PALINDROMICITY OF SHIFTED LOWER BRUHAT INTERVALS: THE (w, x) CENSUS FOR $n \le 6$, THE 3412/4231 SEPARATION, AND THE FAILURE OF THE CARRELL–PETERSON DICTIONARY

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ABSTRACT. For elements w, x of a Coxeter group W, the *shifted lower Bruhat interval* is the right translate $S(w, x) = \{vx^{-1} : v \le w\}$ of the principal order ideal of w, carrying the order induced from the ambient Bruhat order. Dave, Feng, Lesnevich, Oh, Richmond, Wang and Wu introduced it and proved that it has a unique maximum and a unique minimum, that its covering relations are ambient, that it is graded, and that it is EL-shellable. Gradedness makes its rank generating function $F_{w,x}(q)$ well defined, and at $x = e$ it is the Poincaré polynomial of the Schubert variety X_w, palindromic exactly when X_w is rationally smooth. We ask for which pairs (w, x) the polynomial $F_{w,x}$ is palindromic — a question the source does not raise — and settle a first round of it in type A. We give the complete census in $\mathfrak{S}_n$ for $n \le 6$: the palindromic pairs number 4, 36, 464, 8228, 171112; the permutations palindromic for *every* shift number $2, 6, 15, 38, 95$; those palindromic for *no* shift number $0, 0, 1, 4, 63$. We then show that three separate halves of the classical dictionary fail to survive shifting. The two Lakshmibai–Sandhya patterns, interchangeable in every statement of that dictionary, are separated by shifting: 3412 is palindromic for none of the 24 shifts of $\mathfrak{S}_4$ and 4231 for exactly 8. Palindromicity no longer forces unimodality — the source's own worked example is a witness — so the Carrell–Peterson factorisation into q-integers has no analogue. And the permutations palindromic for every shift do not form a pattern-avoidance class. Finally, being palindromic for no shift implies neither containment of 3412 nor containment of 4231, although at $n \le 5$ it appears to imply the first. The counts and witnesses stated below are machine-checked in Lean 4.

1. INTRODUCTION

1.1. **The object.** Let (W, S) be a Coxeter system with length function ℓ and Bruhat order $\le$. For $w, x \in W$, Dave, Feng, Lesnevich, Oh, Richmond, Wang and Wu [1] define the *shifted lower Bruhat interval*

$$S(w, x) \; = \; \{ \, vx^{-1} \; : \; v \le w \, \} \; \subseteq \; W, \tag{1}$$

the right translate by x^{-1} of the principal order ideal $[e, w]$, partially ordered by the restriction of the ambient Bruhat order of W to that set. The object is a hybrid: its ground set is carved out by the untranslated order and its relations are read in the ambient one, so it is neither an interval nor a translate of a poset. At $x = e$ it is the lower interval $[e, w]$; at $w = w_0$ (in the finite case) it is all of W.

The results of [1] are that $S(w, x)$ is nevertheless very well behaved. It has a unique maximum, namely the Demazure product $w \star x^{-1}$ [1, Prop. 4.3], and a unique minimum, described there through an "opposite Demazure" operator [1, Cor. 4.10]. Its covering relations are *ambient*: a cover in $S(w, x)$ is a cover in W [1, Lem. 5.1]. It is graded, by an argument through the weak generalized lifting property of Caselli, D'Adderio and Marietti [1, §4.2]. And it is EL-shellable for every Coxeter group [1, Thm. 5.5, Cor. 5.6]. All numbered references to [1] below are to its version 1.

Ambient covers together with gradedness pin the rank function down completely: writing $\hat{0}_{w,x}$ for the minimum of $S(w, x)$, the rank of $u \in S(w, x)$ is $\ell(u) - \ell(\hat{0}_{w,x})$. So $S(w, x)$ has a well-defined *rank generating function*

$$F_{w,x}(q) \; = \; \sum_{v \le w} q^{\,\ell(vx^{-1}) - \ell(\hat{0}_{w,x})} \; = \; \sum_{j \ge 0} h_j(w, x) \, q^j. \tag{2}$$

1.2. **The question, and what is known about it.** This paper asks:

Question 1.1. For which pairs (w, x) is $F_{w,x}$ palindromic, that is, $h_j = h_{d-j}$ for all j, where $d = \deg F_{w,x}$?

At $x = e$ this is a classical question with a classical answer. Since $S(w, e) = [e, w]$, the polynomial $F_{w,e}$ is the Poincaré polynomial of the Schubert variety X_w, and by Carrell–Peterson [4] it is palindromic exactly when X_w is rationally smooth — a theorem restated in Carrell–Kuttler [5] and in the survey of Billey–Gao–Pawlowski [6] — and in type A, by Lakshmibai–Sandhya [7], exactly when w avoids the two patterns 3412 and 4231. So the $x = e$ column of every table below is a published table, and we use it throughout as a control.

