LOCAL STABILITY OF SHAPIRO–DIANANDA CYCLIC SUMS: EXPLICIT FAILURES AND EXACT DIAGONAL IDENTITIES

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Abstract. For $n \geq 2$, $1 \leq k \leq n-1$ and positive reals $x_1, \dots, x_n$ with cyclic indices, let $S_{n,k}(x) = \sum_{i=1}^{n} x_i/(x_{i+1} + \cdots + x_{i+k})$; $k = 2$ is Shapiro's cyclic sum. At the equal point the sum is n/k, and the Diananda inequality at (n, k) asserts $S_{n,k} \geq n/k$. Sheremet (arXiv:2606.05504) expanded $S_{n,k}(e^u)$ to second order at the equal point, obtaining a circulant quadratic form $Q_{n,k}$ whose minimum $C^{\mathrm{loc}}_{n,k}$ over the zero-mean unit sphere decides whether the equal point is a strict local minimum, quadratically degenerate, or a saddle; he classified $k = 2, 3$ and tabulated $3 \leq k \leq 10$ numerically. We add four things, all of them finite rational arithmetic. (i) Explicit positive-integer vectors refuting $S_{n,k} \geq n/k$ at the least saddle cell of each $3 \leq k \leq 10$; the smallest is $x = (9, 5, 3, 8, 8, 3, 5)$ at $(n, k) = (7, 4)$, where $S_{7,4}(x) = 961/550 < 7/4$. (ii) Five exact identities for $Q_{n,k}$ on the diagonals $n = k+1, k+2, 2k+1, 2k+2, k+3$, valid for every $k \geq 1$: in particular $C^{\mathrm{loc}}_{k+1,k} = (k+1)/k^3$ exactly, the equal point of $S_{2k+1,k}$ is a strict local minimum for every k with $C^{\mathrm{loc}}_{2k+1,k} \geq 1/(2k^2)$, and on $n = k+2$ and $n = 2k+2$ it is never a saddle. (iii) A saddle at $(n, k) = (14, 11)$, beyond the tabulated range, on the diagonal $n = k + 3$ whose symbol factors as $2(k + 2)(c + \frac{1}{k+2})(c + \frac{1}{2})$. (iv) An erratum: three entries of the source's $k = 3$ numerical table are wrong, and the entry -0.005046 printed for $n = 17$ lies below the infimum of the source's own symbol over the whole interval. Every statement is machine-checked in Lean 4. The $k = 2$ local analysis is classical (Searcy and Troesch, 1979) and nothing is claimed for it.

1. Introduction

1.1. **The objects.** Fix $n \geq 2$ and $1 \leq k \leq n-1$. For positive reals $x_1, \dots, x_n$, indices read modulo n, put

$$S_{n,k}(x) = \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2} + \cdots + x_{i+k}}.$$

For $k = 2$ this is the sum of Shapiro's cyclic inequality [11]; for general k the sums were studied by Diananda [3, 4], and after Sadov [7, 8, 9] they are called Shapiro–Diananda sums. At the equal point $x_1 = \cdots = x_n$ one has $S_{n,k} = n/k$, and we call the assertion

$$S_{n,k}(x) \geq \frac{n}{k} \qquad \text{for every positive } x$$

the *Diananda inequality at* (n, k). It is true for some pairs and false for others; for $k = 2$ the exact range of validity is classical (Troesch [13] and the references there), and for $k \geq 3$ the literature is thin.

The sum is homogeneous of degree zero, so its local behaviour at the equal point is naturally studied in logarithmic coordinates $x_i = e^{u_i}$ with $\sum_i u_i = 0$. Sheremet [12, Lemma 3.1] shows that, as $u \to 0$ on that hyperplane,

$$S_{n,k}(e^u) = \frac{n}{k} + Q_{n,k}(u) + O(\|u\|^3), \qquad Q_{n,k}(u) = -\frac{1}{k^2}\sum_{i=1}^{n} u_i L_i + \frac{1}{k^3}\sum_{i=1}^{n} L_i^2, \tag{1}$$

where $L_i = u_{i+1} + \cdots + u_{i+k}$ is the block that sits in the i-th denominator. The form $Q_{n,k}$ is circulant, hence diagonal in the discrete Fourier basis [12, Lemma 4.1]: writing $\lambda_{m,k} = \sum_{j=1}^{k} e^{2\pi i m j/n}$,

$$C^{\mathrm{loc}}_{n,k} := \min\left\{\frac{Q_{n,k}(u)}{\|u\|^2} : \textstyle\sum_i u_i = 0,\ u \neq 0\right\} = \min_{1 \leq m \leq n-1} \gamma_{m,k}, \qquad \gamma_{m,k} = \frac{|\lambda_{m,k}|^2 - k \operatorname{Re}\lambda_{m,k}}{k^3}. \tag{2}$$

The sign of $C^{\mathrm{loc}}_{n,k}$ decides the local trichotomy [12, Theorem 4.2]: if $C^{\mathrm{loc}}_{n,k} > 0$ the equal point is a strict local minimum modulo scaling, if $C^{\mathrm{loc}}_{n,k} = 0$ there is no positive quadratic stability constant (*quadratically degenerate*), and if $C^{\mathrm{loc}}_{n,k} < 0$ the equal point is a saddle, so that $S_{n,k}(x) < n/k$ for positive x arbitrarily close to equality — and the Dianända inequality at (n,k) is false. We call a pair (n,k) a *cell*, and speak of positive, degenerate and saddle cells accordingly.

1.2. **The source.** Numbering below refers to arXiv:2606.05504v1 [12]. Besides (1)–(2) and the trichotomy, the paper proves: the *periodic equality families* (Theorem 5.1: if $d \mid n$ and $d \mid k$ then every d-periodic positive x has $S_{n,k}(x) = n/k$ exactly, so $\gcd(n,k) > 1$ excludes strict stability, Corollary 5.2); a uniform eventual-saddle theorem (Theorem 7.2: $k \geq 3$ and $n > 12k$ imply $C^{\mathrm{loc}}_{n,k} < 0$); a computable finite-grid saddle criterion (Corollary 8.1, with a table of thresholds β_k for $3 \leq k \leq 10$); the complete classification for $k = 2$ (Theorem 10.1: $C^{\mathrm{loc}}_{n,2} = \frac{1}{2}\sin^2(\pi/n)$ for odd n and 0 for even n) and for $k = 3$ (Theorem 11.1: $C^{\mathrm{loc}}_{n,3} > 0$ exactly for $n \in \{4,5,7,8,10,13,16,19\}$, $C^{\mathrm{loc}}_{n,3} = 0$ exactly for $n \in \{6,9,12,\ldots,30\}$, and $C^{\mathrm{loc}}_{n,3} < 0$ for every other $n \geq 4$); and, in a separate global part, the sharp value $\log 2$ of Sadov's constant (Theorem 14.3). Section 12 ("Numerical illustration") prints $C^{\mathrm{loc}}_{n,3}$ to six decimals for $4 \leq n \leq 20$, and Section 13 ("Small fixed values of k") prints, for $3 \leq k \leq 10$, the sets of positive and degenerate n and the threshold past which every n is a saddle. That table is introduced by the sentences

> "The exact finite-grid criterion can be evaluated directly for any prescribed k. The following table gives illustrative positive and zero-quadratic cases for $3 \leq k \leq 10$. It is not used in any proof. The entries were obtained by direct evaluation of the exact finite Fourier criterion; all admissible values $n \geq k+1$ not appearing in the second or third column are saddle cases."

and closed by "This table is included only as an illustration of the exact finite Fourier criterion." So for $4 \leq k \leq 10$ the cells are printed data, not theorems. The source states no closed form on any diagonal $n = f(k)$ and no cell-by-cell information for $k \geq 11$ beyond the uniform Theorem 7.2, and it prints no positive vector with a computed value of the sum below n/k for any $k \geq 3$. At $(11,3)$ it exhibits the direction $u_i = \cos(8\pi i/11)$ and asserts $S_{11,3}(e^{tu}) < 11/3$ for all sufficiently small $t \neq 0$ (Section 11.3); at every other saddle cell its conclusion is the existential one of Theorem 4.2(ii).

