SKIRTING SETS AND THE INVERSE FOOTBALL POOL PROBLEM, WITH $f(8,4) \leq 36$

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Abstract. A tuple $x \in \mathbb{Z}_q^n$ *skirts* $y \in \mathbb{Z}_q^n$ if $x_i \neq y_i$ for every coordinate i, and $S \subseteq \mathbb{Z}_q^n$ is a *skirting set* if every tuple is skirted by some element of S; $f(n,q)$ denotes the least size of a skirting set. This is the *inverse football pool number* $T(n,q)$ of Östergård and Riihonen and of Brink, studied since the 1950s. It was reintroduced in 2026 by Adriaensen, Ihringer, Martin and Villagrán, who named skirting sets and identified $f(n,q)$ with the total domination number of a direct power of a complete graph, and by Kuang and Wang as the number of binary subcubes needed to cover the ternary cube; neither 2026 paper cites the older work, and every interval in the tables they print is superseded by it. Already known are $f(7,3) = 29$ and $f(8,3) = 44$ (Brink), and $f(6,4) = 15$ and $f(7,4) \leq 23$ (Ashik Mathew and Östergård, under the name of blocking sets in a discrete torus). One cell is improved here: $f(8,4) \leq 36$, against the published 40, so that the record for that cell becomes $[27,36]$. We also give explicit machine-checked witnesses at every cell of the 2026 tables — among them 44 tuples of $\mathbb{Z}_3^8$, 29 of $\mathbb{Z}_3^7$ and 15 of $\mathbb{Z}_4^6$ realising three of the known values above — each verified by complete evaluation from the definition, and three of them irredundant. We record besides that the growth constant $C_q = \inf_{n\geq 1} f(n,q)^{1/n}$ equals $q/(q-1)$ for every $q \geq 2$, granted the Johnson–Lovász–Stein covering estimate as it has been printed for general q; the matching lower bound is a counting argument, and for $q = 3$ no unproved estimate is needed. Every theorem below has been machine-checked in Lean 4.

1. Introduction

Let $q \geq 2$ and $n \geq 1$, and write $\mathbb{Z}_q^n$ for the set of q-ary n-tuples. Say that x *skirts* y when x and y differ in *every* coordinate, that is when their Hamming distance is n, and call $S \subseteq \mathbb{Z}_q^n$ a *skirting set* when every tuple of $\mathbb{Z}_q^n$ is skirted by at least one element of S. The *skirting number* $f(n,q)$ is the size of a smallest skirting set. These are the definitions of Adriaensen, Ihringer, Martin and Villagrán [1], who introduced the quantity and observed that it is the total domination number of the graph on $\mathbb{Z}_q^n$ whose edges join tuples at Hamming distance exactly n — equivalently, of the direct (categorical) power $K_q^{\times n}$ of the complete graph on q vertices, so that $f(n,q) = \gamma_t(K_q^{\times n})$ in the usual notation for the total domination number. The condition is total domination and not domination: no tuple skirts itself, so a single tuple is never a skirting set, however it is chosen.

The quantity is much older than that terminology. The set of tuples skirted by x is the sphere $\{y : d(x,y) = n\}$ of maximal radius about x, so a skirting set is a covering of $\mathbb{Z}_q^n$ by such spheres, and $f(n,q)$ is Brink's $T(n,q)$ [3] — the case $p = q-1$ of the function $T(n,q,p)$ introduced by Östergård and Riihonen [4]. Brink defines it exactly this way and proposes to call the problem of determining $T(n,3)$ the *inverse football pool problem*: the least number of bets on n three-outcome games that guarantees one bet with every outcome wrong. The values $T(n,3)$ for $n \leq 6$ were found in the Finnish football pool magazine *Veikkaaja* in the 1950s, together with the bounds $T(7,3) \leq 29$ and $T(8,3) \leq 44$; that material was brought into the mathematical literature by [4] in 2003, and [3] proved in 2011 that those two bounds are equalities. The sequence is OEIS `A086676` [8]. Neither [1] nor [2] cites any of this, and the consequences for their tables are recorded below.

Two regimes are settled by argument. Over a binary alphabet each tuple skirts exactly one tuple, so $f(n,2) = 2^n$ and nothing is left to say; and $f(n,q) = n+1$ whenever $0 < n < q$, which is equation (2) of [3] and is proved again, independently, in [1]. For $3 \leq n < q$ that value can also be read off the lower bound $\gamma(K_{n_1} \times \cdots \times K_{n_t}) \geq t+1$, valid for $t \geq 3$, of Mekiš [9], where γ is the ordinary domination number: $\gamma_t \geq \gamma$ always, and $n+1$ constant tuples already skirt everything. The interesting range is therefore $2 < q \leq n$, and there [1] computes. Its table — reproduced as Table 1 below — gives $f(n,q)$ for $q \in \{3,4\}$ and $2 \leq n \leq 8$, exactly when a single value is printed and as an interval containing $f(n,q)$ otherwise. Five cells are intervals:

$$f(7,3) \in [28,30], \qquad f(8,3) \in [29,54], \qquad f(6,4) \in [13,16],$$
$$f(7,4) \in [14,28], \qquad f(8,4) \in [15,40].$$

The paper says of the whole table only that its values "were found with Gurobi using a standard integer linear program". One later paper has moved one of these cells: for $q = 3$ a skirting set is precisely a covering of the ternary cube by subcubes with every side of size 2, and under that description [2] proves $f(n,3) \geq (3/2)^{n-k} f(k,3)$ and $f(n,3) \leq 2(3/2)^n - 1$, from which they deduce $41 \leq f(8,3) \leq 50$. Their methods are ternary — they write that their proof "uses a parity construction that is specific to the ternary setting" — and their concluding section raises the general-q case, about which it poses its Problem 5.1 (Remark 8.6 below).

Every one of those five intervals is superseded. Brink's Theorem 4 gives the exact values $f(7,3) = 29$ and $f(8,3) = 44$; his Theorem 6 gives $f(6,4) \in [14,16]$ and $f(7,4) \in [19,28]$. The quaternary column then moves again, from a direction further away still. A set of side-2 hypercubes in the discrete torus of width 4 meeting every side-2 hypercube is called a *blocking set*, and the least size $h(d)$ of one in dimension d is $f(d,4)$. Indeed each side-2 hypercube of C_4^d is $\prod_i \{a_i, a_i+1\}$ for a unique label $a \in \mathbb{Z}_4^d$; two of them meet if and only if their labels satisfy $b_i \neq a_i + 2$ in every coordinate, since in C_4 a pair of adjacent vertices misses exactly the opposite pair; so translating every label by the constant 2 tuple carries blocking sets to skirting sets and back. Ashik Mathew and Östergård [5] classified these, and [6, §2.5, §4] records the outcome as $h(6) = 15$ and $h(7) \leq 23$. Iterating the projection inequality $f(n,q) \geq \frac{q}{q-1} f(n-1,q)$ of [3, (1)] from $f(6,4) = 15$ then gives $f(7,4) \geq 20$ and $f(8,4) \geq 27$. So the published record at the five cells is 29, 44, 15, $[20,23]$ and $[27,40]$, and only the last of these is improved below. Table 2 sets it beside what is proved here.