For $x \neq e$ we could find no prior treatment, and the source itself does not raise the question. Precisely: the words *palindromic*, *smooth*, *rationally smooth* and *Poincaré* do not occur anywhere in [1]; its single remark about $F_{w,x}$ is that it "need not be unimodal" [1, Ex. 3.3]; and its stated open problems are CL-shellability of middle intervals, which pairs give thin intervals, the possible Möbius values, analogues of R-polynomials and Kazhdan–Lusztig polynomials, and making its geometric dictionary precise. The nearest prior work on palindromicity of Bruhat intervals that we located is McAlmon–Oh–Yoo [13], by an author of [1]; it treats palindromicity of lower intervals $[\mathrm{id}, v]$ in parabolic quotients W^J, that is, the case $x = e$, and the words *shift* and *translate* do not occur in it. It is present in the bibliography file of the e-print of [1] but is cited nowhere in its text.

The counting sequences produced below — $4, 36, 464, 8228, 171112$ and its complement, the sequence $0, 0, 1, 4, 63$, and the counts of pairs that are palindromic but not unimodal — were searched for in the OEIS [15] on 2026-08-28, each search run in the same session as a positive control: the Lakshmibai–Sandhya counts $1, 2, 6, 22, 88, 366, 1552$, which return A032351. All returned nothing. The one exception is discussed in §10: the sequence $2, 6, 15, 38, 95$ of Theorem 3.2 matches a single OEIS entry, and the match is a coincidence that ends at the next term.

1.3. **Is Question 1.1 a known question in disguise?** The abstract of [1] says that these posets "arise from affine pavings of Richardson varieties", and the $x = e$ case is a smoothness criterion, so the first thing to settle is whether palindromicity of $F_{w,x}$ is some published geometric condition restated. Four points say it is not.

(a) The geometry is not yet available. The motivating work cited by [1] is Lesnevich–Oh–Richmond [2], listed there as in preparation; we found no public version (arXiv author and title searches, OpenAlex and web search, 2026-08-28). The closing sentence of [1] is that making the geometric dictionary precise "would be an interesting topic".

(b) The rank of $S(w, x)$ is not a cell dimension of X_w. In [1, Ex. 3.3] one has $w = x = 2413$ with $\ell(w) = 3$, while $S(w, x)$ has rank 5 (Proposition 9.2 below). If the cells of a paving of X_w were indexed by $S(w, x)$ with dimension equal to the poset rank, the Poincaré polynomial of X_w would depend on x, which it cannot. So $F_{w,x}$ is not the Poincaré polynomial of X_w under any paving, and Question 1.1 is not the rational smoothness of X_w in disguise.

(c) There is no Carrell–Peterson theorem for Richardson varieties. The number of $\mathbb{F}_q$-points of an open Richardson variety $C_z \cap C^y$ — a Schubert cell met with an opposite Schubert cell — is the Kazhdan–Lusztig R-polynomial $R_{y,z}(q)$ ([3, Ch. 5] for R-polynomials), so the point count of the closed Richardson variety $X_v \cap X^u$ is $\sum_{u \le y \le z \le v} R_{y,z}(q)$ and not the rank generating function of $[u, v]$; and the two already disagree in $\mathfrak{S}_4$. For $u = 1324$ and $v = 3412$,

$$\sum_{u \le y \le z \le v} R_{y,z}(q) \;=\; 1 + 3q + 5q^2 + q^3, \qquad \sum_{u \le z \le v} q^{\,\ell(z)-\ell(u)} \;=\; 1 + 4q + 4q^2 + q^3,$$

both of total mass $|[u, v]| = 10$. So a paving of $X_v \cap X^u$ by affine cells indexed by $[u, v]$ cannot have cell dimension equal to the poset rank either. The published criteria for singularities of Richardson varieties — the local-invariant factorisation of Knutson–Woo–Yong [9] and the cohomological vanishing criterion of Billey–Coskun [10] — are not criteria on poset ranks.

(d) Carrell–Peterson is a theorem about lower *intervals.* The general-interval form sometimes quoted — X_w is rationally smooth at e_u if and only if the rank generating function of $[u, w]$ is palindromic — is false. Fan and Guo [8] record the failure in its Kazhdan–Lusztig form: writing $P_{u,v}$ for the Kazhdan–Lusztig polynomial, $P_{e,w} = 1$ if and only if $[e, w]$ is rank-symmetric, but for a general interval this is not so, since $P_{w,w_0} = 1$ for every w while $[w, w_0]$ need not be rank-symmetric. A witness is easy to check by hand: in $\mathfrak{S}_4$ the interval $[1324, 4321]$ has rank sequence $1, 4, 6, 5, 3, 1$, which is not palindromic, while $X_{4321} = G/B$ is smooth. The correct general statement is Carrell's regularity criterion on the Bruhat graph [4].

Points (b) and (d) together say that even the general-interval version of the classical question lies outside the known criterion; the shifted question lies further outside it. We therefore state everything below purely combinatorially — a graded poset and its rank generating function — and assert no geometric interpretation of $F_{w,x}$ for $x \neq e$.

1.4. **Results.** Everything below is in type A: $W = \mathfrak{S}_n$. Write P_n for the number of pairs $(w, x) \in \mathfrak{S}_n \times \mathfrak{S}_n$ with $F_{w,x}$ palindromic, U_n for the number of w that are palindromic for *every* shift x, and Z_n for the number palindromic for *no* shift. The census (Theorems 3.1, 3.2, 3.3) is:

n	pairs $(n!)^2$	P_n	U_n	Z_n
2	4	4	2	0
3	36	36	6	0
4	576	464	15	1
5	14 400	8228	38	4
6	518 400	171 112	95	63

Beyond the counts, four structural findings.

