COMPLETE SOLUTION SETS AND THE EXTREMAL PRODUCT FOR $\sigma_{n-2}(x) = \sigma_n(x)$ AT $n = 6$ AND $n = 7$

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Abstract. For integers $1 \le x_1 \le \cdots \le x_n$ the equation $\sigma_{n-2}(x) = \sigma_n(x)$ between elementary symmetric polynomials is the decomposition of unity $\sum_{1\le i<j\le n} 1/(x_i x_j) = 1$; write $f_{n-2}(n)$ for its number of solutions. Kiss, Sándor and Zakarczemny introduced the sequence $v_1 = 1$, $v_2 = 2$, $v_{n+1} = v_n^2 - v_n + 1 + v_1 \cdots v_{n-1}$, observed that $(v_1, \ldots, v_{n-1}, v_n - 1)$ solves the equation, listed all solutions for $n \le 5$, and asked whether $x_1 x_2 \cdots x_n \le v_1 v_2 \cdots v_{n-1}(v_n - 1)$ holds for every solution. We answer this affirmatively at the two smallest lengths their paper leaves open. For $n = 6$ and $n = 7$ we determine the solution set completely — $f_4(6) = 144$ and $f_5(7) = 3598$ — and show that the maximum of $x_1 \cdots x_n$ is exactly $1\,257\,813\,360$ and $2\,881\,515\,884\,053\,973\,040$ respectively, attained at $(1, 2, 4, 15, 219, 47862)$ and $(1, 2, 4, 15, 219, 47863, 2290845186)$. We also prove, for every n, that the tuple $(v_1, \ldots, v_{n-1}, v_n - 1)$ satisfies the equation and has product exactly the proposed bound, so that the question is in substance whether that one family is the product-maximiser; and we replace the majorization estimate through which the source obtains finiteness by an elementary prefix inequality, which is what makes the two exhaustive searches terminate. Every theorem below is machine-checked in Lean 4.

1. Introduction

For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ let $\sigma_j(x) = \sum_{1\le i_1<\cdots<i_j\le n} x_{i_1} \cdots x_{i_j}$ be the j-th elementary symmetric polynomial, with $\sigma_0(x) = 1$ and $\sigma_j(x) = 0$ for $j < 0$. Kiss, Sándor and Zakarczemny [1] study

$$\sigma_k(x_1, \ldots, x_n) = \sigma_n(x_1, \ldots, x_n), \qquad 1 \le x_1 \le x_2 \le \cdots \le x_n, \quad x_i \in \mathbb{Z}, \quad 1 \le k < n, \tag{1}$$

and write $f_k(n)$ for the number of solutions. This paper is about the top case $k = n - 2$. Dividing (1) by $\sigma_n(x) = x_1 \cdots x_n > 0$ turns it into a decomposition of unity into reciprocals of *pairs*,

$$\sum_{1\le i<j\le n} \frac{1}{x_i x_j} = 1, \tag{2}$$

which is the form used throughout [1] and here. (The neighbouring case $k = n-1$ of (1) is the classical Egyptian-fraction decomposition $\sum_i 1/x_i = 1$. There, as [1, §2.3] recalls, Sylvester's sequence $u_1 = 2$, $u_{n+1} = u_n^2 - u_n + 1$ [4] plays exactly the role that the sequence v below plays here: the source records the identity $\sigma_{n-1}(u_1, \ldots, u_{n-1}, u_n - 1) = \sigma_n(u_1, \ldots, u_{n-1}, u_n - 1)$ and, as a theorem of Erdős, the bound $x_n \le u_n - 1$, of which [1, Theorem 10] below is the $k = n - 2$ analogue.)

Write $\mathcal{S}_n$ for the set of solutions of length n, so that $f_{n-2}(n) = |\mathcal{S}_n|$.

The sequence v and the question. Section 2.4 of [1] introduces a rapidly growing integer sequence. In the source it is *defined* by a division (its Definition 7) and the recursions below are proved from that definition (its Theorem 8); we shall find it more convenient to take one of the recursions as the definition and recover the division form as Proposition 2.3.[1]

Definition 1.1. $v_1 = 1$, $v_2 = 2$, and $v_{n+1} = v_n^2 - v_n + 1 + \prod_{1\le i\le n-1} v_i$ for $n \ge 2$.

[1] All environments of [1] share one counter, so the numbers used here (Remark 6, Definition 7, Theorem 8, …, Problem 16) are the ones printed by compiling the arXiv e-print of v1; they agree with that source's own internal cross-references to "Corollary 19" and "Lemma 20". As of 28 August 2026 arXiv:2601.14057 carries that single version and no journal reference, so the numbering is unambiguous.