1.3. **What was already known.** Three older sources matter for the novelty of what follows; the first two are not cited by [12], whose bibliography has eight items.

Searcy and Troesch, 1979. For Shapiro's sum ($k = 2$, N variables) Searcy and Troesch [10] write $x = x^0 + e$ with x^0 a "nominal" vector — their (1.1): for even N, $x^0_i = (1+\alpha)/2$ for odd i and $(1-\alpha)/2$ for even i, $0 \leq \alpha \leq 1$, all of which give $S = N/2$, the equal point being $\alpha = 0$ — expand the sum to second order in e, and solve the eigenvalue problem of the resulting symmetric matrix: the eigenvectors are sinusoids of frequency $h = 2\pi j/N$, and "The N corresponding eigenvalues are $\lambda = 2\sin h\,(2\sin h - \alpha)$". A negative eigenvalue requires "$0 < \sin(2\pi j/N) < 1/2$, $2\pi j/N < \pi/6$, or finally $N > 12j$"; and at the equal point itself: "$\lambda = 4\sin^2 h \geq 0$ for $\alpha = 0$. This means that for odd N, where the only simple nominal vector x^0 is furnished by $\alpha = 0$, the eigenvalues are all nonnegative, so that the argument given above cannot be applied to odd N. Furthermore, higher order terms in the e-expansion do not alter this conclusion." Taken at $\alpha = 0$ this is the $k = 2$ case of (2): the source's Theorem 10.1 is, up to normalisation, their spectrum in logarithmic coordinates, and their $N > 12j$ has the same shape as the source's $n > 12k$. Nothing in this paper concerns $k = 2$ beyond the fact that our identities are valid there too.

Boarder and Daykin, 1973. Section 10 of [1] reports a 1968 computer search by K. Y. Choong and D. E. Daykin at the University of Malaya for counterexamples to the Dianända inequality, in their notation $\sum a_i/(a_{i+1} + \cdots + a_{i+k}) \geq M/k$ with M variables (their (16)). It prints the list of pairs (k,M) for which "inequality (16) is not proved true by (17) or Nowosad (5) but no example was found which made it false", adding "For $k \leq 12$ the list contains all cases (k,M) for which it is not yet known whether (16) holds for all a_i":

(3)
$(2,11), (2,12), (2,13), (2,15), (2,17), (2,19), (2,21), (2,23);\ (3,9), (3,10), (3,12), (3,13), (3,16), (3,19);$
$(4,12), (4,13), (4,16), (4,17);\ (5,15), (5,16);\ (6,18), (6,19);\ (7,11), (7,21), (7,22);$
$(8,24), (8,25);\ (9,27);\ (10,15), (10,30);\ (12,9), (12,18).$

No counterexample vector, no list of failing M and no criterion is printed; the sufficient condition (17) as printed there misquotes Diananda's 1959 theorem, as Sadov notes [9]. (We read the list from the publisher's PDF and sorted it by k and M; the page prints the same 32 pairs in a different order. The pair $(12, 9)$ has $M < k$ and is presumably a misprint.)

Sadov's surveys. Sadov's 2021 review [8] states that "the author is not aware of any substantial results concerning power sums of Shapiro-Diananda type not mentioned or not referred to in the quoted references" and contains no local result. Three dedicated surveys — Fink [5], Clausing [2], and Chapter 16 of Mitrinović, Pečarić and Fink [6] — could not be read for this paper (publisher paywall), nor could Diananda's 1959 note [3]. They are the remaining risk to the novelty claims of Section 3; see Section 1.5.

1.4. **Results.** Everything below is rational arithmetic: a failure of the Diananda inequality is exhibited by an explicit positive integer vector with the value of the sum computed exactly, and a local statement is an identity or inequality between rational quadratic forms. The Fourier symbol (2) is the motivation, never the object, except in the erratum, where the source's own closed formula for $k = 3$ is the object.

- Theorem 3.1: for each $3 \le k \le 10$, an explicit positive integer vector x with $S_{n,k}(x) < n/k$ at the least n for which the equal point is a saddle. The smallest cell is $(7, 4)$, where $x = (9, 5, 3, 8, 8, 3, 5)$ gives $961/550 < 7/4$ and can be checked by hand. Proposition 3.2 adds seven further cells, including $n = 25$ for $k = 3, 4, 5$.
- Theorems 4.2 to 4.6: exact identities for $Q_{n,k}$ on the diagonals $n = k+1, k+2, 2k+1, 2k+2, k+3$, each valid for every $k \ge 1$. Consequences: $C^{\mathrm{loc}}_{k+1,k} = (k+1)/k^3$; $C^{\mathrm{loc}}_{k+2,k} \ge 0$ with equality exactly for even k; $C^{\mathrm{loc}}_{2k+1,k} \ge 1/(2k^2)$, so the equal point of $S_{2k+1,k}$ is a strict local minimum for every k; $C^{\mathrm{loc}}_{2k+2,k} \ge 0$; and on $n = k+3$ the symbol is $2(k+2)(c + \frac{1}{k+2})(c + \frac{1}{2})$ in $c = \cos\theta$. A universal bound $C^{\mathrm{loc}}_{n,k} \ge -1/(4k)$ (Proposition 4.1) comes from completing the square.
- Theorem 5.1: $Q_{14,11} < 0$ on an explicit zero-sum integer vector, so $(14, 11)$ is a saddle cell — the cell of the diagonal $n = k + 3$ at the first k past the source's table.
- Theorem 6.1: the value -0.005046 printed for $C^{\mathrm{loc}}_{17,3}$ in the source's Section 12 is below the infimum of the source's own symbol $\Gamma_3(c) = 2(1-c)(2c+1)(3c+2)/27$ over $c \in [-1, 1]$; the correct value is $-0.004681049\ldots$. The $n = 19$ and $n = 20$ entries are also wrong.

Computations that go beyond the machine-checked statements — the exact classification of all cells with $3 \le k \le 24$, its comparison with the list (3), and a check that quadratic degeneracy has only the periodic explanation for $k \le 40$ — are reported as remarks in Section 5, clearly separated from the theorems.

1.5. **What is and is not new.** The identities of Section 4 were found in no source, global or local; the $n = k + 1$ inequality itself is the classical n-variable Nesbitt inequality, but its quadratic form $((k+1)/k^3)\|u\|^2$ is not in print as far as we can tell. The cells of Theorem 3.1 need a caveat. All fifteen cells of Theorem 3.1 and Proposition 3.2 are absent from (3); since each of them genuinely fails, none can have been correctly proved true, so (unless the misprinted criterion was applied) each was settled by an unpublished counterexample in 1968. Nothing about that computation was printed. We therefore claim for Theorem 3.1 only that the statements are nowhere in the literature and that the vectors are new, not that the facts were unknown to anyone; and we repeat that the three surveys named in Section 1.3 were not read. The $k = 2$ local analysis belongs to [10]. The cells with $4 \le k \le 10$ are printed in the source's Section 13 as numerical data, and we do not claim them; what Theorem 3.1 adds at those cells is a finite certificate for the inequality itself.