(i) 3412 *and* 4231 *separate under shifting* (Theorem 5.1). In every statement of the classical dictionary the two Lakshmibai–Sandhya patterns are interchangeable. Here 3412 is palindromic for none of the 24 shifts of $\mathfrak{S}_4$ while 4231 is palindromic for exactly 8 of them.

(ii) *Palindromic does not imply unimodal* (Theorem 6.1), so the factorisation half of the classical dictionary has no analogue. The witness is the source's own Example 3.3, which it uses only to record non-unimodality.

(iii) *The universally palindromic permutations are not a permutation class* (Theorem 7.1), so $2, 6, 15, 38, 95$ is not a pattern-avoidance count.

(iv) *Being palindromic for no shift implies neither pattern* (Theorem 8.1), although at $n \leq 5$ every such permutation contains 3412.

Section 4 records the one statement here that holds at every n, and it is classical: shifting by w_0x complements every rank, so F_{w,w_0x} is the reversal of $F_{w,x}$ and palindromicity depends only on the pair $\{x, w_0x\}$. Section 9 records the calibration against published values. Section 10 collects computations that support the picture but are not among the verified theorems, including the $n = 7$ values.

2. Preliminaries

2.1. **The symmetric group and the Bruhat order.** We write $\mathfrak{S}_n$ for the symmetric group on $\{1, \ldots, n\}$ and use one-line notation, so 3412 is the permutation sending $1 \mapsto 3$, $2 \mapsto 4$, $3 \mapsto 1$, $4 \mapsto 2$. The inversion set and length are

$$\operatorname{Inv}(u) = \{(i, j) \ : \ i < j,\ u(i) > u(j)\}, \qquad \ell(u) = |\operatorname{Inv}(u)|.$$

The Bruhat order is generated by its covering relation: $\sigma \lessdot \tau$ when $\ell(\tau) = \ell(\sigma) + 1$ and $\sigma = \tau t$ for a transposition t, and $\sigma \leq \tau$ when there is a chain of covers from σ up to τ (see [3, Ch. 2]). The longest element is $w_0 = n\,(n-1)\cdots 1$, of length $\binom{n}{2}$. We write $[e, w] = \{v : v \leq w\}$ for the principal order ideal.

2.2. **Shifted intervals and their rank sequences.**

Definition 2.1 ([1, Def. 3.1])**.** For $w, x \in \mathfrak{S}_n$, $S(w,x) = \{vx^{-1} : v \le w\}$, partially ordered by the restriction of the Bruhat order of $\mathfrak{S}_n$.

By [1], $S(w,x)$ has a unique minimum and a unique maximum, its covers are ambient covers of $\mathfrak{S}_n$, and it is graded; hence its rank function is $u \mapsto \ell(u) - m(w,x)$ with $m(w,x) = \min_{v \le w} \ell(vx^{-1})$, and, setting $M(w,x) = \max_{v\le w} \ell(vx^{-1})$, its rank sequence is

$$h(w,x) = \big(h_0, h_1, \ldots, h_d\big), \qquad h_j = \#\{v \le w \;:\; \ell(vx^{-1}) = m(w,x)+j\}, \qquad d = M(w,x)-m(w,x).$$

Note that the map $v \mapsto vx^{-1}$ is a bijection from $[e,w]$ onto $S(w,x)$, so $\sum_j h_j = |[e,w]|$ for every x: shifting redistributes a fixed number of elements among the ranks.

Definition 2.2. $S(w,x)$ is *palindromic* when $h(w,x)$ equals its own reversal, i.e. $h_j = h_{d-j}$ for all $0 \le j \le d$. We write

$$\pi_n(w) = \#\{x \in \mathfrak{S}_n \;:\; S(w,x) \text{ is palindromic}\},$$

and call w *universally palindromic* when $\pi_n(w) = n!$ and *never palindromic* when $\pi_n(w) = 0$. Then $P_n = \sum_w \pi_n(w)$, $U_n = \#\{w : \pi_n(w) = n!\}$ and $Z_n = \#\{w : \pi_n(w) = 0\}$.

Example 2.3. Take $w = x = 2413$, the example that Section 6 turns on. The ideal $[e, 2413]$ has eight elements, and the shift $v \mapsto v \cdot 2413^{-1}$ moves them as follows (the second row is the ambient length of the image):

v	2413	2314	1423	1324	1234	2134	1243	2143
vx^{-1}	1234	1243	2134	2143	3142	3241	4132	4231
$\ell(vx^{-1})$	0	1	1	2	3	4	4	5

So $S(2413, 2413)$ has minimum 1234, maximum 4231, and rank sequence $1,2,1,1,2,1$: it is graded of rank 5, although $\ell(w) = 3$. Every element of $[e, 2413]$ is still present; what the shift has changed is how they are distributed among the ranks.

Since $S(w,e) = [e,w]$, the case $x = e$ of Definition 2.2 is the classical palindromicity of the Poincaré polynomial of X_w. In particular a universally palindromic w is smooth and a never palindromic w is singular; we use this repeatedly below and record it as Lemma 2.4.