We have not been able to read [5] itself, which is behind a paywall; the two values quoted from it are taken from [6], the doctoral dissertation of its first author, which summarises it. They should be confirmed against the article. Whether [5] also gives a lower bound for $h(7)$ better than the 20 above, or anything at all about $h(8)$, we do not know.

[1] closes with a numbered list of three problems. Problem 1 asks to "[d]etermine the exact values of the constants C_q", where

$$C_q \;=\; \inf_{n \geq 1} \, f(n,q)^{1/n},$$

"or at least improve the bounds presented in Proposition 3", namely the bracket $1 + \frac{1}{q-1} \leq C_q \leq q^{1/(q-1)}$. Problem 2 asks for constructions of small skirting arrays. Problem 3 asks for better lower bounds on the difference $f(n,q) - f(n-1,q)$, starting from the observation used in the proof of its Lemma 6 that deleting a coordinate from a skirting set of $\mathbb{Z}_q^n$ leaves a skirting set of $\mathbb{Z}_q^{n-1}$ "with the property that none of its elements are essential, in the sense that after deleting any element, we are still left with a skirting set". It is only the notion introduced in that sentence — an element of a skirting set is *essential* when deleting it destroys the skirting property — that recurs below; this paper says nothing about the lower bounds Problem 3 asks for.

Results. This paper improves the best published upper bound at one cell, $f(8,4)$; gives machine-checked explicit witnesses at every cell of Table 1; and answers Problem 1 — for $q = 3$ from theorems proved in [2], and for general q granted in addition the covering estimate that [2] states in its concluding section. Everything rests on explicit sets: each upper bound below is one list of tuples, printed in full in Appendix A, and the skirting condition is then verified by complete evaluation over all q^n tuples.

The quaternary cells $n = 7, 8$ (Section 5). We prove $f(7,4) \leq 24$ and $f(8,4) \leq 36$. Only the second is new: $f(7,4) \leq 23$ is already in [5], so the 24 is beaten by one, and the record at $(7,4)$ stays $[20,23]$.

At $(8,4)$ the published bracket $[27,40]$ becomes $[27,36]$. The published upper ends at these two cells are 28 and 40, and both coincide exactly with what the subadditivity $f(n+m,q) \le f(n,q)f(m,q)$ — equation (3) of [3], and re-proved in [1] — yields from the exact values in the same table, namely $f(4,4)f(3,4) = 7\cdot 4$ and $f(5,4)f(3,4) = 10\cdot 4$; we reproduce both of these product sets explicitly, so that the numbers the improvement is measured against are themselves verified rather than quoted. The only published general-q upper bound that is a closed formula in n and q, the Johnson–Lovász–Stein covering estimate as restated in [2, §5], gives 65.1 and 97.8 at these two cells and so does not compete; the two other published routes are weighed in Remark 5.5. Both sets are irredundant.

The ternary cell $n = 8$ (Section 6). We exhibit a 44-element skirting set of $\mathbb{Z}_3^8$, verified from the definition, and it is irredundant. This is *not* an improvement: $f(8,3) \le 44$ goes back to *Veikkaaja* and $f(8,3) = 44$ is Theorem 4 of [3]. Measured against the 2026 papers alone it would read as one, 44 against the 50 of [2] and the 54 of [1], and it is worth saying plainly that it is not. What the section does contribute is a witness found and checked here rather than quoted, and the price of the neighbouring cell 43 — which is not open either, being excluded by [3].

Two further cells (Section 7). We exhibit a 15-element skirting set of $\mathbb{Z}_4^6$ and a 29-element skirting set of $\mathbb{Z}_3^7$. Both match rather than improve the literature: $f(6,4) = 15$ is [5] as recorded in [6], and $f(7,3) = 29$ is the other half of Theorem 4 of [3].

The growth constant (Section 8). We give a proof of the counting bound $f(n,q) \ge (q/(q-1))^n$ of [1], which yields $C_q \ge q/(q-1)$ unconditionally, and we show that *any* upper bound of the shape $f(n,q) \le A\, n\, (q/(q-1))^n$ forces $C_q \le q/(q-1)$. The Johnson–Lovász–Stein estimate printed in [2, §5] has that shape for every q, and the ternary construction proved in [2] has it for $q = 3$; hence $C_q = q/(q-1)$ for every $q \ge 2$, and $C_3 = 3/2$ with no unproved estimate used at all. Both halves are published; what is new here is only the observation that they meet, and the verification. At $q = 3$ the asymptotics are known more precisely still: Barnes and Jaffe [7] prove $f(n,3) = (\Lambda_3 + o(1))(3/2)^n$ with $7/4 \le \Lambda_3 \le 2$.

Sections 3 and 4 describe the method and record what the exact cells of Table 1 become when they are revisited from the definition. Section 9 says precisely what the machine verification covers and where compiled evaluation, rather than kernel reduction, is used. The neighbouring literature on domination in direct products of complete graphs works in the regime of few, large factors, which is the opposite of the one here; what it offers at these lengths is the submultiplicativity of γ_t over direct products, and that is (1) in graph language, giving exactly the 28 and 40 above. Numbered statements of [1], [2] and [7] are cited by the numbering of the arXiv versions listed in the bibliography; none of the three had appeared in a journal when this was written.

2. Preliminaries

2.1. **Skirting sets.**

Definition 2.1. For $x, y \in \mathbb{Z}_q^n$ say that x *skirts* y if $x_i \ne y_i$ for every $i \in \{1,\dots,n\}$. A finite set $S \subseteq \mathbb{Z}_q^n$ is a *skirting set* if for every $y \in \mathbb{Z}_q^n$ there is an $x \in S$ skirting y, and

$$f(n,q) \;=\; \min\{\, |S| : S \subseteq \mathbb{Z}_q^n \text{ a skirting set} \,\}.$$

For $q \ge 2$ the whole space is a skirting set, so the minimum is over a nonempty set of integers and $f(n,q)$ is well defined and attained. Two elementary consequences are used constantly: exhibiting a skirting set of size m proves $f(n,q) \le m$, and showing that no set of size at most k is skirting proves $f(n,q) > k$. The set of tuples skirted by a single x is the product $\prod_{i=1}^n \bigl(\mathbb{Z}_q \setminus \{x_i\}\bigr)$, of size $(q-1)^n$; for $q = 3$ these are exactly the subcubes of the ternary cube with all sides of size 2, which is the description used in [2].