2. Preliminaries

We use the notation of (1): for $u \in \mathbb{R}^n$ (indices modulo n), $L_i = u_{i+1} + \cdots + u_{i+k}$, $\|u\|^2 = \sum_i u_i^2$, and

$$Q_{n,k}(u) = -\frac{1}{k^2}\sum_{i=1}^{n} u_i L_i + \frac{1}{k^3}\sum_{i=1}^{n} L_i^2.$$

$Q_{n,k}$ is defined for every u, not only on the hyperplane $\sum u_i = 0$; $C^{\mathrm{loc}}_{n,k}$ is its minimum over the zero-mean unit sphere as in (2). Two elementary facts are used throughout.

Lemma 2.1 (window and autocorrelation)**.** *Let $u \in \mathbb{R}^n$ with $\sum_i u_i = 0$ and let $c_u(a) = \sum_i u_i u_{i+a}$ be the cyclic autocorrelation. Then:*

(a) *for every i, $u_i + L_i + (u_{i+k+1} + \cdots + u_{i+n-1}) = 0$: the block, its base point and the complementary block exhaust the cycle;*

(b) *$c_u(-a) = c_u(a)$, and $\sum_i u_i L_i = \sum_{a=1}^{k} c_u(a)$.*

Proof. (a) is $\sum_i u_i = 0$ read from the base point i; (b) is the substitution $i \mapsto i - a$ and the definition of L_i. □

We also record the source's illustrative table (Section 13 of [12]) in the form we refer to it; it is numerical data there, and it is reproduced here only so that the cells of Sections 3 and 4 can be located in it.

k	$\{n : C^{\mathrm{loc}}_{n,k} > 0\}$	$\{n : C^{\mathrm{loc}}_{n,k} = 0\}$	saddle for all $n \geq$
3	4,5,7,8,10,13,16,19	6,9,12,15,18,21,24,27,30	31
4	5,9,13,17	6,8,10,12,16,20,24	25
5	6,7,8,11,12,16	10,15,20	21
6	7,13,19	8,9,12,14,18,24	25
7	8,9,11,15,16,22	14,21	23
8	9,17,25	10,12,16,18,24	26
9	10,11,19,20	12,18,27	28
10	11,21	12,15,20,22,30	31

Reading the table down the columns: for every k the positive cells contain $n = k + 1$ and $n = 2k + 1$, for odd k also $n = k + 2$ and $n = 2k + 2$, while for even k the cells $n = k + 2$ and $n = 2k + 2$ are degenerate. The remaining positive cells lie on $n = 3k + 1$ (for $3 \leq k \leq 8$), $n = 4k + 1$ ($k = 3, 4$), $n = 5k + 1$ and $6k + 1$ ($k = 3$), $n = k + 3$ ($k = 5$) and $n = k + 4$ ($k = 7$); the rows $k = 9, 10$ have none. Section 4 explains the first four diagonals for every k; the other positive cells are not treated here. The least saddle n in each row — $11, 7, 9, 10, 10, 11, 13, 13$ for $k = 3, \ldots, 10$ — are the cells of Theorem 3.1.

3. Explicit failures of the Diananda inequality

Theorem 3.1. *For each row of the following table the vector x of positive integers satisfies $S_{n,k}(x) < n/k$; the third column is the exact value of $S_{n,k}(x)$.*

(n,k)	x	$S_{n,k}(x)$	$n/k - S_{n,k}(x)$
$(11,3)$	$(11,3,5,10,1,8,8,1,10,5,3)$	$56803/15504$	$15/5168$
$(7,4)$	$(9,5,3,8,8,3,5)$	$961/550$	$3/1100$
$(9,5)$	$(8,6,4,5,8,8,5,4,6)$	$135677/75516$	$1259/377580$
$(10,6)$	$(5,4,2,2,4,5,4,2,2,4)$	$1042/627$	$1/209$
$(10,7)$	$(8,5,4,8,7,4,7,8,4,5)$	$17624/12341$	$6/12341$
$(11,8)$	$(8,6,4,7,8,5,5,8,7,4,6)$	$27485/19992$	$1/4998$
$(13,9)$	$(7,6,5,6,7,7,5,5,7,7,6,5,6)$	$4289/2970$	$1/2970$
$(13,10)$	$(8,5,5,8,6,4,7,7,4,6,8,5,5)$	$46771/35990$	$8/17995$

In particular the Diananda inequality is false at each of these eight cells.

Proof. Each row is a finite computation in $\mathbb{Q}$: the n denominators are sums of k of the entries, and the sum of the n fractions is evaluated exactly and compared with n/k. Every row is checked in Lean 4 by exactly this computation (Section 7); the machine-checked statements are the eight strict inequalities, and the exact values in the third column are the evaluations through which each proof passes. For $(7,4)$ the computation fits on one line: the seven denominators are $24, 22, 24, 25, 22, 22, 25$, so

$$S_{7,4}(x) = \frac{9}{24} + \frac{5}{22} + \frac{3}{24} + \frac{8}{25} + \frac{8}{22} + \frac{3}{22} + \frac{5}{25} = \frac{1}{2} + \frac{8}{11} + \frac{13}{25} = \frac{961}{550} < \frac{7}{4} = \frac{962.5}{550}. \qquad \square$$

The cells are the least saddle n of each row of the table in Section 2; for $k = 3$ this is a theorem of the source (Theorem 11.1), for $4 \leq k \leq 10$ it is the source's numerical table, reproduced exactly by an independent exact computation (Remark 5.3). Nothing is asserted about the Diananda inequality at the smaller n of each row: a strict local minimum does not decide a global inequality. The vectors were found by rounding a multiple of a negative Fourier mode of $Q_{n,k}$ to small integers and testing exactly; how they were found plays no role in the statement. The gaps are small — of the order of the Hessian eigenvalue that produces them — and the next proposition shows they are not artefacts.

Proposition 3.2. *The Diananda inequality is also false at the cells* $(14,3)$*,* $(17,3)$*,* $(20,3)$*,* $(11,4)$*,* $(25,3)$*,* $(25,4)$ *and* $(25,5)$*, with the explicit witnesses*

$(14,3):\ (9,4,5,9,3,7,8,3,8,7,3,9,5,4),\ S = 4516/969;$

$(17,3):\ (9,4,5,9,3,6,8,3,7,7,3,8,6,3,9,5,4),\ S = 577/102;$

$(20,3):\ (8,5,5,8,4,6,8,4,7,7,4,7,7,4,8,6,4,8,5,5),\ S = 6458/969;$

$(11,4):\ (8,6,4,7,8,5,5,8,7,4,6),\ S = 10709/3900;$

$(25,3):\ (9,4,5,9,3,7,8,3,8,6,4,9,5,5,9,4,6,8,3,8,7,3,9,5,4),\ S = 48421/5814;$

$(25,4):\ (8,6,4,7,7,4,5,8,6,4,7,8,5,5,8,7,4,6,8,5,4,7,7,4,6),\ S = 1120501/179400;$

$(25,5):\ (10,6,2,5,10,7,2,4,10,8,3,3,9,9,3,3,8,10,4,2,7,10,5,2,6),\ S = 6475833019/1295854560.$

Proof. The same computation as in Theorem 3.1, machine-checked cell by cell. □

The first three are the next saddle cells of Theorem 11.1 after $n = 11$; $(11,4)$ is the second saddle cell of the $k = 4$ row. The three cells with $n = 25$ sit at the threshold the source derives from its finite-grid criterion for $k = 4$ and $k = 5$ ("for $k = 4$ the general criterion already implies saddle behaviour for all $n \geq 25$, and for $k = 5$ it implies saddle behaviour for all $n \geq 25$", Section 8); $(25,3)$ is a saddle cell by Theorem 11.1. At $(25,5)$ one has $\gcd(25,5) = 5$, so the equal point already lies on a periodic equality family (Theorem 5.1 of the source); the point is that the sum goes strictly below 5.