Lemma 2.4. *If w is universally palindromic then w avoids* 3412 *and* 4231*; if w is never palindromic then w contains* 3412 *or* 4231*.*

Proof. $x = e$ is one of the $n!$ shifts and $S(w,e) = [e,w]$, so $F_{w,e}$ is the Poincaré polynomial of X_w. By Carrell–Peterson [4] it is palindromic exactly when X_w is rationally smooth, in type A exactly when X_w is smooth, and by Lakshmibai–Sandhya [7] exactly when w avoids 3412 and 4231. ☐

2.3. **The computation of $h(w,x)$.**

The tables below rest on an enumeration of $[e,w]$, and it is worth saying exactly which enumeration, because the entire content of the new results is the correctness of a finite search. We use no rank-matrix (Ehresmann) criterion and no subword property: the ideal is produced by walking the covering relation downwards from w.

Lemma 2.5. *For $w \in \mathfrak{S}_n$ and $k \ge 0$ let $D_k(w)$ be the set of words reached from w by k successive steps down the covering relation. Then*

$$D_k(w) = \{v \;:\; v \le w \text{ and } \ell(v) + k = \ell(w)\},$$

so $[e,w]$ is the disjoint union of $D_0(w), \ldots, D_{\ell(w)}(w)$, and for every x and every j,

$$h_j(w,x) = \#\big\{v \in \textstyle\bigsqcup_k D_k(w) \;:\; \ell(vx^{-1}) = m(w,x) + j\big\}.$$

Proof. Induction on k. One step down from τ produces exactly the σ with $\sigma \lessdot \tau$, so a k-fold step produces exactly the σ joined to w by a chain of k covers; since every cover raises ℓ by exactly one, such a chain exists precisely when $\sigma \le w$ and $\ell(\sigma) + k = \ell(w)$. Disjointness of the levels is the same length grading. The last display is the definition of h_j rewritten over the computed list. ☐

The following reformulation is what the independent implementations used, and it explains why the question has a metric flavour.

Remark 2.6. For all $v, x \in \mathfrak{S}_n$, $\ell(vx^{-1}) = |\operatorname{Inv}(v) \triangle \operatorname{Inv}(x)|$, the Kendall tau distance between v and x. Indeed $\ell(vx^{-1})$ counts the pairs $\{i, j\}$ on which v and x induce opposite orders, and so does the symmetric difference. Hence $F_{w,x}$ is the distance distribution from the centre x to the ideal $[e, w]$, viewed inside the hypercube $\{0,1\}^{\binom{n}{2}}$ under the inversion-set embedding, and Question 1.1 asks for which centres that distribution is symmetric.

3. The census

Theorem 3.1. *The number P_n of pairs $(w, x) \in \mathfrak{S}_n \times \mathfrak{S}_n$ for which $S(w, x)$ is palindromic is*

$$P_2 = 4, \qquad P_3 = 36, \qquad P_4 = 464, \qquad P_5 = 8228, \qquad P_6 = 171\,112.$$

Theorem 3.2. *The number U_n of universally palindromic $w \in \mathfrak{S}_n$ is*

$$U_2 = 2, \qquad U_3 = 6, \qquad U_4 = 15, \qquad U_5 = 38, \qquad U_6 = 95.$$

Theorem 3.3. *The number Z_n of never palindromic $w \in \mathfrak{S}_n$ is*

$$Z_2 = Z_3 = 0, \qquad Z_4 = 1, \qquad Z_5 = 4, \qquad Z_6 = 63,$$

and the unique never palindromic element of $\mathfrak{S}_4$ is 3412.

Proof of Theorems 3.1, 3.2 and 3.3. All three numbers at a given n are read off one exhaustive evaluation. For each of the $n!$ elements w the ideal $[e, w]$ is produced as the repetition-free list of Lemma 2.5; then for each of the $n!$ shifts x the multiset $\{\ell(vx^{-1}) : v \le w\}$ is histogrammed between its minimum and its maximum and the histogram is compared with its reversal. This yields $\pi_n(w)$ for every w, and P_n, U_n, Z_n are its sum, the number of w attaining $n!$, and the number attaining 0. The search is finite and complete: it examines every one of the $(n!)^2$ pairs. Its size is $\left(\sum_w |[e, w]|\right) \cdot n!$ evaluations of $\ell(vx^{-1})$, namely $213 \cdot 24 = 5112$ at $n = 4$, $3781 \cdot 120 = 453\,720$ at $n = 5$ and $98\,407 \cdot 720 = 70\,853\,040$ at $n = 6$. Nothing in these three statements is proved by a general argument; each is the outcome of that exhaustive computation, carried out inside the proof assistant (§11). The final assertion of Theorem 3.3 combines $Z_4 = 1$ with $\pi_4(3412) = 0$ from Theorem 5.1. $\square$

Remark 3.4. The counts are not close to trivial in either direction: $P_n/(n!)^2$ falls from 1 at $n \le 3$ through 0.806, 0.571 to 0.330 at $n = 6$. By Lemma 2.4 the universally palindromic permutations are strictly contained in the smooth ones, and strictly: $15 < 22$, $38 < 88$, $95 < 366$. So $2, 6, 15, 38, 95$ counts a new and strictly smaller subset of the Lakshmibai–Sandhya class.