Proposition 2.2 ([1]). *$f(n,2) = 2^n$ for every n, and $f(n,q) = n+1$ whenever $0 < n < q$.*

Proof. For $q = 2$ each tuple skirts exactly one tuple, its complement, so a skirting set must be all of $\mathbb{Z}_2^n$. For $n < q$: the $n+1$ constant tuples with $n+1$ distinct values form a skirting set, since an n-tuple has at most n distinct entries and therefore misses one of the $n+1$ values in every coordinate;

conversely, given $x_1, \dots, x_n$, the diagonal tuple y with $y_i = (x_i)_i$ is skirted by none of them. Both halves are those of [1]; for $3 \le n < q$ the lower half is also implied by the bound of Mekiš [9] recalled in Section 1. $\square$

Proposition 2.3 (the counting bound, [1])**.** *For $q \ge 2$ and every skirting set $S \subseteq \mathbb{Z}_q^n$ one has $q^n \le |S|\,(q-1)^n$. In particular $f(n,q) \ge \big(q/(q-1)\big)^n$.*

Proof. The sets $T_x = \prod_i (\mathbb{Z}_q \setminus \{x_i\})$, for $x \in S$, cover $\mathbb{Z}_q^n$ by the definition of a skirting set, and each has $(q-1)^n$ elements; so $q^n = |\mathbb{Z}_q^n| \le \sum_{x \in S} |T_x| = |S|(q-1)^n$. This is the argument printed in [1]; we reproduce it because it is the one estimate of that paper used in Section 8 in a proved rather than a quoted form. $\square$

We use one further result of [1], equation (3) of [3], quoted and not reproved: if $S_n \subseteq \mathbb{Z}_q^n$ and $S_m \subseteq \mathbb{Z}_q^m$ are skirting sets then the set of concatenations $S_n \times S_m \subseteq \mathbb{Z}_q^{n+m}$ is a skirting set, whence

$$f(n+m,q) \;\le\; f(n,q)\,f(m,q). \tag{1}$$

Only two instances of (1) occur below, and both are re-derived from explicit sets in Section 5 rather than used as quoted numbers.

n	2	3	4	5	6	7	8
$f(n,3)$	3	5	8	12	18	$[28,30]$	$[29,54]$
$f(n,4)$	3	4	7	10	$[13,16]$	$[14,28]$	$[15,40]$

Table 1. The table of [1]. A single entry is the exact value; an interval contains the value. For $f(8,3)$ the interval was later narrowed to $[41,50]$ in [2]. All five intervals are superseded by [3]; see Table 2.

2.2. **The published table.** Table 1 is the object this paper works on. Its exact cells are revisited here from Definition 2.1 (Section 4), five of the nine from both sides; each of its five bracketed cells gets an explicit witness (Sections 5–7), and at one of them the witness is smaller than anything published. Table 2 collects the outcome and sets it beside the published record.

cell	[1]	best published	source	here	record
$f(7,3)$	$[28,30]$	29	[3], Thm. 4	≤ 29	29
$f(8,3)$	$[29,54]$	44	[3], Thm. 4	≤ 44	44
$f(6,4)$	$[13,16]$	15	[5], via [6]	≤ 15	15
$f(7,4)$	$[14,28]$	$[20,23]$	[5], via [6]	≤ 24	$[20,23]$
$f(8,4)$	$[15,40]$	$[27,40]$	derived, see text	≤ 36	$[27,36]$

Table 2. The five cells that [1] leaves as intervals: what [1] prints, what was already published, what is proved here, and the resulting record. Only at $f(8,4)$ does the “here” column improve the “best published” one; at $(7,4)$ it is beaten by one, and at the other three cells it meets a known value. The lower ends 20 at $(7,4)$ and 27 at $(8,4)$, and the upper end 40, are the projection inequality and subadditivity of [3] applied to $f(6,4) = 15$ and to $f(5,4)f(3,4)$; no lower end is reproved here.

3. Method

3.1. **From tuples to integers.** Every statement proved below is a statement about $f(n,q)$, but the objects that a machine evaluates are lists of integers. Identify $x \in \mathbb{Z}_q^n$ with its base-q code $\sum_{i=1}^n x_i q^{i-1} \in \{0,\dots,q^n-1\}$, so that $x_i = \lfloor x/q^{i-1} \rfloor \bmod q$, and let $\mathrm{skirts}_{n,q}$ and $\mathrm{isSkirting}_{n,q}$ be the corresponding Boolean predicates on codes and on lists of codes.

Proposition 3.1. *Let L be a list of integers, all less than q^n.*

1. *If* $\mathrm{isSkirting}_{n,q}(L)$ *holds then* $f(n,q) \le |L|$.
2. *If* $\mathrm{isSkirting}_{n,q}(L)$ *fails for* every *such list L of length at most k, and $q > 1$, then $f(n,q) > k$.*

Proof. The base-q code is a bijection $\mathbb{Z}_q^n \to \{0,\dots,q^n-1\}$ carrying the skirting relation to $\mathrm{skirts}_{n,q}$ coordinatewise, so a passing list yields a skirting set whose size is at most the length of the list (repetitions can only shorten it), and conversely a skirting set of size at most k yields a passing list of length at most k. □

Part (1) is how every upper bound in this paper is obtained: one list, and one evaluation of the predicate $\mathrm{isSkirting}_{n,q}$, which is a loop over all q^n tuples asking whether some element of the list skirts it. Part (2) is how the exact values of Section 4 are obtained, through the two devices of the next two subsections.

3.2. **Normalisation.**

Proposition 3.2. *Let $\sigma_1,\dots,\sigma_n$ be permutations of $\mathbb{Z}_q$ and let $\Sigma(x) = (\sigma_1(x_1),\dots,\sigma_n(x_n))$. Then Σ carries skirting sets to skirting sets of the same size. Consequently, for $q \ge 2$, every skirting set can be replaced by a skirting set of no larger size containing both the constant* 0 *tuple and the constant* 1 *tuple.*

Proof. Coordinatewise relabelling preserves "differs in every coordinate", and it is a bijection, which gives the first claim. For the second, pick $x_0 \in S$ skirting the constant 0 tuple and apply the relabelling $\sigma_i = (x_{0,i}\ 0)$, which moves x_0 onto the constant 0 tuple; in the image pick x_1 skirting the constant 0 tuple, so $x_{1,i} \ne 0$ for all i, and apply $\tau_i = (x_{1,i}\ 1)$, which moves x_1 onto the constant 1 tuple and fixes the constant 0 tuple because τ_i moves only the two nonzero symbols $x_{1,i}$ and 1. □

The effect on an exhaustive search is large: at $n = 4$, $q = 3$ it takes a branch-and-bound from $2.9 \cdot 10^8$ nodes to $1.1 \cdot 10^6$, a factor of about 260.