Proposition 3.3 (controls)**.**

(a) $S_{7,4}(1,\dots,1) = 7/4$*,* $S_{11,3}(1,\dots,1) = 11/3$*,* $S_{13,4}(1,\dots,1) = 13/4$*; and the* 4*-periodic vector* $(5,2,9,1,5,2,9,1,5,2,9,1)$ *gives* $S_{12,4} = 3 = 12/4$ *exactly, an instance of the source's Theorem 5.1.*

(b) $S_{7,4}(9,5,3,8,8,3,5) = 7/4-3/1100$*, and it is* not *the case that* $S_{7,4}(9,5,3,8,8,3,5) < 7/4-1/300$*;* $S_{11,3}(11,3,5,10,1,8,8,1,10,5,3) = 11/3 - 15/5168$*.*

(c) $S_{7,4}(8,5,3,8,8,3,5) = 809/462 > 7/4$ *and* $S_{7,4}(9,5,4,8,8,3,5) = 431789/246675 > 7/4$*: changing one entry of the* $(7,4)$ *witness by one destroys it.*

(d) *At the positive cells* $(13,4)$ *and* $(9,4)$ *the vectors produced by the same construction do not go below* n/k*:* $S_{13,4}(101,99,101,99,101,100,100,100,100,101,99,101,99) = 521303/160400 > 13/4$ *and* $S_{9,4}(101,100,100,101,100,100,101,100,100) = 60451/26867 > 9/4$*.*

These are the checks one wants before believing a table of strict inequalities all pointing the same way: the equal point really gives n/k, the gaps are exactly what they are and not larger, the witnesses are fragile in the way a 10^{-3} Hessian eigenvalue predicts, and the construction does not fire at cells where the equal point is a strict local minimum.

4. Five exact identities on diagonals

All identities in this section hold for every $k \geq 1$; n is a function of k and u ranges over $\mathbb{R}^n$ with $\sum_i u_i = 0$ unless stated otherwise. They are exact identities of rational quadratic forms, proved by elementary manipulation of Lemma 2.1, and every one of them is machine-checked as an identity valid for all k and all u (with $\mathbb{Q}$ in place of $\mathbb{R}$, which is immaterial for a polynomial identity). We begin with the one bound that needs no hypothesis at all.

Proposition 4.1 (completing the square)**.** *For every* n*, every* $k \geq 1$ *and* every $u \in \mathbb{R}^n$ *(no zero-mean hypothesis),*

$$Q_{n,k}(u) + \frac{1}{4k}\|u\|^2 = \frac{1}{k^3}\sum_{i=1}^{n}\Big(L_i - \frac{k}{2}u_i\Big)^2.$$

Consequently $C^{\mathrm{loc}}_{n,k} \geq -1/(4k)$ *for every cell* (n,k)*.*

Proof. Expand the right-hand side: $\sum_i (L_i - \frac{k}{2}u_i)^2 = \sum_i L_i^2 - k\sum_i u_i L_i + \frac{k^2}{4}\|u\|^2$, and divide by k^3. $\square$

The bound is uniform in both n and k but far from sharp: for $k=3$ it gives $-1/12$, while the infimum of $C^{\mathrm{loc}}_{n,3}$ over n is $-0.00489\ldots$ (Section 6). It is recorded because it is the only bound here that holds at every cell.

Theorem 4.2 ($n = k+1$). *For every $k \geq 1$ and every $u \in \mathbb{R}^{k+1}$ with $\sum_i u_i = 0$,*

$$Q_{k+1,k}(u) = \frac{k+1}{k^3}\,\|u\|^2.$$

Hence $C^{\mathrm{loc}}_{k+1,k} = (k+1)/k^3$ exactly, and the equal point of $S_{k+1,k}$ is a strict local minimum for every k.

Proof. With $n = k+1$ the block L_i is the whole cycle minus u_i, so $L_i = -u_i$ by Lemma 2.1(a). Then $\sum_i u_i L_i = -\|u\|^2$ and $\sum_i L_i^2 = \|u\|^2$, and $Q = (1/k^2 + 1/k^3)\|u\|^2$. $\square$

For $k=3$ this is the entry $C^{\mathrm{loc}}_{4,3} = 4/27 = 0.148148\ldots$ of the source's Section 12, and for $k=2$ it is $C^{\mathrm{loc}}_{3,2} = 3/8 = \frac{1}{2}\sin^2(\pi/3)$ from Theorem 10.1.

Theorem 4.3 ($n = k+2$). *For every $k \geq 1$ and every $u \in \mathbb{R}^{k+2}$ with $\sum_i u_i = 0$,*

$$Q_{k+2,k}(u) = \frac{k+2}{2k^3}\sum_{i=1}^{k+2}(u_i + u_{i-1})^2.$$

Hence $C^{\mathrm{loc}}_{k+2,k} \geq 0$ for every k: the equal point of $S_{k+2,k}$ is never a saddle. It is quadratically degenerate exactly when k is even, and a strict local minimum when k is odd.

Proof. Now $L_i = -(u_i + u_{i-1})$ by Lemma 2.1(a). Writing $N = \|u\|^2$ and $P = \sum_i u_i u_{i-1}$, one gets $\sum_i u_i L_i = -(N+P)$ and $\sum_i L_i^2 = 2N + 2P = \sum_i (u_i + u_{i-1})^2$, whence $Q = (1/k^2 + 2/k^3)(N+P)$. For the consequence: if $k+2$ is even the alternating vector $(1,-1,1,-1,\ldots)$ has zero sum and $u_i + u_{i-1} = 0$ for all i, so $Q = 0$; if $k+2$ is odd, $u_i = -u_{i-1}$ around an odd cycle forces $u = 0$, so $Q > 0$ on every nonzero zero-mean u. $\square$

The parity here is exactly the parity visible in the table of Section 2: for even k the cell $n = k+2$ sits in the degenerate column (it is an instance of the source's Corollary 5.2, since $2 \mid \gcd(k+2,k)$), for odd k in the positive one.

Theorem 4.4 ($n = 2k+1$). *For every $k \geq 1$ and every $u \in \mathbb{R}^{2k+1}$ with $\sum_i u_i = 0$,*

$$Q_{2k+1,k}(u) = \frac{1}{2k^2}\|u\|^2 + \frac{1}{k^3}\sum_{i=1}^{2k+1} L_i^2, \qquad \textit{so} \qquad Q_{2k+1,k}(u) \geq \frac{1}{2k^2}\|u\|^2.$$

Hence $C^{\mathrm{loc}}_{2k+1,k} \geq 1/(2k^2) > 0$: the equal point of $S_{2k+1,k}$ is a strict local minimum for every $k \geq 1$.

Proof. On a cycle of length $2k+1$ the complement of $\{i\} \cup \{i+1,\ldots,i+k\}$ is the block $M_i = u_{i+k+1} + \cdots + u_{i+2k}$ of the same length k, and Lemma 2.1(a) says $u_i + L_i + M_i = 0$. Multiply by u_i and sum: $\|u\|^2 + \sum_i u_i L_i + \sum_i u_i M_i = 0$. By Lemma 2.1(b), $\sum_i u_i L_i = \sum_{a=1}^{k} c_u(a)$ and $\sum_i u_i M_i = \sum_{a=k+1}^{2k} c_u(a) = \sum_{a=1}^{k} c_u(-a) = \sum_{a=1}^{k} c_u(a)$, because $a \mapsto 2k+1-a$ maps $\{k+1,\ldots,2k\}$ onto $\{1,\ldots,k\}$ and c_u is even. So $\sum_i u_i L_i = -\frac{1}{2}\|u\|^2$ whatever u is, and the remaining term $k^{-3}\sum_i L_i^2$ is nonnegative. $\square$

For $k \leq 10$ the cell $n = 2k+1$ is a positive entry of the source's table; the theorem says why, and that it never stops. The bound is not the exact value: at $k=3$ the source's Section 12 gives $C^{\mathrm{loc}}_{7,3} = 0.066962\ldots$ against $1/18$, and at $k=2$ Theorem 10.1 gives $C^{\mathrm{loc}}_{5,2} = \frac{1}{2}\sin^2(\pi/5)$ against $1/8$.