The first terms are

$$(v_n)_{n\ge 1} = (1,\ 2,\ 4,\ 15,\ 219,\ 47863,\ 2290845187,\ 5247971669768059423,\ \dots),$$

and $v_{n+1} \sim v_n^2$, so the growth is doubly exponential. The source observes, just below its Definition 7 and without proof, that the tuple

$$w(n) \ :=\ (v_1, v_2, \dots, v_{n-1},\, v_n - 1)$$

satisfies $\sigma_{n-2}(w(n)) = \sigma_n(w(n))$; its product is

$$B(n) \ :=\ v_1 v_2 \cdots v_{n-1}(v_n - 1),$$

and the open question of [1] is whether this family is extremal:

Problem 1.2 ([1, Problem 16])**.** Is it true that if $\sigma_{n-2}(x_1, x_2, \dots, x_n) = \sigma_n(x_1, x_2, \dots, x_n)$, where $0 < x_1 \le x_2 \le \cdots \le x_n$, then $x_1 x_2 \cdots x_n \le v_1 v_2 \cdots v_{n-1}(v_n - 1)$?

What [1] establishes around this question is the following. Its Remark 6 lists the complete solution sets for $n = 2, 3, 4, 5$ — respectively $1, 1, 2$ and 27 solutions, printed explicitly — and prints nothing for $n \ge 6$. Its Theorem 3 gives the lower bound $x_1 \cdots x_n \ge \binom{n}{k}^{n/(n-k)}$ for solutions of (1), which at $k = n - 2$ reads $x_1 \cdots x_n \ge \binom{n}{2}^{n/2}$. Its Theorem 10 gives $x_n \le v_n - 1$, so the solution set is finite for each n; its Theorem 11 gives $x_{n-k} \le 2(k+1)v_{n-k}$ and its Corollary 12 gives $x_1 \cdots x_n \le 2^{\,n-1} n!\, v_1 \cdots v_n$, so that Problem 1.2 is known to within the factor $2^{\,n-1} n!\, v_n/(v_n - 1)$. Its Theorem 15 gives $f_{n-2}(n) \ge n - 3$ for $n \ge 4$. The dual case $k = 2$ of (1) is the subject of Miska and Ulas [2]; it is a different equation, and nothing below bears on it.

Results. Our first theorem answers Problem 1.2 at the two smallest lengths for which [1] records nothing.

Theorem 1.3. *Let $n \in \{6, 7\}$. Every $x \in \mathcal{S}_n$ satisfies $x_1 x_2 \cdots x_n \le B(n)$, and the bound is attained. Explicitly,*

$$w(6) = (1, 2, 4, 15, 219, 47862) \in \mathcal{S}_6, \qquad B(6) = 1\,257\,813\,360 = 1 \cdot 2 \cdot 4 \cdot 15 \cdot 219 \cdot 47862,$$
$$w(7) = (1, 2, 4, 15, 219, 47863, 2290845186) \in \mathcal{S}_7,$$
$$B(7) = 2\,881\,515\,884\,053\,973\,040 = 1 \cdot 2 \cdot 4 \cdot 15 \cdot 219 \cdot 47863 \cdot 2290845186.$$

The bound is obtained from complete knowledge of the two solution sets, which is the second theorem.

Theorem 1.4. *$f_4(6) = 144$ and $f_5(7) = 3598$. More precisely, for $n \in \{6, 7\}$ the list $E(n, 1, 1, 0, 0)$ produced by the enumerator of Section 4 has pairwise distinct entries, every entry of it lies in $\mathcal{S}_n$, and every element of $\mathcal{S}_n$ occurs in it; its length is 144 for $n = 6$ and 3598 for $n = 7$.*

Table 1 collects the resulting picture. The counts and solutions for $n \le 5$ are those of [1, Remark 6], recomputed here as a control (Proposition 5.1), and the maximal products in those rows follow from the printed lists; the rows $n = 6$ and $n = 7$ are new.

Two further results are needed to make sense of the first two, and are of independent interest. Theorem 3.2 proves, for *every* n and with no computation, that $w(n)$ solves the equation and that its product is exactly $B(n)$ — this is the assertion of [1] quoted above, there stated without proof — so that Problem 1.2 is in substance the question of whether the source's own family is the product-maximiser, and so that the appearance of $w(n)$ in the last column of Table 1 is not an accident of small n: that tuple has product exactly $B(n)$ at every length. (Membership $w(n) \in \mathcal{S}_n$ requires in addition that the entries of $w(n)$ be in nondecreasing order; Remark 3.3 says why that clause is kept separate.) Proposition 4.1 replaces the finiteness bound of [1] by an elementary one. The source deduces its $x_n \le v_n - 1$ from its Theorem 9, whose proof rests in turn on a majorization lemma [1, Lemma 20] obtained from Muirhead's theorem [3]; we instead derive, from (2) alone, a quadratic inequality that a prefix $(x_1, \dots, x_m)$ of a solution must satisfy, which is both elementary and sharper in the sense that matters: it is exactly the test a depth-first search performs at every node, and with it the search at $n = 7$ makes about 2.5 million descents (Section 7 fixes the count).

n	$f_{n-2}(n)$	$\max_{x\in\mathcal{S}_n} x_1 \cdots x_n \;=\; B(n)$	attained at $w(n)$
2	1	1	$(1, 1)$
3	1	6	$(1, 2, 3)$
4	2	112	$(1, 2, 4, 14)$
5	27	26 160	$(1, 2, 4, 15, 218)$
6	144	1 257 813 360	$(1, 2, 4, 15, 219, 47862)$
7	3598	2 881 515 884 053 973 040	$(1, 2, 4, 15, 219, 47863, 2290845186)$

TABLE 1. Solution counts and maximal products. The last two rows are new. That $w(n)$ satisfies the equation and has product exactly $B(n)$ holds at every n (Theorem 3.2); that $w(n)$ lies in $\mathcal{S}_n$, so that the maximum is at least $B(n)$, and that the maximum is at most $B(n)$, are proved here only for $n \le 7$.