3.3. **A search whose completeness is proved.** For a list S of codes let $U(S)$ be the list of tuples that S does not skirt. Define $\mathrm{search}(k,U)$ to be true if U is empty; false if U is nonempty and $k = 0$; and otherwise the disjunction, over the $(q-1)^n$ codes x that skirt the first element of U, of $\mathrm{search}(k-1,U')$ where U' is U with the tuples skirted by x removed.

Proposition 3.3. *If some list T of at most k codes, all less than q^n, skirts every element of U, then* $\mathrm{search}(k,U)$ *is true. Consequently, if $q \ge 2$, $n \ge 1$ and* $\mathrm{search}\big(k-2,\, U([\underline{0},\underline{1}])\big)$ *is false, then $f(n,q) > k$.*

Proof. Induction on k. If U is nonempty, some $x \in T$ skirts its first element y; that x is one of the branches, and $T \setminus \{x\}$ is a list of at most $k-1$ codes skirting everything left in U', so the branch succeeds by induction. The contrapositive says that a false search refutes every completion of the current partial set by at most k further codes. For the second statement, apply Proposition 3.2: a skirting set of size at most k may be assumed to contain the two constant tuples $\underline{0}, \underline{1}$, and the remaining at most $k-2$ elements would have to skirt everything those two miss. □

Proposition 3.3 is the only route by which a lower bound is obtained in this paper, and the search is run to exhaustion; nothing about it is heuristic. It is the branching, not the value, that the induction certifies, so a search returning false is a proof of nonexistence.

3.4. **How the new sets were found.** The upper bounds of Sections 5–7 are not the output of an exhaustive search; they are single sets found by local search and then verified. Two heuristics were used. The cells $f(7,3)$, $f(6,4)$ and $f(8,3)$ fell to simulated annealing on the $m \times n$ array of candidate tuples with cost the number of unskirted tuples and a min-conflicts move (pick an unskirted tuple, a row, and a coordinate where the row agrees with it, and change that entry). The quaternary cells $n = 7, 8$ did not: at $q = 4$, $n = 7$ a uniformly random row skirts a given tuple with probability $(3/4)^7 \approx 0.134$, so a random 26-row array leaves about 393 tuples unskirted and pure coverage counting asks for about 68 rows; every good set is highly structured, and an annealer restarting from a random array has to rediscover the structure each time. Both new quaternary sets came instead from a weighted local search that (i) starts *warm*, from the product set of (1), and deletes one row at a time, and (ii) gives each tuple a weight that rises each time it is selected while still unskirted, so that the objective changes under a stalled search. A control run with the warm start but without the weighting stalls at $n = 8$, $q = 3$, $m = 43$ after $4.59 \cdot 10^6$ moves; the weighting is what moves the quaternary cells.

None of this enters any proof. The heuristics produce a candidate list; what is proved is that the list works, and that is a complete evaluation over all q^n tuples.

4. The exactly known cells, revisited

Before any bound is measured against Table 1, its nine exact cells were revisited here from Definition 2.1. Five of them are settled from both sides below. For the other four — $f(5,3)$, $f(6,3)$, $f(4,4)$ and $f(5,4)$ — the table's value is matched here by an explicit skirting set, and that the value cannot be lowered is quoted from [1] — equivalently from [3], whose appendix also prints witnesses — and not reproved.

Proposition 4.1. *$f(2,3) = 3$, $f(3,3) = 5$, $f(2,4) = 3$ and $f(3,4) = 4$.*

Proof. The upper bounds are the four explicit sets $\{11, 20, 02\}$ for $(2,3)$; $\{010, 021, 111, 120, 202\}$ for $(3,3)$; $\{20, 11, 02\}$ for $(2,4)$; and $\{330, 111, 022, 203\}$ for $(3,4)$, written with $x_i = \lfloor x/q^{i-1} \rfloor \bmod q$ as above. The lower bounds are four exhaustive searches in the sense of Proposition 3.3, each of which returns false: with the two constant tuples fixed, no $k - 2$ further tuples finish the job, for $k = 2, 4, 2, 3$ respectively. The three values with $n < q$ agree with Proposition 2.2, by a route with nothing in common. □

Proposition 4.2. $f(4,3) = 8$.

Proof. The upper bound is the 8-element set read off the first matrix displayed in [1]; see Remark 4.4. The lower bound is the search of Proposition 3.3 at $k = 7$: with the two constant tuples fixed by Proposition 3.2 it visits 1 118 481 nodes and returns false. This is the largest cell settled from below here. □

Proposition 4.3. *There are explicit skirting sets realising $f(5,3) \le 12$, $f(6,3) \le 18$, $f(4,4) \le 7$ and $f(5,4) \le 10$, and the second matrix displayed in [1] is a skirting set of $\mathbb{Z}_3^6$ of size 21.*

Remark 4.4. [1] announces witnesses for $f(5,3)$, $f(6,3)$ and $f(5,4)$ and then displays a 4×8, a 6×21 and a 5×10 matrix. Read as columns, all three are genuine skirting sets — we have verified all three — but only the third matches its label. The first has 8 columns of length 4 and the second 21 columns of length 6: these are the values 8 and 21 of the even-n construction of [1] at $n = 4$ and $n = 6$, not the table's $f(5,3) = 12$ and $f(6,3) = 18$. At $n = 4$ that construction happens to be optimal, so the first matrix is a witness for $f(4,3) = 8$ and is used as such in Proposition 4.2; at $n = 6$ it is not, since $21 > 18$. Witnesses for $f(5,3) = 12$, $f(6,3) = 18$, $f(4,4) = 7$ and $f(5,4) = 10$ are nevertheless on record: the appendix of [3] prints the lexicographically least covering matrix at each of these cells, and at $f(7,3) = 29$ as well. We have checked all five of those matrices against Definition 2.1 too, outside the formal development. The sets behind Proposition 4.3 are different sets, found here by search; they are printed in Appendix A.

Remark 4.5. One of those cells has a witness with structure, which we record because the sets found by local search have none. Every element of the wreath product $S_q \wr S_n$ — permute the coordinates, then relabel the symbols of each coordinate — is an automorphism of the skirting relation, since "differ in every coordinate" is preserved by both operations. Searching for a union of orbits of a prescribed cyclic group therefore makes sense, and at $(6,3)$ it produces a skirting set of size 18, the value Table 1 gives for that cell, invariant under the cyclic shift of the six coordinates: the union of the five shift-orbits of `111111`, `202020`, `100100`, `211200` and `122210`, of sizes 1, 2, 3, 6 and 6. Neither the witnesses of Proposition 4.3 nor those of the sections below have any such symmetry. This computation lies outside the formal development; nothing else in the paper depends on it.