Theorem 4.5 ($n = 2k+2$). *For every $k \geq 1$ and every $u \in \mathbb{R}^{2k+2}$ with $\sum_i u_i = 0$,*

$$Q_{2k+2,k}(u) = \frac{1}{4k^2}\sum_{i=1}^{2k+2}(u_i + u_{i+k+1})^2 + \frac{1}{k^3}\sum_{i=1}^{2k+2} L_i^2.$$

Hence $C^{\mathrm{loc}}_{2k+2,k} \geq 0$ for every k: the equal point of $S_{2k+2,k}$ is never a saddle.

Proof. Now the cycle is $\{i\} \cup \{i+1, \dots, i+k\} \cup \{i+k+1\} \cup \{i+k+2, \dots, i+2k+1\}$, so $u_i + L_i + u_{i+k+1} + M_i = 0$ with M_i the last block; the same reflection argument gives $\sum_i u_i M_i = \sum_i u_i L_i$, hence $\sum_i u_i L_i = -\frac{1}{2}\big(\|u\|^2 + \sum_i u_i u_{i+k+1}\big)$, and $\|u\|^2 + \sum_i u_i u_{i+k+1} = \frac{1}{2}\sum_i (u_i + u_{i+k+1})^2$ because $i \mapsto i+k+1$ is an involution of the cycle. $\square$

For odd k the first sum vanishes only on vectors with $u_{i+k+1} = -u_i$, and the source's table lists $n = 2k+2$ as positive for odd $k \le 9$; for even k the cell is degenerate (Corollary 5.2 of the source, $2 \mid \gcd(2k+2, k)$). We have not proved positivity for odd k in general, and do not claim it.

Theorem 4.6 ($n = k+3$)**.** *For every $k \ge 1$ and every $u \in \mathbb{R}^{k+3}$ with $\sum_i u_i = 0$,*

$$k^3\, Q_{k+3,k}(u) = (k+3)\|u\|^2 + (k+4)\sum_{i=1}^{k+3} u_i u_{i-1} + (k+2)\sum_{i=1}^{k+3} u_i u_{i-2}.$$

Proof. Here $L_i = -(u_i + u_{i-1} + u_{i-2})$ by Lemma 2.1(a). With $N = \|u\|^2$, $P = \sum_i u_i u_{i-1}$ and $R = \sum_i u_i u_{i-2}$ one finds $\sum_i u_i L_i = -(N + P + R)$ and $\sum_i L_i^2 = 3N + 4P + 2R$, so $k^3 Q = k(N+P+R) + 3N + 4P + 2R$. $\square$

Corollary 4.7. *On the Fourier mode $u_j = \cos(j\theta)$, $\theta = 2\pi m/(k+3)$, the right-hand side of Theorem 4.6 has symbol*

$$(k+3) + (k+4)\cos\theta + (k+2)\cos 2\theta = 2(k+2)\Big(c + \frac{1}{k+2}\Big)\Big(c + \frac{1}{2}\Big), \qquad c = \cos\theta,$$

which is negative exactly for $-\frac{1}{2} < c < -\frac{1}{k+2}$. Hence $(k+3, k)$ is a saddle cell if and only if some $\cos(2\pi m/(k+3))$, $1 \le m \le k+2$, lies in the open arc $(-\frac{1}{2}, -\frac{1}{k+2})$.

Proof. $\cos 2\theta = 2c^2 - 1$ and the factorisation is one line; the equivalence is the symbol formula (2) applied to the identity. (This corollary is not machine-checked; it is the only place in this section where the symbol is used.) $\square$

For $k = 4$ the arc is $(-\frac{1}{2}, -\frac{1}{6})$ and contains $\cos(4\pi/7) = -0.2225\dots$, which is the cell $(7,4)$ of Theorem 3.1; for $k = 3$, 6, 9 the grid contains $c = -\frac{1}{2}$ itself (the degenerate cells $(6,3)$, $(9,6)$, $(12,9)$ of the table) and misses the open arc. Section 5 returns to this diagonal.

Proposition 4.8 (the zero-mean hypothesis is essential)**.** *For every n and $k \ge 1$, $Q_{n,k}(1,\dots,1) = 0$. In particular the identity of Theorem 4.2 fails on the constant vector, whose right-hand side is $(k+1)^2/k^3 > 0$.*

Proof. On the constant vector $L_i = k$, so $Q = -n/k + n/k = 0$; this is the scaling invariance of $S_{n,k}$, and it is why $C^{\mathrm{loc}}_{n,k}$ is a minimum over the zero-mean subspace only. $\square$

5. Beyond the source's table

Theorem 5.1 ($(n,k) = (14,11)$ is a saddle cell)**.** *Let $u = (2,0,-2,1,1,-2,0,2,0,-2,1,1,-2,0) \in \mathbb{Z}^{14}$. Then $\sum_i u_i = 0$ and*

$$Q_{14,11}(u) = -\frac{10}{1331} < 0,$$

so $C^{\mathrm{loc}}_{14,11} < 0$ and the equal point of $S_{14,11}$ is a saddle. Moreover, in agreement with Theorem 4.6, $11^3\, Q_{14,11}(u) = 14\|u\|^2 + 15\sum_i u_i u_{i-1} + 13\sum_i u_i u_{i-2}$.

Proof. Direct evaluation of the rational quadratic form on a fixed vector; the last identity is Theorem 4.6 at $k = 11$. Since $\|u\|^2 = 28$, the Rayleigh quotient gives $C^{\mathrm{loc}}_{14,11} \le -5/18634$. By the source's Theorem 4.2(ii) there are positive vectors arbitrarily close to equality with $S_{14,11} < 14/11$; we have not written one down. $\square$

This is the cell of the diagonal $n = k+3$ of Corollary 4.7 at $k = 11$, the first k past the source's $k \le 10$ table: for $k = 11$ the arc is $(-\frac{1}{2}, -\frac{1}{13})$ and the grid point $m = 4$, $\cos(2\pi \cdot 4/14) = \cos(4\pi/7) = -0.2225\dots$, lies in it. The vector u above is $2\cos(4\pi j/7)$, $j = 0, \dots, 13$, rounded to integers; it has period 7 because $\gcd(4, 14) = 2$.

The source's Section 13 table records three further saddle cells of Theorem 3.1 on the same diagonal, $k = 7, 8, 10$, and Theorem 3.1 certifies each of them by an explicit failure of the inequality; the corresponding local certificates for three of the source's printed signs are the following.

Proposition 5.2 (saddle directions at printed cells)**.** *The zero-sum integer vectors*

$$u^{(11,3)} = (8, -5, -1, 6, -7, 3, 3, -7, 6, -1, -5), \qquad u^{(7,4)} = (4, -1, -3, 2, 2, -3, -1),$$
$$u^{(9,5)} = (2, 0, -2, -1, 2, 2, -1, -2, 0)$$

satisfy $Q_{11,3}(u^{(11,3)}) = -\frac{1}{3}$, $Q_{7,4}(u^{(7,4)}) = -\frac{3}{32}$ *and* $Q_{9,5}(u^{(9,5)}) = -\frac{3}{25}$. *Hence* $C^{\mathrm{loc}}_{11,3} \leq -1/912$, $C^{\mathrm{loc}}_{7,4} \leq -3/1408$ *and* $C^{\mathrm{loc}}_{9,5} \leq -3/550$.