4. Shifting by $w_0 x$, and the parity of π_n

This section records the only statement in this paper that holds at every n. It is classical, and the order-dual form of it is already in [1]: in the unnumbered paragraph following [1, Cor. 4.10] it is recorded that right multiplication by w_0 carries $S(w, x)$ onto $S(w, w_0 x)$ and, being an anti-automorphism of the Bruhat order, reverses the induced order. We give the rank-level form, which is what the tables use.

Proposition 4.1. *Let $n \ge 1$. For every $u \in \mathfrak{S}_n$, $\ell(u) + \ell(uw_0) = \binom{n}{2}$. Consequently, for all $w, x \in \mathfrak{S}_n$ and all $v \le w$,*

$$\ell\big(v(w_0x)^{-1}\big) = \binom{n}{2} - \ell\big(vx^{-1}\big),$$

so $h(w, w_0x)$ is the reversal of $h(w, x)$. In particular $S(w, x)$ is palindromic if and only if $S(w, w_0x)$ is, and palindromicity depends only on the pair $\{x, w_0x\}$. If moreover $n \ge 2$, then $\pi_n(w)$ is even for every w.

Proof. In one-line notation uw_0 is the reversal of the word of u, so the first identity says that for a word with no repeated letter the inversions of the word and of its reversal partition the $\binom{n}{2}$ pairs of positions. This is an induction on the word: appending a letter a to l creates exactly the inversions with the entries of l that exceed a, and for $a \notin l$ the trichotomy $\#\{y \in l : y < a\} + \#\{y \in l : y > a\} = |l|$ closes the step. For the second identity, $(w_0x)^{-1} = x^{-1}w_0$, and right multiplication by w_0 sends the word of vx^{-1} to its reversal; apply the first identity to $u = vx^{-1}$. Since the same constant $\binom{n}{2}$ is subtracted from every rank, the whole sequence $h(w, \cdot)$ is reversed, and a sequence is palindromic if and only if its reversal is. The parity statement follows because for $n \geq 2$ the map $x \mapsto w_0x$ is a fixed-point-free involution of $\mathfrak{S}_n$ — $w_0x = x$ would force $w_0 = e$, which fails once $n \geq 2$ — and it preserves palindromicity. The hypothesis $n \geq 2$ is needed: at $n = 1$ one has $w_0 = e$ and $\pi_1(e) = 1$. □

We claim nothing new in Proposition 4.1; in Coxeter language the length identity is the standard $\ell(u) + \ell(uw_0) = \ell(w_0)$. It is stated because it is the only general argument available here, because it halves any future search over x, and because its parity consequence is a nontrivial constraint on the tables: every value of π_n in §5 is even, as it must be.

5. The two Lakshmibai–Sandhya patterns separate

In the classical dictionary 3412 and 4231 are interchangeable. Both are exactly the obstructions to rational smoothness of X_w in type A [7]. Both are the two elements of the smallest nonempty case of Pucinskaite's criticality [14], where they carry the same degree and identical displayed data. No statement we know of in the smoothness literature distinguishes them. Shifting does.

Theorem 5.1. *In $\mathfrak{S}_4$,*

$$\pi_4(3412) = 0, \qquad \pi_4(4231) = 8.$$

That is, $S(3412, x)$ is palindromic for none of the 24 shifts $x \in \mathfrak{S}_4$, while $S(4231, x)$ is palindromic for exactly 8 of them. As a companion, $\pi_n(w_0) = n!$ for $n = 4, 5$.

Proof. Exhaustive: for each of the 24 (respectively 120) shifts the rank sequence of the relevant shifted interval is computed by Lemma 2.5 and compared with its reversal. The ideal $[e, 3412]$ has 14 elements and $[e, 4231]$ has 20, so each of the two counts is decided by 24 histogram comparisons. The companion statement is a control rather than a discovery: $S(w_0, x) = \mathfrak{S}_n$ for every x, so the rank sequence is the Mahonian distribution, with generating function $[n]_q! = \prod_{i=1}^{n}(1+q+\cdots+q^{i-1})$, which is palindromic. □

Remark 5.2. The eight shifts that work for 4231 are, in one-line notation,

$$1423,\ 1432,\ 2314,\ 2341,\ 3214,\ 3241,\ 4123,\ 4132,$$

and they fall into the four pairs $\{x, w_0x\}$ predicted by Proposition 4.1: $\{1423, 4132\}$, $\{1432, 4123\}$, $\{2314, 3241\}$, $\{2341, 3214\}$. This list is read off the same enumeration and is not among the separately verified statements.

The two values are extreme in their table: by Theorem 3.3 the value $\pi_4(w) = 0$ occurs for 3412 alone, and 8 is the smallest positive value taken by π_4, whose range is $\{0, 8, 12, 16, 24\}$ (§10). What we do *not* know is any structural reason for the asymmetry. Nothing in [7], in [4], or in [14] predicts that the two patterns should behave differently, and the shifted setting is where they first do.

6. Palindromic without unimodal

Call a sequence $(h_0, \dots, h_d)$ *unimodal* when it has no strict interior valley, i.e. there are no $i < j < k$ with $h_i > h_j$ and $h_k > h_j$.

For a rationally smooth Schubert variety in type A the Poincaré polynomial factors,

$$P_w(q) = \prod_i \bigl(1 + q + \cdots + q^{d_i}\bigr) \qquad \text{for suitable integers } d_i \geq 0,$$

by Gasharov [11]; see also Billey [12]. Such a product is symmetric *and* unimodal. In the classical picture, therefore, palindromicity carries unimodality with it. It does not survive shifting.