Remark 4.6. Several checks guard against a mis-stated predicate. The values of Propositions 4.1 and 4.2 are refuted in both directions ($f(4,3) \neq 7$, $f(4,3) \neq 9$, $f(3,3) \neq 4$, $f(3,3) \neq 6$), so neither side is vacuous. Deleting either the first or the last element of the 8-element set of Proposition 4.2 destroys the skirting property, so "is a skirting set" is not satisfied by every list. The search of Proposition 3.3 returns *true* when given one more unit of budget than a lower bound allows, so a false verdict is not an artefact of the budget bookkeeping. No tuple skirts itself and no singleton is a skirting set, which is the one definitional trap: this is total domination, not domination. The 5-element skirting set of $\mathbb{Z}_3^3$ used above does not contain the constant 0 tuple, so the normalised search would miss it and is sound only through Proposition 3.2. Finally $f(3,2) = 8 = 2^3$ exhibits the degenerate binary case of Proposition 2.2 from both sides.

5. The quaternary cells $n = 7$ and $n = 8$

The two widest cells of Table 1 are $f(7,4) \in [14, 28]$ and $f(8,4) \in [15, 40]$; the published records are $f(7,4) \in [20, 23]$, the upper end from [5] and the lower from the projection inequality on $f(6,4) = 15$, and $f(8,4) \in [27, 40]$, one projection step further down and subadditivity above. Both of the table's upper ends are exactly what (1) gives from the exact quaternary cells of the same table, and we begin by reproducing them as explicit sets, so that the baseline is verified rather than quoted.

Proposition 5.1. *Let $W_{4,4}$ and $W_{5,4}$ be the skirting sets of $\mathbb{Z}_4^4$ and $\mathbb{Z}_4^5$ of sizes* 7 *and* 10 *from Proposition 4.3, and $W_{3,4}$ the* 4*-element skirting set of $\mathbb{Z}_4^3$ from Proposition 4.1. Then the concatenation products $W_{4,4} \times W_{3,4}$ and $W_{5,4} \times W_{3,4}$ are skirting sets of $\mathbb{Z}_4^7$ and $\mathbb{Z}_4^8$ of sizes* 28 *and* 40*. In particular $f(7,4) \leq 28$ and $f(8,4) \leq 40$.*

Proof. On base-q codes, concatenating a tuple of $\mathbb{Z}_q^a$ with one of $\mathbb{Z}_q^b$ is $(x, y) \mapsto x + q^a y$, so the two products are explicit lists of 28 and 40 integers built from the three sets named, with no numerals of their own. That each is a skirting set is a complete evaluation over the $4^7 = 16\,384$ resp. $4^8 = 65\,536$ tuples; both pass. The values agree with (1) and with the entries of Table 1. □

Theorem 5.2. $f(7,4) \leq 24$.

Proof. The set $W_{7,4}$ of 24 tuples of $\mathbb{Z}_4^7$ printed in Appendix A is a skirting set. The verification is a complete evaluation of the skirting condition over all $4^7 = 16\,384$ tuples of $\mathbb{Z}_4^7$: for each tuple, some element of $W_{7,4}$ differs from it in all seven coordinates. Proposition 3.1(1) then gives the bound. The set was found as described in Section 3.4, by weighted local search descending from the 28-element product set of Proposition 5.1; it was verified twice, once by an independent implementation working directly from Definition 2.1 (zero unskirted tuples out of 16 384) and once in the formal development. □

Theorem 5.2 does not reach the published record: [5] gives $f(7,4) \leq 23$, one better. It is kept for its witness, which is explicit and checked here from the definition, and because the search that produced it is the same one that moves the $n = 8$ cell below.

Theorem 5.3. $f(8,4) \leq 36$.

Proof. The same argument for the set $W_{8,4}$ of 36 tuples of $\mathbb{Z}_4^8$ printed in Appendix A: a complete evaluation over all $4^8 = 65\,536$ tuples finds each of them skirted, with zero exceptions, and Proposition 3.1(1) applies. The set descends by weighted local search from the 40-element product set of Proposition 5.1. □

Proposition 5.4. *Both sets are irredundant: for each of the* 24 *elements of* $W_{7,4}$ *and each of the* 36 *elements of* $W_{8,4}$*, deleting that element leaves a set that is not skirting. Deleting the first element of* $W_{7,4}$ *leaves* 23 *tuples of* $\mathbb{Z}_4^7$ *unskirted, and deleting the first element of* $W_{8,4}$ *leaves* 132 *tuples of* $\mathbb{Z}_4^8$ *unskirted.*

Proof. Sixty complete evaluations, one per deletion, all failing. Each failing evaluation stops at its first unskirted tuple, so the whole probe costs about what one successful evaluation costs. The two counts in the second sentence are complete evaluations run to the end. □

So every element of both sets is *essential* in the sense of [1], and neither bound is one deletion away from the next: a 23-element skirting set of $\mathbb{Z}_4^7$ or a 35-element skirting set of $\mathbb{Z}_4^8$, if one exists, is a different set.

Remark 5.5. Three published *general* routes bear on these two cells, and none of them reaches 24 or 36; the classification of [5] does reach 23 at $(7,4)$, by a computation of a different kind, and says nothing about $(8,4)$. The subadditivity (1), which [1] records inside the proof of its Theorem 1 rather than as a numbered statement, gives 28 and 40, which are the table entries. The Johnson–Lovász–Stein estimate $f(n,q) \leq (1 + n\ln(q-1))(q/(q-1))^n$ restated in [2, §5] evaluates to 65.10 at $(7,4)$ and 97.78 at $(8,4)$. Proposition 8 of [1] bounds $f(n,q)$ by $q - v + \mathrm{SAN}(n+v-q, n, v)$ for $1 \leq v < q$, where $\mathrm{SAN}(t,m,v)$ is the least number of rows of a skirting array of strength t on m columns over a v-letter alphabet; at $(7,4)$ this is $2 + \mathrm{SAN}(5,7,2)$ for $v = 2$, and $\mathrm{SAN}(t,m,2)$ is the binary covering array number $\mathrm{CAN}(t,m,2)$, as [1] notes just before its Lemma 7, so $\mathrm{SAN}(5,7,2) \geq 2^5 = 32$ and that route cannot give better than 34. For $v = 3$ it is $1 + \mathrm{SAN}(6,7,3)$, a strength-6 skirting array over $\mathbb{Z}_3$ on 7 columns, which [1] does not evaluate; $v = 1$ is vacuous, no array over a one-letter alphabet having positive strength.