The signs at these three cells are the source's ($k = 3$ by Theorem 11.1, where the mode $u_i = \cos(8\pi i/11)$ is written down; $k = 4, 5$ in the Section 13 table); the rational directions and their exact values are ours. The first vector is the source's mode at $n = 11$ scaled by 8 and rounded.

Remark 5.3 (the exact classification for $3 \leq k \leq 24$; computation, not formalised). The Fourier symbol (2) is a polynomial in $y = 2\cos(2\pi m/n)$ with integer coefficients, and y is a root of the minimal polynomial of $2\cos(2\pi/d)$ for a divisor $d > 1$ of n. So the sign of $C^{\mathrm{loc}}_{n,k}$ can be decided exactly, with no floating point, by Sturm sequences over $\mathbb{Z}$. We did this for every cell with $3 \leq k \leq 24$ and $n \leq 12k$ (twenty CPU-minutes), which by the source's Theorem 7.2 classifies every cell with $3 \leq k \leq 24$. The rows $k = 3, \ldots, 10$ reproduce the source's table in Section 2 exactly, including the thresholds. For $11 \leq k \leq 24$ the result is: $C^{\mathrm{loc}}_{n,k} > 0$ exactly for $n \in \{k+1, 2k+1\}$ when k is even and for $n \in \{k+1, k+2, 2k+1, 2k+2\}$ when k is odd; the least saddle n is $k + 3$ for every $11 \leq k \leq 24$; and the degenerate cells are

k	$\{n : C^{\mathrm{loc}}_{n,k} = 0\}$	k	$\{n : C^{\mathrm{loc}}_{n,k} = 0\}$	k	$\{n : C^{\mathrm{loc}}_{n,k} = 0\}$
11	22, 33	16	18, 32, 34	21	42
12	14, 18, 24, 26, 36	17	34	22	24, 44, 46
13	26, 39	18	20, 36, 38	23	46
14	16, 28, 30	19	38	24	26, 48, 50
15	30	20	22, 40, 42		

so that past $k = 10$ only the four diagonals of Section 4 survive as positive cells. In terms of Corollary 4.7: the arc $(-\frac{1}{2}, -\frac{1}{k+2})$ widens towards $(-\frac{1}{2}, 0)$ while the grid spacing $2\pi/(k+3)$ shrinks, so from $k = 11$ on the grid never misses it; for $k \leq 10$ it meets the arc exactly for $k \in \{4, 7, 8, 10\}$, the cells $(7, 4), (10, 7), (11, 8), (13, 10)$ of Theorem 3.1. None of this paragraph is machine-checked; the formal development covers the four diagonals for all k and the single cell $(14, 11)$.

Remark 5.4 (what the 1968 search was seeing; computation, not formalised). Compared with the classification of Remark 5.3, every cell on the list (3) is a positive or a degenerate cell (the pair $(12, 9)$ has $M < k$ and is not a cell), and every saddle cell with $3 \leq k \leq 12$ is absent from the list (for $k = 2$ there are no saddle cells). In other words the 1968 search, hunting numerically for counterexamples, reproduced the local trichotomy cell for cell, without ever printing a failing cell, a vector or a criterion. This comparison is not in the literature; Sadov cites [1] only for its asymptotic numerics. It is not formalised: it would need positivity certificates for the twenty-three cells of (3) with $3 \leq k \leq 12$, most of which no identity of Section 4 reaches.

Remark 5.5 (degeneracy has only the periodic explanation, $k \leq 40$; computation). The source proves (Theorem 5.1 and Corollary 5.2) that $d \mid \gcd(n, k)$, $d > 1$, forces a zero Fourier coefficient, and remarks that "several zero Fourier modes have a global equality explanation". The converse — that $\gamma_{m,k} = 0$ only when $\lambda_{m,k} = 0$, i.e. only when $n/\gcd(m, n)$ divides k — is not in the source. Since $\gamma_{m,k} = 0$ means that the minimal polynomial of $2\cos(2\pi/d)$, $d = n/\gcd(m, n)$, divides the integer polynomial $W_k(y) = 2(|\lambda|^2 - k \operatorname{Re} \lambda)$ of degree k, the order d is constrained by $\varphi(d) \leq 2k$, hence $d \leq 8k^2$, and finitely many divisibility tests settle every n at once. We ran them for every $k \leq 40$: no zero other than the periodic ones occurs. We record the general statement as a conjecture.

Conjecture 5.6. For all $1 \leq k \leq n - 1$ and $1 \leq m \leq n - 1$: $\gamma_{m,k} = 0$ if and only if $\lambda_{m,k} = 0$, if and only if $n/\gcd(m, n)$ divides k.

6. An erratum for the source's numerical table

Section 11 of the source proves the closed form (its boxed display)

$$\gamma_{m,3} = \frac{2(1-\cos\theta_m)(2\cos\theta_m+1)(3\cos\theta_m+2)}{27}, \qquad \theta_m = \frac{2\pi m}{n},$$

and Section 12 tabulates $C^{\mathrm{loc}}_{n,3} = \min_m \gamma_{m,3}$ for $4 \le n \le 20$ to six decimals. Fourteen of the seventeen entries are correct. Three are not:

n	printed	correct
17	-0.005046	$-0.004681049\ldots$
19	0.002959	$0.001402823\ldots$
20	-0.004826	$-0.004886585\ldots$

The corrected values were obtained by four independent computations (exact Sturm sequences over $\mathbb{Z}$; a floating-point evaluation of $Q_{n,3}$ built from its definition rather than from the symbol; an 80-digit evaluation; and the source's boxed formula at 60 digits), which agree on all seventeen entries except these three. The signs, and therefore the classification column of the table and the theorem it illustrates (Theorem 11.1), are unaffected.

The $n = 17$ entry can be refuted without any arithmetic on roots of unity, and that is the machine-checked part. Write

$$\Gamma_3(c) = \frac{2(1-c)(2c+1)(3c+2)}{27}, \qquad c \in [-1,1],$$

for the source's symbol as a function of $c = \cos\theta$, so that $\gamma_{m,3} = \Gamma_3(\cos\theta_m)$ for every n and m.

Theorem 6.1. *For every $c \in [-1,1]$, $\Gamma_3(c) \ge -\dfrac{49}{10000}$. Consequently $\Gamma_3(c) > -0.005046$ for every $c \in [-1,1]$: the value printed for $C^{\mathrm{loc}}_{17,3}$ is not the value of $\gamma_{m,3}$ for any n and any m. The bound is not vacuous: $\Gamma_3(-\frac{3}{5}) < -\frac{1}{250}$.*

Proof. $(1-c)(2c+1)(3c+2) = -6c^3 - c^2 + 5c + 2$ has derivative $-(18c^2 + 2c - 5)$, so on $[-1,1]$ the minimum of Γ_3 is attained at the root $c^* = (-1-\sqrt{91})/18 = -0.58552\ldots$ of $18c^2 + 2c - 5$, where $\Gamma_3(c^*) = -0.004890210835\ldots$. The machine-checked proof avoids c^*: on $[0,1]$ the cubic is positive, and on $[-1,0]$ the inequality $\Gamma_3(c) + 49/10000 \ge 0$ is a consequence of the nonnegativity of $(1+c)(c+\frac{1171}{2000})^2$, $(1-c)(c+\frac{1171}{2000})^2$, $(1-c)(1+c)$ and $-c\,(c+\frac{1171}{2000})^2$ (the point $-1171/2000$ is c^* to four decimals), which a polynomial-arithmetic tactic combines. The constant $-49/10000$ is within $9.8 \cdot 10^{-6}$ of the true minimum. □

The entries for $n = 19$ and $n = 20$ lie inside the range of Γ_3, so refuting them needs the specific grid $\{\cos(2\pi m/n)\}$; they are corrected above by computation only, and no machine-checked statement is claimed for them. The source's statement that "the last strict local minimum occurs at $n = 19$" stands.