What rests on computation. Theorem 3.2 and everything in Sections 2–4 are proved for all n by ordinary induction and algebra; no case analysis and no computer search enters them. Theorems 1.3 and 1.4 rest on an exhaustive search which is finite *because* of Proposition 4.1, and the exhaustion — the statement that the search misses no solution — is itself proved (Theorem 4.5); what is delegated to computation is only the evaluation of the search at $n = 6$ and $n = 7$ and of four Boolean checks over its output. Section 8 says precisely which.

2. Notation and preliminaries

Symmetric polynomials indexed from the top. Because n occurs in (1) both as the length of the tuple and as an index of σ, it is convenient to index elementary symmetric polynomials by *codegree*. For a finite sequence x of length n put

$$E_j(x) \;:=\; \sigma_{n-j}(x) \qquad (j \ge 0),$$

so $E_j(x) = 0$ for $j > n$. Reading x as a list, E obeys a recursion in which n does not appear:

$$E_0(\,) = 1, \qquad E_{j+1}(\,) = 0, \qquad E_0(a, x) = a\,E_0(x), \qquad E_{j+1}(a, x) = a\,E_{j+1}(x) + E_j(x), \tag{3}$$

where (a, x) denotes the sequence x with a prepended. Thus $E_0(x) = x_1 \cdots x_n$, and equation (1) at $k = n - 2$ is simply $E_2(x) = E_0(x)$, an identity with no free occurrence of n at all. All of the formal development is written in these terms.

Lemma 2.1. *E_j is invariant under permutation of the entries: if x and y are rearrangements of one another then $E_j(x) = E_j(y)$ for every j.*

Proof. By (3) the transposition of the first two entries gives, for $j \ge 2$, $E_j(a, b, x) = ab\,E_j(x) + (a + b)E_{j-1}(x) + E_{j-2}(x)$, visibly symmetric in a and b; the cases $j = 0, 1$ are the same computation with the last term or the last two terms absent. Since adjacent transpositions generate all rearrangements, and E_j commutes with prepending a common entry, the general case follows by induction on a chain of adjacent transpositions. □

Lemma 2.2. *Let x be a finite sequence of positive integers. Then*

$$E_1(x) = \Big(\textstyle\prod_i x_i\Big) \sum_i \frac{1}{x_i}, \qquad E_2(x) = \Big(\textstyle\prod_i x_i\Big) \sum_{i<j} \frac{1}{x_i x_j},$$

and consequently $E_2(x) = E_0(x)$ if and only if $\sum_{i<j} 1/(x_i x_j) = 1$. In particular (1) *at $k = n - 2$ and* (2) *have the same solutions.*

Proof. Both identities are immediate inductions on the length using (3), the positivity being needed only so that the reciprocals are defined; the equivalence follows on dividing by $\prod_i x_i > 0$. □

We record the object of study once more in this notation:

$$\mathcal{S}_n = \big\{\, x \in \mathbb{Z}^n :\ 1 \le x_1 \le \cdots \le x_n,\ E_2(x) = E_0(x) \,\big\}, \qquad f_{n-2}(n) = |\mathcal{S}_n|.$$

All three hypotheses — the ordering, the positivity and the length — are load-bearing; see Section 6.

The sequence v. Definition 1.1 is the recursion [1, Theorem 8(2)]. That it defines the same sequence as [1, Definition 7], which is by division, is the following statement; it is worth proving rather than assuming, since two recursions agreeing on the printed initial terms need not agree thereafter.

Proposition 2.3. *For every $n \ge 0$,*

$$v_{n+1} \;=\; 1 + \frac{\displaystyle\sum_{1 \le i \le n} \frac{1}{v_i}}{1 - \displaystyle\sum_{1 \le i < j \le n} \frac{1}{v_i v_j}},$$

the denominator being nonzero; for $n \ge 2$ this is the defining formula of [1, Definition 7].

The proof is given in Section 3, where the two identities that drive it are established.

3. The family $w(n)$, at every n

Throughout this section $k \ge 0$ and $L_k := (v_1, \dots, v_k)$, with $P_k := v_1 v_2 \cdots v_k$ and $P_0 = 1$. Everything rests on two identities.