Theorem 6.1. *For $w = x = 2413$ in $\mathfrak{S}_4$,*

$$F_{w,x}(q) = 1 + 2q + q^2 + q^3 + 2q^4 + q^5,$$

which is palindromic and not unimodal. Hence a palindromic $F_{w,x}$ need not be a product of q-integers, and the factorisation half of the classical dictionary has no analogue for shifted lower Bruhat intervals.

Proof. $[e, 2413]$ has 8 elements; the eight values $\ell(v \cdot 2413^{-1})$ are computed by Lemma 2.5 and histogrammed, giving the rank sequence $1, 2, 1, 1, 2, 1$ between ranks 0 and 5. That sequence equals its reversal, and taking $(i, j, k) = (1, 2, 4)$ gives $h_1 = 2 > 1 = h_2$ and $h_4 = 2 > 1 = h_2$, a strict interior valley. For the last sentence: a product $\prod_i (1 + \cdots + q^{d_i})$ is a product of unimodal symmetric polynomials with nonnegative coefficients, hence unimodal, so a non-unimodal polynomial is not such a product. □

This pair (w, x) is Example 3.3 of [1] and its Figure 2, where it is introduced to show that the rank generating function of a shifted lower interval "need not be unimodal". That its rank sequence is at the same time palindromic is not remarked on there, and it is exactly what separates the two halves of the classical dictionary. The phenomenon is not isolated: over the full enumeration the palindromic-but-not-unimodal pairs number $0, 4, 152, 3560$ for $n = 3, 4, 5, 6$ (§10).

7. The universally palindromic permutations are not a class

The first guess about the sequence $2, 6, 15, 38, 95$ of Theorem 3.2 is that it counts a pattern-avoidance class — Lemma 2.4 places it inside one, and the classical case $x = e$ *is* a class. It is not.

Theorem 7.1. *In $\mathfrak{S}_5$ the permutations 35421 and 54213 are universally palindromic: $\pi_5(35421) = \pi_5(54213) = 120$. In $\mathfrak{S}_4$,*

$$\pi_4(2431) = 12, \qquad \pi_4(4312) = 12,$$

so neither 2431 nor 4312 is universally palindromic; and 35421 contains the pattern 2431 while 54213 contains the pattern 4312. Consequently the set of universally palindromic permutations is not closed under pattern containment: it is not a permutation class, and $2, 6, 15, 38, 95$ is not a pattern-avoidance count.

Proof. The four values of π are exhaustive computations as in Theorem 5.1, over the 120 shifts of $\mathfrak{S}_5$ and the 24 of $\mathfrak{S}_4$ respectively. The two containments are finite checks over the subsequences of a five-letter word: the subsequence 3542 of 35421 standardises to 2431, and the subsequence 5423 of 54213 standardises to 4312. A permutation class is by definition closed downwards under containment, so a universal permutation containing a non-universal pattern refutes closure. □

Two independent witnesses are given because a single one could be a coincidence of the small case; note that the two use different patterns. Theorem 7.1 closes the cheapest route to a formula for U_n.

8. Never palindromic implies neither pattern

At $n = 4$ the unique never palindromic permutation *is* 3412 (Theorem 3.3), and at $n = 5$ the four never palindromic permutations are

$$14523, \quad 24513, \quad 34125, \quad 35124,$$

all of which contain 3412. (This list is read off the enumeration of §3; what is separately verified there is the count $Z_5 = 4$.) The natural guess is therefore that 3412 is the obstruction. It is false, and so is the guess with 4231 in place of 3412.

Theorem 8.1. *In $\mathfrak{S}_6$,*

$$\pi_6(523641) = 0 \quad \textit{and} \quad 523641 \textit{ avoids } 3412,$$

$$\pi_6(125634) = 0 \quad \textit{and} \quad 125634 \textit{ avoids } 4231.$$

Hence being palindromic for no shift implies neither the containment of 3412 *nor the containment of* 4231*.*

Proof. Each of the two values $\pi_6(\cdot) = 0$ is an exhaustive computation over the 720 shifts of $\mathfrak{S}_6$, with the ideal of the permutation produced by Lemma 2.5. Each avoidance is a finite check over the 2^6 subsequences of a six-letter word. □

By Lemma 2.4 no never palindromic permutation can avoid *both* patterns, so Theorem 8.1 is as strong a separation as the setting allows. The two witnesses are the lexicographically smallest of their kind; over the 63 never palindromic elements of $\mathfrak{S}_6$, exactly 13 avoid 3412 and exactly 32 avoid 4231, and none avoids both. Those three counts are read off the enumeration of §3 and are not among the separately verified statements; what is verified is the pair of witnesses above and the $x = e$ column for $n \leq 6$ (§9).

We record that no characterisation of the never palindromic permutations is known to us. Theorems 7.1 and 8.1 together close the pattern-avoidance route to one, from both ends.

9. Calibration and controls

Since the new statements are outcomes of a search, the search has to be calibrated against published values, and the predicate has to be shown to be neither vacuous nor trivial. Both are done here. Nothing in this section is new.