Corollary 5.6. $f(8,4)^6 < 15^8$*; equivalently* $f(8,4)^{1/8} < 15^{1/6}$.

Proof. $36^6 = 2\,176\,782\,336 < 2\,562\,890\,625 = 15^8$. □

Corollary 5.6 improves the best finite certificate this paper has for the growth constant of Section 8: $f(8,4)^{1/8} < 1.5651$, against $f(6,4)^{1/6} < 1.5705$ from Theorem 7.2 and $f(5,4)^{1/5} < 1.5849$ from Table 1. It says nothing about C_4 that Theorem 8.3 does not say better. The $n = 7$ cell gives no such certificate: $24^6 = 191\,102\,976 > 170\,859\,375 = 15^7$, so $f(7,4)^{1/7} > 15^{1/6}$.

6. The ternary cell $n = 8$

This cell is closed, and has been since 2011: $f(8,3) = 44$, by Theorem 4 of [3], whose lower half comes from the projection inequality applied to $f(7,3) = 29$ and whose upper half is a covering published in *Veikkaaja* and reported in [4]. Neither 2026 paper knows this. [1] prints $f(8,3) \in [29, 54]$; [2] proves $f(n,3) \geq (3/2)^{n-k} f(k,3)$ in its Lemma 4.1 and $f(n,3) \leq 2(3/2)^n - 1$ in its Theorem 1.2, and, taking $k = 6$ in the first with $f(6,3) = 18$ from Table 1 and $n = 8$ in the second, records $41 \leq f(8,3) \leq 50$. What this section adds is therefore not a bound but a witness: a 44-element skirting set of $\mathbb{Z}_3^8$ found by local search and checked from the definition, independent of the one behind [4], which we have not seen.

Proposition 6.1. *There is a skirting set of* $\mathbb{Z}_3^8$ *of size* 45*; hence* $f(8,3) \leq 45$.

Proof. The set is the one named at the end of Appendix A, and the verification is a complete evaluation over all $3^8 = 6\,561$ tuples. It is what simulated annealing produced first, from a cold start, in about two minutes; the next theorem removes its one redundant element. □

Theorem 6.2. $f(8,3) \le 44$.

Proof. Simulated annealing produced the 45-element set of Proposition 6.1, and exactly one of its elements is removable: deleting the entry with base-3 code 2966 leaves the 44-element set $W_{8,3}$ printed in Appendix A, which is again a skirting set. Both the deletion identity and the skirting property are verified: the former is an identity between two explicit lists, the latter a complete evaluation over all $3^8 = 6\,561$ tuples of $\mathbb{Z}_3^8$, with zero unskirted. Proposition 3.1(1) gives the bound. The 44-element set was also verified independently, from Definition 2.1, by a separate implementation. □

Proposition 6.3. *$W_{8,3}$ is irredundant: all 44 single deletions fail to be skirting. Deleting its first element leaves 76 tuples of $\mathbb{Z}_3^8$ unskirted.*

Proof. Forty-four complete evaluations, all failing, plus one run to the end for the count. □

Theorem 6.2 matches $f(8,3) = 44$ and does not improve it; nothing in this paper bears on the lower half, which is [3]'s. Irredundance is not minimality, and Proposition 6.3 on its own says only that greedy deletion stops at 44 along this particular chain — that $f(8,3) \le 43$ is false is [3]'s theorem, not ours. What can be reported is the price of not knowing that. Weighted local search at $m = 43$ was run for 2100 seconds of CPU time from two starting sets with no element in common — $W_{8,3}$ itself and the 54-element product set $W_{6,3} \times W_{2,3}$ — and both runs bottomed out at exactly 68 unskirted tuples, which is also the minimum, over the 44 single deletions of $W_{8,3}$, of the number of tuples the deletion leaves unskirted. Neither run ever improved on what one good deletion already gives. The second run did produce a *second* 44-element skirting set of $\mathbb{Z}_3^8$, disjoint from $W_{8,3}$. A search for a skirting set that is a union of orbits of a prescribed cyclic group is negative at this size as far as it was run: there is no skirting set of $\mathbb{Z}_3^8$ of size at most 44 invariant under an 8-cycle of the coordinates together with a 3-cycle of the symbols applied uniformly in every coordinate, nor under a 7-cycle of the coordinates with any non-identity uniform symbol permutation. The cases with a smaller group, where the number of orbits is three to four times larger, were not resolved in the time allowed, and the sweep was stopped at coordinate cycle type $6+1+1$.

7. Two further cells

Both remaining bracketed cells of Table 1 are closed in the literature, and the two theorems below meet the known values without improving them. At $(7,3)$ the value is $f(7,3) = 29$, by [3, Thm. 4]; at $(6,4)$ it is $f(6,4) = 15$, by [5] as recorded in [6]. Measured against [1] alone the two would read as narrowings of $[28,30]$ to $[28,29]$ and of $[13,16]$ to $[13,15]$, and they are neither. What they add is a pair of explicit witnesses, found here and checked from the definition.

Theorem 7.1. $f(7,3) \le 29$.

Theorem 7.2. $f(6,4) \le 15$.

Proof of Theorems 7.1 and 7.2. The sets $W_{7,3}$ (29 tuples of $\mathbb{Z}_3^7$) and $W_{6,4}$ (15 tuples of $\mathbb{Z}_4^6$) of Appendix A are skirting sets; the verifications are complete evaluations over the $3^7 = 2\,187$ and $4^6 = 4\,096$ tuples respectively. Both were found by simulated annealing, in each case by several independent seeds returning different sets. Nothing is proved from below; the matching lower bounds are quoted. □

The 16 that [1] prints at $(6,4)$ is what every published *general* route gives. Subadditivity (1) gives $f(3,4)^2 = 16$, which is exactly the entry; Proposition 8 of [1] is vacuous at $v = 1$, gives at best $2 + \mathrm{CAN}(4,6,2) \ge 2 + 2^4 = 18$ at $v = 2$, and at $v = 3$ needs a strength-5 skirting array over $\mathbb{Z}_3$ on 6 columns that [1] does not evaluate; the Johnson–Lovász–Stein estimate of Remark 5.5 gives 42.66; and [2] is ternary. It took the classification of [5] to get to 15, and 15 is the value. Deleting the first element of $W_{6,4}$ destroys the skirting property, in agreement with $f(6,4) > 14$.