Lemma 3.1. *For every $k \ge 0$,*

$$E_1(L_k) = \sigma_{k-1}(v_1, \dots, v_k) = v_{k+1} - 1, \qquad E_2(L_k) = \sigma_{k-2}(v_1, \dots, v_k) = P_k - 1.$$

Equivalently $\sum_{i \le k} 1/v_i = (v_{k+1} - 1)/P_k$ and $\sum_{i<j\le k} 1/(v_i v_j) = 1 - 1/P_k$. Moreover $v_k + 1 \le v_{k+1}$ for every $k \ge 1$.

Proof. Both identities are inductions on k. They are immediate at $k = 0$, where $E_1 = E_2 = 0$, $v_1 = 1$ and $P_0 = 1$, and at $k = 1$, where $E_1(L_1) = \sigma_0 = 1 = v_2 - 1$ and $E_2(L_1) = \sigma_{-1} = 0 = P_1 - 1$. For the step, let $k \ge 1$ and bring v_{k+1} to the front of L_{k+1} using Lemma 2.1; then (3) gives $E_1(L_{k+1}) = v_{k+1}E_1(L_k) + E_0(L_k) = v_{k+1}(v_{k+1} - 1) + P_k$ by the induction hypothesis, and Definition 1.1 with $n = k + 1$ says exactly that this is $v_{k+2} - 1$; likewise $E_2(L_{k+1}) = v_{k+1}E_2(L_k) + E_1(L_k) = v_{k+1}(P_k - 1) + (v_{k+1} - 1) = P_{k+1} - 1$. For the last assertion, $v_2 - v_1 = 1$, and for $k \ge 2$ Definition 1.1 gives $v_{k+1} - v_k = (v_k - 1)^2 + P_{k-1} \ge 1$. □

Proof of Proposition 2.3. By Lemma 2.2 and Lemma 3.1, $\sum_{i \le n} 1/v_i = E_1(L_n)/P_n = (v_{n+1} - 1)/P_n$ and $1 - \sum_{i<j\le n} 1/(v_i v_j) = 1 - E_2(L_n)/P_n = 1/P_n$, which is nonzero. Dividing the first by the second gives $v_{n+1} - 1$. □

Theorem 3.2. *Let $n \ge 2$ and $w(n) = (v_1, \dots, v_{n-1}, v_n - 1)$. Then, as four separate assertions, each holding at every such n:*

(a) *$w(n)$ has length n and every entry of it is ≥ 1;*
(b) *$\sigma_{n-2}(w(n)) = \sigma_n(w(n))$;*
(c) *the product of $w(n)$ is $v_1 v_2 \cdots v_{n-1}(v_n - 1) = B(n)$;*
(d) *$v_k + 1 \le v_{k+1}$ for every $k \ge 1$.*

Proof. Write $n = k + 1$ and $t = v_{k+1} - 1$, so that $w(n)$ is L_k with t appended. By Lemma 2.1 we may instead prepend t, and then (3) gives

$$E_2(t, L_k) = t\,E_2(L_k) + E_1(L_k), \qquad E_0(t, L_k) = t\,E_0(L_k) = t\,P_k.$$

By Lemma 3.1 the equation $E_2 = E_0$ therefore reads $t(P_k - 1) + (v_{k+1} - 1) = tP_k$, i.e. $v_{k+1} - 1 = t$, which holds by the choice of t. (The displayed equivalence shows more: given the prefix L_k, the value $t = v_{k+1} - 1$ is the *only* completion of L_k to a solution of length $k + 1$.) The product is $t\,E_0(L_k) = P_k\,(v_{k+1} - 1) = B(k + 1)$ by Lemma 2.1 again, which is (c). For (a), the length is n by

inspection and every entry is ≥ 1 because $v_i \geq 1$ for $i \geq 1$ and $v_n - 1 \geq v_{n-1} \geq 1$ for $n \geq 2$; (d) is the last assertion of Lemma 3.1. □

Remark 3.3. Parts (a)–(d) are exactly the assertions the formal development carries at general n, and the theorem is stated in parts for that reason. Assembling (d) into the single statement "the entries of $w(n)$ are listed in nondecreasing order" — from $v_1 \leq v_2 \leq \cdots \leq v_{n-1}$ and $v_{n-1} \leq v_n - 1$, both instances of (d) — and hence into $w(n) \in \mathcal{S}_n$ and $\max_{x \in \mathcal{S}_n} x_1 \cdots x_n \geq B(n)$, is an evident induction, but it is a step the formal development does not take at general n: there the ordering of $w(n)$ is checked one length at a time, and in this paper only for $n \leq 7$. Every use of $w(n) \in \mathcal{S}_n$ below is therefore at $n \in \{6, 7\}$, where it is checked. With that caveat, Problem 1.2 is the question of whether this one family is extremal.

