not* part of that development: the stabilizer-state counts of Remark 2, and the restricted pool search of Remark 24. The development is deliberately kernel-only: no compiled evaluation (`native_decide`) and no external oracle is used anywhere, so every finite computation named above — the 125-code orbit count of Lemma 9, the 256-case core of Proposition 17, the class-size checks feeding Theorem 13, and the exact identities on 2^m basis indices behind Proposition 20, Theorem 22 and Proposition 23 — is reduced by the proof kernel itself. An audit of the axiom set of all 64 declarations of the three modules written for the present results found nothing beyond `propext`, `Classical.choice` and `Quot.sound`, and in particular no incomplete proof; 56 use exactly those three, 6 use fewer, and 2 use none at all. Two conventions of the formalisation are worth restating, since they bound what the theorems mean: stabilizer states are the states (1), whose identification with the Pauli stabilizer states is cited rather than proved (Remark 2), and rank is defined projectively, over rays. Every numerical value asserted in Lean was recomputed beforehand in independent exact arithmetic, including the orbit count $125 = 1 + 4 \cdot 31$, the weight-class amplitude tables, and every witness identity; one transcription error — coefficients expressed against raw state vectors rather than against the canonical-form amplitudes, which corrupts two of the 64 indices at $m = 6$ — was caught by exactly that check.

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[4] S. Peleg, A. Shpilka and B. L. Volk, *Lower Bounds on Stabilizer Rank*, Quantum **6** (2022), 652; arXiv:2106.03214. Lemma 3.1 is the one carrying the constant 1/100 and Theorem 3.2 its application to the magic states; the numbering is that of v2, the version accepted by the journal.

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