9.1. The $x = e$ column.

Proposition 9.1. *The number of $w \in \mathfrak{S}_n$ with $F_{w,e}$ not palindromic is* 2*,* 32 *and* 354 *for $n = 4, 5, 6$.*

These are the published values: the two elements at $n = 4$ are 3412 and 4231; the value 32 at $n = 5$ goes back to Jantzen and is quoted in [14]; the value 354 at $n = 6$ is that paper's count of critical permutations. The complements $24 - 2 = 22$, $120 - 32 = 88$, $720 - 354 = 366$ are the Lakshmibai–Sandhya counts of smooth permutations, A032351 in [15]. The agreement is a genuine control here, because the enumeration used for Proposition 9.1 shares no code with the enumeration used for those published counts, and it walks the covering relation rather than applying a rank-matrix criterion. As a second control, the total ideal sizes $\sum_w |[e, w]|$ produced by the same walk are 213, 3781, 98 407 for $n = 4, 5, 6$, matching independently measured values; those three numbers are measurements, not verified statements.

9.2. The two worked examples of the source.

Proposition 9.2. *The poset $S(2413, 2413)$ has rank sequence $1, 2, 1, 1, 2, 1$, so it is graded of rank* 5*; and $S(2341, 3214)$ has rank sequence $1, 2, 2, 1, 1, 1$, so it is graded of rank* 5 *with a unique coatom. The minimum of the latter is* 1324 *and its maximum is* 4321*.*

These reproduce Examples 3.3 and 3.2 of [1] — including the endpoints of the increasing maximal chain $1324 \lessdot 3124 \lessdot 3214 \lessdot 4213 \lessdot 4312 \lessdot 4321$ displayed there — from the definitions used in this paper, and they are the reason to believe that Definition 2.1 has been transcribed correctly. In particular they exhibit [1, Prop. 4.3] and [1, Cor. 4.10] at one cell. Gradedness itself was checked over all pairs with $n \leq 7$ — no rank strictly between the minimum and the maximum was ever empty — but that check is a measurement of the kind collected in §10 and is not among the verified statements.

9.3. **Negative controls.**

Proposition 9.3. *The palindromicity predicate is not vacuous and not trivial:* $S(1234, 1234)$ *is palindromic, while* $S(3412, e)$ *is not — its rank sequence is* $1, 3, 5, 4, 1$*. Every one of the* 4 *pairs of* $\mathfrak{S}_2 \times \mathfrak{S}_2$ *and every one of the* 36 *pairs of* $\mathfrak{S}_3 \times \mathfrak{S}_3$ *is palindromic, so the phenomenon of this paper begins at* $n = 4$*. Moreover each headline count is straddled:* $P_4 \notin \{463, 465\}$*,* $P_5 \notin \{8227, 8229\}$*,* $P_6 \notin \{171\,111, 171\,113\}$*,* $U_4 \notin \{14, 16\}$*,* $U_6 \notin \{94, 96\}$*,* $Z_5 \notin \{3, 5\}$*,* $Z_6 \notin \{62, 64\}$ *and* $\pi_4(4231) \notin \{7, 9\}$*.*

The straddles are stated because a count produced by a search is worth exactly as much as the evidence that an off-by-one in the search would have been caught.

Proposition 9.4. *The word-level form of the length identity used in the proof of Proposition 4.1 genuinely needs its no-repeated-letter hypothesis, and is not an identity between equal terms. For the two-letter word* 11 *the identity fails:* $0+0 \neq 1 = \binom{2}{2}$*. For* $u = 2413$ *the two sides are different numbers put together:* $\ell(u) = 3 = \ell(uw_0)$ *with* $\binom{4}{2} = 6$*. And in* $\mathfrak{S}_4$ *every shift of every element of every principal ideal is a genuine permutation word, so the hypothesis holds throughout the computation.*

10. Computations outside the verified development

The statements in this section were produced by an independent implementation in C and are *not* among the machine-checked theorems. They are recorded because they support the picture above and because one of them decides a question that the verified range leaves open. They should be read as measurements.

10.1. $n = 7$.

The full sweep at $n = 7$ is $\left(\sum_w |[e, w]|\right) \cdot 7! = 3\,550\,919 \cdot 5040 = 1.79 \cdot 10^{10}$ evaluations of $\ell(vx^{-1})$. In compiled C this takes 32 seconds; in the proof assistant's evaluator it was measured at roughly 69 hours, well past the budget we allow a single check, so the $n = 7$ row is a measurement and not a theorem. Its values are

$$P_7 = 4\,167\,896, \qquad U_7 = 238, \qquad Z_7 = 515.$$

These have the same calibration available to them as the verified rows. The same program, on the same run, reports $\sum_w |[e, w]| = 3\,550\,919$ at $n = 7$, and its $x = e$ column gives 3488 non-palindromic permutations with asymmetry-degree split {1:3379, 2:96, 3:12, 4:1}; both agree with independently measured values, and the complement $5040 - 3488 = 1552$ is the next Lakshmibai–Sandhya count, A032351.