4. A derived finiteness bound, and an exhaustive search

Fix n and let $x \in \mathcal{S}_n$. For $0 \leq m < n$ call $p = (x_1, \ldots, x_m)$ a *prefix* of x and put

$$D = E_0(p) = \sigma_m(p), \qquad S = E_1(p) = \sigma_{m-1}(p), \qquad Q = E_2(p) = \sigma_{m-2}(p),$$

so that $(D, S, Q) = (1, 0, 0)$ for the empty prefix. Appending one entry y transforms the triple by

$$(D, S, Q) \longmapsto (Dy,\ Sy + D,\ Qy + S), \tag{4}$$

directly from (3) and Lemma 2.1, and the solution condition at the end of the tuple, $E_2 = E_0$, is $Q = D$.

Proposition 4.1 (the prefix inequality)**.** *Let p be a sequence of positive integers, let $y \geq 1$, and let u be a nonempty sequence of r entries, each $\geq y$, such that the concatenation $p\,u$ satisfies* (2)*. Then, with (D, S, Q) the triple of p,*

$$Q < D \qquad \text{and} \qquad 2Dy^2 \ \leq\ 2Qy^2 + 2Sry + r(r-1)D.$$

Proof. Splitting the pair sum of $p\,u$ into pairs inside p, cross terms, and pairs inside u,

$$1 = \sum_{i<j \text{ in } p} \frac{1}{x_i x_j} + \Big(\sum_{i \text{ in } p} \frac{1}{x_i}\Big)\Big(\sum_{j \text{ in } u} \frac{1}{x_j}\Big) + \sum_{i<j \text{ in } u} \frac{1}{x_i x_j}.$$

Multiply by $D = \prod_{i \in p} x_i$ and use Lemma 2.2 on the first two sums: $D = Q + S\,T + D\,U$ where $T = \sum_{j \in u} 1/x_j$ and $U = \sum_{i<j \in u} 1/(x_i x_j)$. Since every entry of u is at least $y > 0$ we have $T \leq r/y$ and $U \leq \binom{r}{2}/y^2$, whence $D \leq Q + Sr/y + D\,r(r-1)/(2y^2)$; multiplying by $2y^2$ gives the displayed inequality. For the strict inequality $Q < D$: if p is empty then $Q = 0 < 1 = D$, and otherwise $T > 0$ and $S > 0$, so $D = Q + ST + DU > Q$. □

For the empty prefix, $(D, S, Q) = (1, 0, 0)$ and $r = n$, so Proposition 4.1 reads $2x_1^2 \leq n(n-1)$, i.e. $x_1 \leq \lfloor \sqrt{\binom{n}{2}} \rfloor$; at $n = 6$ this already forces $x_1 \leq 3$. Write

$$T(D, S, Q, r, y): \qquad Q < D \ \text{ and } \ 2Dy^2 \leq 2Qy^2 + 2Sry + r(r-1)D$$

for the test. Two properties make it usable as the pruning rule of a search.

Lemma 4.2 (downward closure)**.** *If $T(D, S, Q, r, y)$ holds and $1 \leq z \leq y$, then $T(D, S, Q, r, z)$ holds.*

Proof. This is the convexity of $A\zeta^2 - B\zeta - C$ with $A = 2(D - Q) \geq 0$, $B = 2Sr$, $C = r(r-1)D \geq 0$ and nonpositive value at $\zeta = 0$, but it can be done without leaving the nonnegative integers: from $Ay^2 \leq By + C$ and $1 \leq z \leq y$,

$$Az^2 y = (Azy)z \leq (Azy)y = Ay^2 z \leq (By + C)z = B\,yz + Cz \leq B\,yz + Cy = (Bz + C)y,$$

and cancelling $y \geq 1$ gives $Az^2 \leq Bz + C$. □

Lemma 4.3 (scan bound)**.** *If $y \geq 1$ and $T(D, S, Q, r, y)$ holds, then $y \leq 2Sr + r(r-1)D$.*

Proof. With A, B, C as above, $Q < D$ gives $A \geq 2$, so $2y^2 \leq Ay^2 \leq By + C \leq By + Cy = (B + C)y$; cancelling y gives $2y \leq B + C$. □

Finally, when exactly one entry remains the equation is linear and no scan is needed: by (4) the condition $Q' = D'$ after appending y is $Qy + S = Dy$, i.e.

$$(D - Q)\, y = S. \tag{5}$$

So the last entry is determined by the prefix, and exists only if $D - Q$ divides S. This is the same phenomenon as the uniqueness noted in the proof of Theorem 3.2, and it is what makes the search at $n = 7$ practical rather than merely finite.

Definition 4.4. For $r \geq 0$, a lower bound $\ell \geq 1$ and a triple (D, S, Q) of nonnegative integers, define a finite list $E(r, \ell, D, S, Q)$ of nondecreasing r-tuples by

$$E(0, \ell, D, S, Q) = \begin{cases} [\,(\,)\,] & \text{if } Q = D, \\ [\,] & \text{otherwise,} \end{cases} \qquad E(1, \ell, D, S, Q) = \begin{cases} [\,(y)\,] & \text{if } Q < D,\ (D - Q) \mid S,\ y \geq \ell, \\ [\,] & \text{otherwise,} \end{cases}$$

where $y = S/(D - Q)$ in the second case; and, for $r \geq 2$,

$$E(r, \ell, D, S, Q) = \big[\,(y, z)\ :\ y \in C,\ z \in E(r - 1,\, y,\, Dy,\, Sy + D,\, Qy + S)\,\big],$$

where $C = (\ell, \ell + 1, \dots)$ is truncated at the first y failing $T(D, S, Q, r, y)$.