The searches that failed are worth recording, because both were looking for something that does not exist. The exhaustive route of Proposition 3.3 is out of reach at these lengths, and the corresponding satisfiability instances — "is there a skirting set of size 28 in $\mathbb{Z}_3^7$", "of size 14 in $\mathbb{Z}_4^6$" — were each run under a one-hour limit and returned no answer, at $1.25 \cdot 10^7$ and $1.21 \cdot 10^7$ conflicts respectively. They were encoded with 61 974 variables and 432 612 clauses, resp. 57 735 variables and 349 376 clauses, and with only two symmetry breakers, both provably sound: fixing the two constant tuples as in Proposition 3.2, and ordering the remaining rows lexicographically, which is legitimate because those rows are still freely permutable. Local search agrees from the other side: at $m = 28$ it came no closer than 35 unskirted tuples in 900 seconds from $W_{7,3}$ and in a further 1200 seconds from the 40-element product set $W_{4,3} \times W_{3,3}$, and at $m = 14$ no closer than 17 unskirted tuples in 900 seconds. Both instances are unsatisfiable: the first by [3, Thm. 4], the second because $f(6,4) = 15$. Brink's own refutation of a 7×28 covering matrix took 28 seconds, by a canonical-row search rather than by propositional reasoning; against an hour with no answer here, that is a fair measure of what the encoding above costs, and of why the canonical-augmentation searches of [3] and [5] settled these cells and a general-purpose solver did not.

8. The growth constant

[1] defines $C_q = \inf_{n \ge 1} f(n,q)^{1/n}$ and shows that $f(n,q) = C_q^{(1+o(1))n}$ in its Theorem 1, proves the bracket $1 + \frac{1}{q-1} \le C_q \le q^{1/(q-1)}$ in its Proposition 3, and asks for the exact value in its Problem 1. We record that the bracket collapses to its left endpoint, and that both halves of the collapse are already in the literature.

Proposition 8.1. *For every $q \ge 2$, $C_q \ge q/(q-1)$.*

Proof. Proposition 2.3 gives $f(n,q) \ge (q/(q-1))^n$, hence $f(n,q)^{1/n} \ge q/(q-1)$ for every $n \ge 1$; take the infimum. □

Proposition 8.2. *Let $q \ge 2$ and $A > 0$, and suppose $f(n,q) \le A\, n\, (q/(q-1))^n$ for every $n \ge 1$. Then $C_q \le q/(q-1)$.*

Proof. For each $n \ge 1$, $C_q \le f(n,q)^{1/n} \le A^{1/n} n^{1/n} \cdot q/(q-1)$, and both $A^{1/n} \to 1$ and $n^{1/n} \to 1$; pass to the limit. Only the shape of the sub-exponential factor matters: an n-th root annihilates any polynomial factor. □

Theorem 8.3. *Suppose that for a given $q \ge 2$ the Johnson–Lovász–Stein covering estimate*

$$f(n,q) \;\le\; \big(1 + n\ln(q-1)\big)\Big(\frac{q}{q-1}\Big)^n \qquad (n \ge 1), \tag{2}$$

as printed for general q in [2, §5], *holds. Then $C_q = q/(q-1)$. Moreover $C_3 = 3/2$, with* (2) *replaced by the construction $f(n,3) \le 2(3/2)^n - 1$ proved in* [2].

Proof. Since $q \ge 2$ gives $\ln(q-1) \ge 0$, the right-hand side of (2) is at most $(1+\ln(q-1))\, n\, (q/(q-1))^n$ for $n \ge 1$, so Proposition 8.2 applies with $A = 1+\ln(q-1)$ and yields $C_q \le q/(q-1)$; Proposition 8.1 gives the reverse. For $q = 3$ the hypothesis is not needed: the construction $f(n,3) \le 2(3/2)^n - 1$ proved in [2] is itself of the shape required by Proposition 8.2, with $A = 2$. □

Theorem 8.3 answers Problem 1 of [1]: $C_q = q/(q-1)$, the left endpoint of its own bracket, for every $q \ge 2$ — conditionally on the estimate (2) for general q, and for $q = 3$ using nothing beyond what is proved in [2]. Neither half is new; (2) is a greedy covering bound and is not proved here, and the lower half is the counting argument of [1], which also follows by iterating the projection inequality of [3, (1)] from $f(1,q) = 2$. At $q = 3$ the conclusion $C_3 = 3/2$ is subsumed by [7], which determines $f(n,3)$ up to the constant factor $\Lambda_3 \in [7/4, 2]$. What is new here is only that the two halves are recorded together for general q, and that the collapse is not merely asymptotic:

Proposition 8.4. *Granting* (2) *at $q = 4$: $f(24,4) \le 48\,834$, and $48\,834^6 < 15^{24}$, i.e. $f(24,4)^{1/24} < 15^{1/6}$.*

Proof. At $n = 24$, $q = 4$ the estimate (2) reads $f(24,4) \leq (1 + 24 \ln 3)(4/3)^{24}$, and $\ln 3 \leq 2$ by $\ln x \leq x - 1$, so $f(24,4) \leq 49 \cdot (4/3)^{24} < 48\,835$. The displayed integer inequality is arithmetic. With the true value of $\ln 3$ the first n at which the estimate beats $15^{1/6}$ is $n = 19$, where it gives $f(19,4) \leq 5173$ and $5173^6 < 15^{19}$; $n = 24$ is the first value reachable from the crude bound on $\ln 3$, which is what keeps the verification elementary. □

So the certificate $C_4 \leq 15^{1/6} = 1.5704\ldots$ of Theorem 7.2, and the sharper $C_4 \leq 36^{1/8} = 1.5650\ldots$ of Corollary 5.6, are superseded by $C_4 = 4/3 = 1.3333\ldots$ at a finite length, not merely in the limit.

Remark 8.5. Four checks bound what Theorem 8.3 says. The counting bound of Proposition 2.3 is strict at the computed cells — $3^3 = 27 < 40 = f(3,3) \cdot 2^3$ and $4^3 = 64 < 108 = f(3,4) \cdot 3^3$ — so no single cell attains $q/(q-1)$ and Theorem 8.3 is genuinely a statement about the infimum. The hypothesis (2) is not vacuous: at the cell $f(4,3) = 8$ settled in Proposition 4.2 it reads $8 \leq (1 + 4 \ln 2)(3/2)^4 = 19.09\ldots$, true with room to spare. Too small is refuted unconditionally, $C_4 > 1.33$, by Proposition 8.1 alone; too large is refuted under (2), $C_4 < 1.5705$, so the value is not $15^{1/6}$.

Remark 8.6. [2, §5] also defines a constant it calls C_q, namely $\sup_{n \geq 1} f(n,q)/(q/(q-1))^n$, and its Problem 5.1 asks whether that is finite for every fixed $q \geq 3$ and whether it can be bounded independently of q. That is a different quantity from the $\inf_n f(n,q)^{1/n}$ of [1], and (2) does not answer it: the factor $1 + n \ln(q-1)$ is unbounded. Theorem 8.3 bears on the constant of [1] only.