7. Verification

Every theorem and proposition above that carries a statement about specific vectors or an identity in k — Theorems 3.1, 4.2 to 4.6, 5.1 and 6.1 and Propositions 3.2, 3.3, 4.1, 4.8 and 5.2 — is formalised and checked in Lean 4 with Mathlib, in six modules built on two rational definitions: the cyclic sum $S_{n,k}$ on lists of rationals, and $Q_{n,k}$ on functions $\mathbb{Z}/n \to \mathbb{Q}$. The exact Lean statement of every checked declaration is reproduced verbatim in Appendix A. `#print axioms` reports `[propext, Classical.choice, Quot.sound]` — the standard axioms of Mathlib, no `sorry` — for every declaration except eight in the module `Local.lean`: the four evaluations `Qf_11_3_neg`, `Qf_7_4_neg`, `Qf_9_5_neg` and `Qf_14_11_neg` of Theorem 5.1 and Proposition 5.2 and the four zero-sum checks `sum_u_11_3`, `sum_u_7_4`, `sum_u_9_5`, `sum_u_14_11` of the same vectors go through `native_decide`, because a concrete sum over $\mathbb{Z}/n$ of rational numbers does not reduce in the kernel, and therefore additionally carry one auxiliary axiom each, generated by `native_decide` and displayed as `<decl>._native.native_decide.ax_1_1`, which records that the compiled evaluation of the decision procedure returned `true`; `kadd3_at_11` inherits the one of `sum_u_14_11`. Everything else — in particular all fifteen counterexamples of Section 3, the six identities of Section 4 and the erratum — is checked by the kernel through `simp`, `norm_num`, `field_simp`, `ring`, `linarith` and `nlinarith`. The finite computations of Remarks 5.3 to 5.5 are outside the formal development. Repository reference to be added at submission.

Appendix A. The Lean statements

The statements below are copied verbatim from the source files (proofs omitted); `Qf k u` is $Q_{n,k}(u)$ with n inferred from the type of `u`, `bsum k u i` is L_i, `nrm2 u` is $\|u\|^2$, and `Ssum k x` is $S_{n,k}(x)$ with n the length of the list `x`.

Definitions. The cyclic sum lives on lists of rationals (`n` is the length); the quadratic form lives on functions $\mathbb{Z}/n \to \mathbb{Q}$. `Defs.lean`

```
def cyc (x : List ℚ) (i : ℕ) : ℚ := x.getD (i % x.length) 0
def blk (k : ℕ) (x : List ℚ) (i : ℕ) : ℚ := ∑ j ∈ range k, cyc x (i + 1 + j)
def Ssum (k : ℕ) (x : List ℚ) : ℚ := ∑ i ∈ range x.length, cyc x i / blk k x i
def bsum (k : ℕ) (u : ZMod n → ℚ) (i : ZMod n) : ℚ := ∑ j ∈ range k, u (i + ((j : ZMod n) + 1))
def bsumR (k : ℕ) (u : ZMod n → ℚ) (i : ZMod n) : ℚ := ∑ j ∈ range k, u (i - ((j : ZMod n) + 1))
def Qf (k : ℕ) (u : ZMod n → ℚ) : ℚ :=
  -(1 / (k : ℚ) ^ 2) * ∑ i, u i * bsum k u i + (1 / (k : ℚ) ^ 3) * ∑ i, (bsum k u i) ^ 2
def nrm2 (u : ZMod n → ℚ) : ℚ := ∑ i, (u i) ^ 2
```

`Diagonal.lean`

```
def corr (u : ZMod n → ℚ) (a : ZMod n) : ℚ := ∑ i, u i * u (i + a)
```

`Local.lean`

```
def u_11_3 : ZMod 11 → ℚ := show Fin 11 → ℚ from ![8, -5, -1, 6, -7, 3, 3, -7, 6, -1, -5]
def u_7_4 : ZMod 7 → ℚ := show Fin 7 → ℚ from ![4, -1, -3, 2, 2, -3, -1]
def u_9_5 : ZMod 9 → ℚ := show Fin 9 → ℚ from ![2, 0, -2, -1, 2, 2, -1, -2, 0]
def u_14_11 : ZMod 14 → ℚ :=
  show Fin 14 → ℚ from ![2, 0, -2, 1, 1, -2, 0, 2, 0, -2, 1, 1, -2, 0]
```

`Erratum.lean`

```
noncomputable def Gamma3 (c : ℝ) : ℝ := 2 * (1 - c) * (2 * c + 1) * (3 * c + 2) / 27
```

Behind Theorem 3.1 (9 declarations). `Failures.lean`

```
theorem Ssum_7_4 : Ssum 4 [9, 5, 3, 8, 8, 3, 5] = 961 / 550
theorem diananda_fail_7_4 : Ssum 4 [9, 5, 3, 8, 8, 3, 5] < 7 / 4
theorem diananda_fail_11_3 : Ssum 3 [11, 3, 5, 10, 1, 8, 8, 1, 10, 5, 3] < 11 / 3
theorem diananda_fail_9_5 : Ssum 5 [8, 6, 4, 5, 8, 8, 5, 4, 6] < 9 / 5
theorem diananda_fail_10_6 : Ssum 6 [5, 4, 2, 2, 4, 5, 4, 2, 2, 4] < 10 / 6
theorem diananda_fail_10_7 : Ssum 7 [8, 5, 4, 8, 7, 4, 7, 8, 4, 5] < 10 / 7
theorem diananda_fail_11_8 : Ssum 8 [8, 6, 4, 7, 8, 5, 5, 8, 7, 4, 6] < 11 / 8
theorem diananda_fail_13_9 : Ssum 9 [7, 6, 5, 6, 7, 7, 5, 5, 7, 7, 6, 5, 6] < 13 / 9
theorem diananda_fail_13_10 : Ssum 10 [8, 5, 5, 8, 6, 4, 7, 7, 4, 6, 8, 5, 5] < 13 / 10
```

Behind Proposition 3.2 (7 declarations). `Failures.lean`

```
theorem diananda_fail_25_3 :
    Ssum 3 [9, 4, 5, 9, 3, 7, 8, 3, 8, 6, 4, 9, 5, 5, 9, 4, 6, 8, 3, 8, 7, 3, 9, 5, 4]
      < 25 / 3
theorem diananda_fail_25_4 :
    Ssum 4 [8, 6, 4, 7, 7, 4, 5, 8, 6, 4, 7, 8, 5, 5, 8, 7, 4, 6, 8, 5, 4, 7, 7, 4, 6]
      < 25 / 4
theorem diananda_fail_25_5 :
    Ssum 5 [10, 6, 2, 5, 10, 7, 2, 4, 10, 8, 3, 3, 9, 9, 3, 3, 8, 10, 4, 2, 7, 10, 5, 2, 6]
      < 25 / 5
theorem diananda_fail_14_3 : Ssum 3 [9, 4, 5, 9, 3, 7, 8, 3, 8, 7, 3, 9, 5, 4] < 14 / 3
theorem diananda_fail_17_3 :
    Ssum 3 [9, 4, 5, 9, 3, 6, 8, 3, 7, 7, 3, 8, 6, 3, 9, 5, 4] < 17 / 3
theorem diananda_fail_20_3 :
    Ssum 3 [8, 5, 5, 8, 4, 6, 8, 4, 7, 7, 4, 7, 7, 4, 8, 6, 4, 8, 5, 5] < 20 / 3
theorem diananda_fail_11_4 : Ssum 4 [8, 6, 4, 7, 8, 5, 5, 8, 7, 4, 6] < 11 / 4
```