The list C is finite by Lemma 4.3, and by Lemma 4.2 truncating at the first failure discards no feasible value: C contains every $y \geq \ell$ passing the test. The recursion terminates because r decreases.

Theorem 4.5 (completeness of the search)**.** *Let p be a sequence of positive integers with triple (D, S, Q), let $\ell \geq 1$, and let u be a nondecreasing sequence of r entries, all $\geq \ell$, such that $p\,u$ satisfies* (2)*. Then $u \in E(r, \ell, D, S, Q)$. In particular, taking p empty and $\ell = 1$,*

$$\mathcal{S}_n \subseteq E(n, 1, 1, 0, 0) \qquad \textit{for every } n.$$

Proof. Induction on r. For $r = 0$ the hypothesis is that p itself satisfies (2), which by Lemma 2.2 is $Q = D$, and the empty tuple is listed. For $r = 1$, say $u = (y)$; the condition $Q' = D'$ is (5), so $D - Q$ divides S — note $Q < D$ by Proposition 4.1 — and $y = S/(D - Q) \geq \ell$, which is what the second clause of Definition 4.4 lists. For $r \geq 2$ write $u = (y, u')$. Every entry of u is $\geq y$ because u is nondecreasing, so Proposition 4.1 gives $T(D, S, Q, r, y)$; by Lemma 4.2 the truncated scan C therefore reaches y, and $y \in C$. By (4) the prefix $p\,(y)$ has triple $(Dy, Sy + D, Qy + S)$, and u' is a nondecreasing $(r - 1)$-tuple with entries $\geq y$ completing it to a solution, so $u' \in E(r - 1, y, Dy, Sy + D, Qy + S)$ by the induction hypothesis and (y, u') is listed. The last statement follows since the empty prefix has triple $(1, 0, 0)$ and every element of $\mathcal{S}_n$ is nondecreasing with entries ≥ 1. □

Remark 4.6. Theorem 4.5 makes no use of the bound $x_n \leq v_n - 1$ of [1, Theorem 10], nor of the inequality [1, Theorem 9] behind it, nor of the majorization lemma [1, Lemma 20] and Muirhead's theorem [3] on which that in turn rests: finiteness here comes from Proposition 4.1 alone, which is three lines of elementary estimation. This is not a criticism of the source's bound, which is a statement about solutions rather than about prefixes; but a search needs a test at every node, and that is what Proposition 4.1 supplies.

5. The solution sets

Theorem 4.5 makes $E(n, 1, 1, 0, 0)$ a finite list containing $\mathcal{S}_n$. It does not by itself say that every listed tuple is a solution, because the test T is only necessary; that is settled by evaluating one Boolean per length. The pattern of every statement below is therefore: an exhaustive containment, proved; plus a finite check on the list produced, evaluated.

Proposition 5.1 (the source's Remark 6, recomputed)**.** $E(2, 1, 1, 0, 0) = [(1, 1)]$, $E(3, 1, 1, 0, 0) = [(1, 2, 3)]$, $E(4, 1, 1, 0, 0) = [(1, 2, 4, 14), (2, 2, 2, 6)]$, *and* $E(5, 1, 1, 0, 0)$ *is a list of* 27 *five-tuples without repetitions. Every entry of each of these lists lies in $\mathcal{S}_n$. Hence $\mathcal{S}_2 = \{(1, 1)\}$, $\mathcal{S}_3 = \{(1, 2, 3)\}$, $\mathcal{S}_4 = \{(1, 2, 4, 14), (2, 2, 2, 6)\}$ and $f_3(5) = 27$; these are exactly the lists printed in* [1, Remark 6]*, and the* 27 *five-tuples agree with the source's entry by entry and in the source's own order.*

Proof. Containment $\mathcal{S}_n \subseteq E(n, 1, 1, 0, 0)$ is Theorem 4.5. The four lists were evaluated, and for each the Boolean "some listed tuple fails to be a solution" was evaluated and found false; the $n = 5$ list was additionally checked to have length 27 and to be repetition-free. Agreement with [1, Remark 6] was checked by a mechanical comparison against the 27 tuples parsed out of the source. □

Proposition 5.1 is a control rather than a result: it reproduces the only data the source prints, and it is what licenses reading the next two statements as more terms of the same computation.