9. Verification

Every definition, proposition, theorem and corollary in this paper has been formally verified in Lean 4 against *Mathlib*, apart from the exceptions listed in the next sentence, as has the mathematical content of Remarks 4.4, 4.6 and 8.5; the development contains no `sorry`, and the repository reference is to be added at submission. What lies outside the formal development is labelled where it occurs, and nothing in the paper rests on it: Remark 4.5; Remarks 5.5 and 8.6, which read published statements of [1] and [2] and evaluate their estimates numerically; every sentence anywhere in the paper reporting what a source prints, including the check of the five covering matrices of [3] reported in Remark 4.4; Section 3.4, which reports only how the candidate lists were produced; the accounts of *unsuccessful* searches in Sections 6 and 7, that is the local-search runs at $m = 43$, the prescribed-automorphism sweep and the two satisfiability runs that reached their time limit; the three exact counts of unskirted tuples (23, 132 and 76) reported in Propositions 5.4 and 6.3, where what the formal development proves is that each deletion fails to be skirting and not by how much; and all timing and memory measurements.

The layer at which the mathematics is stated is $f(n,q)$ of Definition 2.1, defined as an infimum over finite subsets of $\mathbb{Z}_q^n$; the base-q encoding of Section 3 appears only inside proofs, through Proposition 3.1, which is itself proved. Propositions 2.2, 2.3, 3.1, 3.2, 3.3, 8.1, 8.2, Theorem 8.3 and Proposition 8.4 depend on no axioms beyond the three standard ones (propositional extensionality, choice, and quotient soundness); in particular the completeness of the search is proved rather than assumed, so a search returning false is a proof of nonexistence and not a report about a program.

What is delegated to computation is exactly the finite evaluations, and they are of two kinds. Compiled evaluation, through Lean's `native_decide`, replaces reduction in the kernel by execution of compiled code, and so adds to the statement it proves one axiom asserting that the compiler evaluated that Boolean correctly. It is used precisely where kernel reduction exceeds a 4 GiB working-memory budget, and covers exactly the following: the 1 118 481-node search that gives the lower bound of Proposition 4.2, and hence also the two neighbouring-value refutations of Remark 4.6 that quote $f(4,3) = 8$; the four evaluations behind Proposition 5.1, Theorem 5.2, Theorem 5.3 and, through the last of them, Corollary 5.6; the evaluations behind Proposition 6.1 and Theorem 6.2; and the six irredundance and deletion probes of Propositions 5.4 and 6.3. Everything else is kernel evaluation with no compiled code trusted — in particular the four exact values of Proposition 4.1, all witnesses of Propositions 4.1, 4.2 and 4.3, both bounds of Section 7, the remaining controls of

Remark 4.6, and all of Section 8 including Remark 8.5. For scale: kernel reduction of the 24-element quaternary evaluation of Theorem 5.2 was measured at 7 minutes 22 seconds and 11.3 GB of peak memory, against 1 minute 31 seconds and no measurable marginal memory for the compiled route on the whole file. Every set printed in Appendix A was in addition checked outside the formal development, by an implementation reading Definition 2.1 directly, before it was written down: zero unskirted tuples out of 16 384, 65 536, 6 561, 2 187 and 4 096 respectively. The two product sets of Proposition 5.1 were checked the same way.

Appendix A. The sets

Each row below is one tuple, written as the string $x_1x_2\cdots x_n$ of its coordinates; the tuple with base-q code c has $x_i = \lfloor c/q^{i-1}\rfloor \bmod q$. Within each set the order of the tuples is immaterial.

$W_{7,4}$, 24 tuples of $\mathbb{Z}_4^7$ (Theorem 5.2):

3000000	1133330	1200331	2201132
3300110	1331111	2000002	3131203
2210310	3112111	2203202	0012203
0100220	0023111	1111022	1323203
3011330	2100321	3022022	2220213
0322330	2230131	0333022	2202023

$W_{6,4}$, 15 tuples of $\mathbb{Z}_4^6$ (Theorem 7.2):

010000	313011	021122
320010	123221	033232
310310	132331	203203
110020	002302	222323
101101	311012	231133

$W_{8,4}$, 36 tuples of $\mathbb{Z}_4^8$ (Theorem 5.3):

33110010	11111130	23332131	11220132
00000110	32332011	33110331	33001132
22013110	00221111	22103112	32332332
23332310	11220311	11111312	22210203
11220020	33001311	23332022	33331203
33001020	11111021	33110122	22122203
32332120	22103321	00000322	33222203
00221320	00000031	22013322	11003203
22103030	22013031	00221032	00113203

$W_{8,3}$, 44 tuples of $\mathbb{Z}_3^8$ (Theorem 6.2):

02210000	21222020	22010021	12110012
11111000	20211220	10112021	01211012
21001100	01012220	12100121	20012012
00202100	20221001	01201121	22000112
22020200	01022001	20002121	10102112
10122200	00011201	02220221	21021212
11220110	21212201	11121221	00222212
02121110	10000011	01120102	20100022
01100210	22102011	12221102	12002022
12201210	20110111	11200202	10010122
00021020	12012111	02101202	22112122

$W_{7,3}$, 29 tuples of $\mathbb{Z}_3^7$ (Theorem 7.1):

0012100	1011220	2221111	1000112
1202200	1020001	1212121	0210212
1112010	1001101	0002221	2111212
2022210	2110201	0100002	2012022
0220020	0211201	2201002	1122222
2121020	2200011	2220102	
2100120	0101011	0121102	
0201120	0120111	1021012	

The sets of Propositions 4.1, 4.2 and 4.3 are, in the same notation and as lists of base-q codes $x \mapsto \sum_i x_i q^{i-1}$:

$$W_{2,3} = \{4, 6, 2\}, \qquad W_{3,3} = \{3, 7, 13, 15, 20\},$$
$$W_{2,4} = \{2, 5, 8\}, \qquad W_{3,4} = \{15, 21, 40, 50\},$$
$$W_{4,3} = \{0, 4, 17, 35, 36, 40, 73, 75\}, \qquad W_{4,4} = \{0, 13, 58, 85, 172, 202, 243\},$$

$$W_{5,3} = \{0, 51, 59, 70, 101, 116, 121, 156, 169, 176, 208, 225\},$$
$$W_{5,4} = \{0, 95, 165, 271, 341, 416, 759, 765, 954, 1002\},$$
$$W_{6,3} = \{33, 77, 106, 144, 164, 202, 261, 313, 338, 352, 458, 462, 490, 524, 615, 629, 663, 721\}.$$

$W_{4,3}$ is the first matrix displayed in [1], and $W_{5,4}$ the third, both read as columns; see Remark 4.4. The 45-element set of Proposition 6.1 is $W_{8,3}$ together with the tuple of code 2966, that is `21210011`.

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