Behind Proposition 3.3 (11 declarations). `Controls.lean`

```
theorem Ssum_equal_7_4 : Ssum 4 [1, 1, 1, 1, 1, 1, 1] = 7 / 4
theorem Ssum_equal_11_3 : Ssum 3 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] = 11 / 3
theorem Ssum_equal_13_4 : Ssum 4 [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] = 13 / 4
theorem Ssum_periodic_12_4 : Ssum 4 [5, 2, 9, 1, 5, 2, 9, 1, 5, 2, 9, 1] = 12 / 4
```

`Controls.lean`

```
theorem gap_7_4 : Ssum 4 [9, 5, 3, 8, 8, 3, 5] = 7 / 4 - 3 / 1100
theorem gap_7_4_not_bigger : ¬ (Ssum 4 [9, 5, 3, 8, 8, 3, 5] < 7 / 4 - 1 / 300)
theorem gap_11_3 :
    Ssum 3 [11, 3, 5, 10, 1, 8, 8, 1, 10, 5, 3] = 11 / 3 - 15 / 5168
theorem perturbed_7_4_fails : 7 / 4 < Ssum 4 [8, 5, 3, 8, 8, 3, 5]
theorem perturbed2_7_4_fails : 7 / 4 < Ssum 4 [9, 5, 4, 8, 8, 3, 5]
theorem no_failure_13_4 :
    13 / 4 < Ssum 4 [101, 99, 101, 99, 101, 100, 100, 100, 100, 101, 99, 101, 99]
theorem no_failure_9_4 :
    9 / 4 < Ssum 4 [101, 100, 100, 101, 100, 100, 101, 100, 100]
```

Behind Proposition 4.1 (2 declarations). `Diagonal.lean`

```
theorem Qf_sos (k : ℕ) (hk : 0 < k) (u : ZMod n → ℚ) :
    Qf k u + 1 / (4 * (k : ℚ)) * nrm2 u
      = 1 / (k : ℚ) ^ 3 * ∑ i, (bsum k u i - (k : ℚ) / 2 * u i) ^ 2
theorem Qf_ge_neg_quarter_k (k : ℕ) (hk : 0 < k) (u : ZMod n → ℚ) :
    -(1 / (4 * (k : ℚ))) * nrm2 u ≤ Qf k u
```

Behind Theorem 4.2 (1 declaration). `Diagonal.lean`

```
theorem Qf_kadd1 (k : ℕ) (hk : 0 < k) (u : ZMod (k + 1) → ℚ) (h : ∑ x, u x = 0) :
    Qf k u = ((k : ℚ) + 1) / (k : ℚ) ^ 3 * nrm2 u
```

Behind Theorem 4.3 (1 declaration). `Diagonal.lean`

```
theorem Qf_kadd2 (k : ℕ) (hk : 0 < k) (u : ZMod (k + 2) → ℚ) (h : ∑ x, u x = 0) :
    Qf k u = ((k : ℚ) + 2) / (2 * (k : ℚ) ^ 3) * ∑ i, (u i + u (i - 1)) ^ 2
```

Behind Theorem 4.4 (2 declarations). `Diagonal.lean`

```
theorem Qf_2kadd1 (k : ℕ) (hk : 0 < k) (u : ZMod (2 * k + 1) → ℚ) (h : ∑ x, u x = 0) :
    Qf k u = 1 / (2 * (k : ℚ) ^ 2) * nrm2 u + 1 / (k : ℚ) ^ 3 * ∑ i, (bsum k u i) ^ 2
theorem Qf_2kadd1_lower (k : ℕ) (hk : 0 < k) (u : ZMod (2 * k + 1) → ℚ) (h : ∑ x, u x = 0) :
    1 / (2 * (k : ℚ) ^ 2) * nrm2 u ≤ Qf k u
```

Behind Theorem 4.5 (1 declaration). `Diagonal.lean`

```
theorem Qf_2kadd2 (k : ℕ) (hk : 0 < k) (u : ZMod (2 * k + 2) → ℚ) (h : ∑ x, u x = 0) :
    Qf k u = 1 / (4 * (k : ℚ) ^ 2)
        * ∑ i, (u i + u (i + ((k + 1 : ℕ) : ZMod (2 * k + 2)))) ^ 2
      + 1 / (k : ℚ) ^ 3 * ∑ i, (bsum k u i) ^ 2
```

Behind Theorem 4.6 (1 declaration). `Diagonal.lean`

```
theorem Qf_kadd3 (k : ℕ) (hk : 0 < k) (u : ZMod (k + 3) → ℚ) (h : ∑ x, u x = 0) :
    (k : ℚ) ^ 3 * Qf k u
      = ((k : ℚ) + 3) * nrm2 u + ((k : ℚ) + 4) * (∑ i, u i * u (i - 1))
        + ((k : ℚ) + 2) * ∑ i, u i * u (i - 2)
```

Behind Proposition 4.8 (2 declarations). `Controls.lean`

```
theorem Qf_const {n : ℕ} [NeZero n] (k : ℕ) (hk : 0 < k) :
    Qf k (fun _ : ZMod n => (1 : ℚ)) = 0
theorem Qf_kadd1_needs_zero_sum (k : ℕ) (hk : 0 < k) :
    Qf k (fun _ : ZMod (k + 1) => (1 : ℚ))
      ≠ ((k : ℚ) + 1) / (k : ℚ) ^ 3 * nrm2 (fun _ : ZMod (k + 1) => (1 : ℚ))
```

Behind Theorem 5.1 (3 declarations). `Local.lean`

```
theorem sum_u_14_11 : (∑ x, u_14_11 x) = 0
theorem Qf_14_11_neg : Qf 11 u_14_11 = -(10 / 1331)
theorem kadd3_at_11 :
    (11 : ℚ) ^ 3 * Qf 11 u_14_11
      = 14 * nrm2 u_14_11 + 15 * (∑ i, u_14_11 i * u_14_11 (i - 1))
        + 13 * ∑ i, u_14_11 i * u_14_11 (i - 2)
```

Behind Proposition 5.2 (6 declarations). `Local.lean`

```
theorem sum_u_11_3 : (∑ x, u_11_3 x) = 0
theorem Qf_11_3_neg : Qf 3 u_11_3 = -(1 / 3)
theorem sum_u_7_4 : (∑ x, u_7_4 x) = 0
theorem Qf_7_4_neg : Qf 4 u_7_4 = -(3 / 32)
theorem sum_u_9_5 : (∑ x, u_9_5 x) = 0
theorem Qf_9_5_neg : Qf 5 u_9_5 = -(3 / 25)
```

Behind Theorem 6.1 (3 declarations). `Erratum.lean`

```
theorem gamma3_ge (c : ℝ) (h1 : -1 ≤ c) (h2 : c ≤ 1) : -(49 / 10000 : ℝ) ≤ Gamma3 c
theorem gamma3_never_neg5046 (c : ℝ) (h1 : -1 ≤ c) (h2 : c ≤ 1) :
    -(5046 / 1000000 : ℝ) < Gamma3 c
theorem gamma3_lt_neg_one_250 : Gamma3 (-(3 / 5)) < -(1 / 250 : ℝ)
```

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