Proof of Theorem 1.4. Again the containment is Theorem 4.5, so that it is enough to evaluate the list $E(n, 1, 1, 0, 0)$ at the two lengths and then to check three Booleans for each of them: that no listed tuple fails to be a solution, that the list carries no repetitions, and that its length is the value stated. All six of these evaluations returned the expected verdict. The two searches are small, since the test of Proposition 4.1 cuts the tree down to 5 236 descents at the first length and 2 513 644 at the second (Section 7 fixes that count), while the shortcut (5) removes the last level of the tree altogether. □

Proof of Theorem 1.3. Let $n \in \{6, 7\}$. By Theorem 4.5 every $x \in \mathcal{S}_n$ occurs in $E(n, 1, 1, 0, 0)$, so it suffices to evaluate the single Boolean "some tuple in $E(n, 1, 1, 0, 0)$ has product exceeding $B(n)$". It is false for $n = 6$ and for $n = 7$; this is one finite check over the 144, respectively 3598, listed tuples, and it is the only computation on which the inequality of Theorem 1.3 rests. The values $B(6) = 1\,257\,813\,360$ and $B(7) = 2\,881\,515\,884\,053\,973\,040$ are evaluations of Definition 1.1. Attainment needs no search: $w(n)$ satisfies the equation and has product exactly $B(n)$ by Theorem 3.2, evaluating $w(6)$ and $w(7)$ gives the two tuples displayed in the statement, and the one clause Theorem 3.2 leaves open at these two lengths — that those entries are in nondecreasing order, so that $w(n) \in \mathcal{S}_n$ — is a further finite check. □

Remark 5.2. The same finite check confirms the bound of Problem 1.2 at $n \leq 5$, where by Proposition 5.1 the solution sets are the source's own; the maximal products there, read off the lists of that proposition, are 1, 6, 112 and 26 160, equal to $B(n)$ in each case, as part (c) of Theorem 3.2 predicts. The bound at these four lengths is not offered as a result; it is recorded as a control on the two lengths that are.

Remark 5.3. The resulting sequence $f_{n-2}(n)$ for $n = 2, \ldots, 7$ is $1, 1, 2, 27, 144, 3598$. Searches of the OEIS [4] on 28 August 2026 for this sequence, and for each of its subwords of length at least four, returned no entry; the one hit anywhere in that family of searches was A129710, a triangle of Fibonacci binary words in whose flattened data the four terms $1, 2, 27, 144$ happen to occur. Searches for the sequence v itself and for its subwords returned nothing, while the control search for $2, 3, 7, 43, 1807$ returned Sylvester's sequence A000058 as it should. This is a statement about what is catalogued, not a claim of novelty: the values at $n = 6, 7$ are within reach of a short program, and [1] may well have computed them without printing them.

6. Sharpness, and the necessity of the hypotheses

The following statements are small, but each of them rules out a way in which Theorems 1.3 and 1.4 could be true for an uninteresting reason.

Corollary 6.1. *For $n \in \{6, 7\}$ the bound of Theorem 1.3 cannot be lowered: there is an $x \in \mathcal{S}_n$ with $x_1 \cdots x_n > B(n) - 1$. Equivalently, $B(n)$ is exactly $\max_{x \in \mathcal{S}_n} x_1 \cdots x_n$.*

Proof. Immediate from the attainment half of Theorem 1.3. □

Proposition 6.2. *The lower bound $\binom{n}{2}^{n/2} \leq x_1 \cdots x_n$ of* [1, Theorem 3] *is, at $n = 5$, $10^{5/2} = 316.22\ldots$, and it cannot be raised past 324: the tuple $(3, 3, 3, 3, 4)$ lies in $\mathcal{S}_5$ and has product 324, and no element of $\mathcal{S}_5$ has product smaller than 324.*

Proof. That $(3, 3, 3, 3, 4) \in \mathcal{S}_5$ with product 324 is a direct evaluation. Minimality is one Boolean over the 27 tuples of Proposition 5.1: no listed tuple has product below 324. □

Proposition 6.3. *Each hypothesis defining $\mathcal{S}_n$ is load-bearing.*

(i) Ordering. $(2,1,4,15,218)$ *has positive entries and satisfies $\sigma_3 = \sigma_5$, but is not nondecreasing and so is not in $\mathcal{S}_5$; dropping the ordering therefore changes the solution set, and with it every count above. Already at $n = 4$,* $(2,4,1,14)$ *satisfies the equation and is not in* $E(4,1,1,0,0)$.

(ii) Positivity. $(0,0,0)$ *is nondecreasing and satisfies $\sigma_1 = \sigma_3$, both sides being 0; it is not a solution in the sense used here, its entries not being positive.*

(iii) Length. $(1,2,3) \in \mathcal{S}_3$ *but* $(1,2,3) \notin \mathcal{S}_4$.

(iv) The equation is not vacuous. $(1,2,4,15,219,47863)$*, which differs from $w(6)$ in its last entry only, is not in $\mathcal{S}_6$; neither is* $(1,2,3,4,5)$ *in $\mathcal{S}_5$, nor* $(1,1,2,4,15,218)$ *in $\mathcal{S}_6$.*

(v) The pruning test is not vacuous. *At the empty prefix and $n = 6$ the test $T(1,0,0,6,y)$ holds for $y = 3$ and fails for $y = 4$, so it really does force $x_1 \le 3$; and $T(6,5,6,2,1)$ fails, so the guard $Q < D$ rejects prefixes that have already exhausted* (2).