10.2. **The sequence** $2, 6, 15, 38, 95$ **and A153122.**

Of all the sequences produced here, exactly one has a match in the OEIS [15], and it is a coincidence. The universal counts $2, 6, 15, 38, 95$ match A153122 after stripping signs and shifting the index. That entry is the signed sequence $1, -2, 6, -15, 38, -95, 237, \ldots$, a constant-coefficient linear recurrence of signature $(-2, 2, 1, -2, 1)$ named by a rational generating function, whose only comment records that the ratio of consecutive terms tends to an approximation to a Feigenbaum constant; it has no combinatorial content, lists no references, and links only to a page on that constant. It continues with 237, while the measured value is $U_7 = 238$; and $2, 6, 15, 38, 95, 238$ matches nothing in the OEIS. The five-term coincidence therefore breaks at the sixth term, by one.

10.3. **Further measurements.**

The pairs that are palindromic but not unimodal (§6) number 0, 4, 152, 3560, 86 984 for $n = 3, \ldots, 7$. At $n = 4$ and at $n = 5$ *every* non-unimodal rank sequence is palindromic; the first non-palindromic non-unimodal pair appears only at $n = 6$, where there are 1440 of them. The universally palindromic permutations with a non-universal pattern of size $n - 1$, as in Theorem 7.1, number 2 at $n = 5$, 7 at $n = 6$ and 22 at $n = 7$. At $n = 4$ the values taken by π_4 are $\{0, 8, 12, 16, 24\}$; Proposition 4.1 explains why they are even, but not why they are all divisible by 4.

Finally, refining the non-palindromic pairs by their *asymmetry degree* — the least i with $h_i \neq h_{d-i}$, the grading used in [14] for the case $x = e$ — gives

$$\{1{:}112\}, \quad \{1{:}5432,\ 2{:}572,\ 3{:}168\}, \quad \{1{:}270816,\ 2{:}58552,\ 3{:}15448,\ 4{:}2408,\ 5{:}64\}$$

for $n = 4, 5, 6$. The $x = e$ sub-column of these splits reproduces the published $\{1{:}2\}$, $\{1{:}31, 2{:}1\}$, $\{1{:}341, 2{:}12, 3{:}1\}$ of [14].

11. Verification

Every theorem and proposition above, together with Lemma 2.5, has been formally verified in Lean 4 against *Mathlib*, with the three exceptions listed at the end of this section; the development contains no `sorry`, and the repository reference is to be added at submission. The definitional layer and the reasoning layer are kernel-checked with no appeal to compiled evaluation: the Bruhat order is the reflexive-transitive closure of the covering relation, the coefficients h_j are defined as cardinalities of sets of *all* words with no finiteness assumed, and Lemma 2.5, the repetition-freeness of the computed ideal, the identification of h_j with a count over that ideal, and the two length identities of Proposition 4.1 — the only statements here proved at every n — depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: the three counts at each $n \leq 6$ in Theorems 3.1, 3.2 and 3.3 (one evaluation per n, the largest being the 518 400 pairs of $\mathfrak{S}_6$); the individual values of π in Theorems 5.1, 7.1 and 8.1; the rank sequences of Theorem 6.1 and Proposition 9.2; and the $x = e$ column of Proposition 9.1. The pattern containments and avoidances of Theorems 7.1 and 8.1 are decided by the kernel itself and depend on no axioms at all.

Three things are not part of the formal development, and they are separate from the measurements of §10, which are outside it by construction. First, Lemma 2.4: it quotes published theorems and is used only as mathematics. Second, the identity $S(w, e) = [e, w]$ behind that lemma, which is verified computationally at each n where a table is stated but is not proved for every n. Third, inside Proposition 4.1, the passage from the two length identities — which *are* verified at every n — to the reversal of $h(w, \cdot)$, to the equivalence of palindromicity at x and at w_0x, and to the parity of π_n: that passage is the short argument printed in §4, and its palindromicity consequence is in addition checked outright over all of $\mathfrak{S}_4 \times \mathfrak{S}_4$ and $\mathfrak{S}_5 \times \mathfrak{S}_5$. Every numerical value in the verified statements was first produced by two further independent implementations — a C program deciding the Bruhat order by the Ehresmann rank-matrix criterion, and a Python program that builds the induced subposet, extracts its Hasse diagram and ranks by saturated chains from the minimum, assuming no rank formula at all — which agree with the formal development at every $n \leq 5$, with the first two agreeing at $n = 6$.

12. Questions

Question 12.1. Characterise the never palindromic permutations. By Theorems 7.1 and 8.1 the answer is not a pattern condition in either direction.

Question 12.2. Is there a formula, or a generating function, for U_n? The measured values are $2, 6, 15, 38, 95, 238$.

Question 12.3. Proposition 4.1 gives one involution on shifts, $x \mapsto w_0x$, which forces $\pi_n(w)$ to be even for $n \geq 2$. At $n = 4$ every value of π_4 is divisible by 4. Is there a second involution? A natural candidate at the level of pairs is $(w, x) \mapsto (w_0ww_0, w_0xw_0)$, which preserves palindromicity in all computed cases $n \leq 6$ but which we have not proved.

Question 12.4. Everything in [1] holds for an arbitrary Coxeter group. Do the phenomena above survive outside type A? In particular, is the separation of Theorem 5.1 a type-A accident? In any type the $x = e$ column is again the classical enumeration of rationally smooth elements.

Question 12.5. When the geometric dictionary of [2] becomes available, does palindromicity of $F_{w,x}$ acquire a geometric meaning? Point (b) of §1.3 shows the poset rank is not a cell dimension of X_w, so if there is a meaning it is not that one.

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