Proof. Each item is a direct evaluation of the equation, of the ordering predicate, or of the test T at the stated tuple; the second sentence of (i) additionally uses the $n = 4$ list of Proposition 5.1. □

7. The next length

The following observations lie outside the formal development and are recorded only as a guide; none of them is used above.

The obstruction at $n = 8$ is not the size of the answer but the shape of the search. Fix a counting convention: a *descent* is one value of y at which $E(r,\ell,D,S,Q)$ with $r \ge 2$ passes the test T and recurses, so that descents count the edges of the search tree that issue from a scanning node, and exclude both the one failing test that ends each scan and the divisor test (5) at $r = 1$. By this count the two complete searches of Section 5 take 5 236 descents at $n = 6$ and 2 513 644 at $n = 7$.

An independent implementation of Definition 4.4 in C, instrumented with a per-level descent counter, was run at $n = 8$ and stopped after $2 \cdot 10^9$ descents. It had emitted no solution, and its profile by the number r of entries still to be chosen was

r	8	7	6	5	4	3	2
descents	1	2	3	12	205	47 645	1 999 952 132

— more than 99.99% of them at the single level $r = 2$, with the search still inside its first choice of x_1. The reason is visible in Proposition 4.1: at $r = 2$ the test reads $(D-Q)y^2 \le 2Sy + D$, so the scan runs to about $2S/(D-Q)$, which is enormous on the prefixes with $D - Q = 1$. The remedy is one level up from (5): with two entries left, (2) can be written as a factorization. In the notation of Section 4, [1] records — with $A = \sigma_{n-2}$, $C = \sigma_{n-3}$, $B = \sigma_{n-4}$ of the prefix $(x_1,\dots,x_{n-2})$, that is with $A = D$, $C = S$, $B = Q$ — the identity

$$A(A-B) + C^2 = \big((A-B)x_{n-1} - C\big)\big((A-B)x_n - C\big), \qquad \text{i.e.} \qquad m := D(D-Q) + S^2 = uv,$$

so the last pair ranges over the divisor pairs of the single integer m instead of over a scan. Over the length-6 prefixes the search reaches at $n = 8$ the largest such m is about $5 \cdot 10^{19}$, so the search becomes a factoring problem. For scale,

$$v_8 = 5\,247\,971\,669\,768\,059\,423, \qquad B(8) = 15\,122\,113\,732\,103\,021\,601\,031\,372\,056\,773\,070\,960.$$

8. Verification

Every numbered theorem, proposition, lemma and corollary of this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [5, 6], in the directory `LeanProblemSpec/SigmaTuples/` of a development that is free of `sorry`; repository reference to be added at submission. The remarks are not part of that development: Remark 3.3 records what it does not contain, Remark 4.6 compares proof routes, Remark 5.2 is a control at lengths the paper does not claim, Remark 5.3 reports a database search, and Section 7 says of itself that it lies outside. The division of labour between proof and computation is exactly as described above, and can be read off the axioms each statement depends on. Lemma 2.1, Lemma 2.2, Proposition 2.3, Lemma 3.1, Theorem 3.2, Proposition 4.1,

Lemma 4.2, Lemma 4.3 and Theorem 4.5 — that is, the whole reasoning layer, including the assertion that the search misses no solution — depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, while Lemma 2.1 and part (c) of Theorem 3.2 depend on `propext` alone, and the assertion of item (i) of Proposition 6.3 about $(2, 1, 4, 15, 218)$ on no axioms at all. What is delegated to compiled evaluation, through Lean's `native_decide`, is only the finite checks named in the proofs: for each of $n = 2, \ldots, 7$ the evaluation of the list $E(n, 1, 1, 0, 0)$ together with the Boolean "some listed tuple is not a solution", the explicit lists and counts of Proposition 5.1 and Theorem 1.4, the two products-exceed-the-bound Booleans of Theorem 1.3, the evaluation of $w(6)$ and $w(7)$, the minimality check of Proposition 6.2, and, in Proposition 6.3, the appeal to the $n = 4$ list in item (i) and the six-entry tuple of item (iv); the remaining items there are small enough to be decided by the kernel outright. In particular the inequality of Theorem 1.3 at $n = 6$ rests on exactly one compiled evaluation — that a single filter over 144 tuples returned the empty list — with every other step, above all the exhaustiveness of the enumeration, checked by the kernel. The counts 144 and 3598, the maximal products, and the extremal tuples were each reproduced independently by two implementations written outside the formal development, in C and in Python.

References

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[2] P. Miska and M. Ulas, *On the Diophantine equation* $\sigma_2(\mathbf{X}_n) = \sigma_n(\mathbf{X}_n)$, Int. J. Number Theory **20** (2024), no. 5, 1287–1306. (The dual case $k = 2$; not used here.)

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1934 (Muirhead's theorem; used in [1], not here).

[4] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entry A000058 (Sylvester's sequence).

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[6] The mathlib Community, *The Lean Mathematical Library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.