SUM-INTERSECTING FAMILIES OF INTEGERS: EXACT VALUES THROUGH $n = 15$, AND THE DISTINCT-SUM AND k-SUM VARIANTS

KOREA SUPERINTELLIGENCE LABS

Abstract. A family $\mathcal{F}$ of subsets of $[n] = \{1, \dots, n\}$ is *sum-intersecting* if for all $A, B \in \mathcal{F}$ (the case $A = B$ included) the intersection $A \cap B$ contains elements x, y, z, not necessarily distinct, with $x + y = z$; write $f(n)$ for the largest size of such a family. Berger and Mani conjectured $f(n) = 2^{n-2}$ and proved $f(n) \leq 0.32 \cdot 2^n$. The partition method of Chung, Graham, Frankl and Shearer settles the conjecture only as far as the Schur number $S(3) = 13$.

We determine $f(n)$ for every $n \leq 15$; in particular $f(14) = 4096$ and $f(15) = 8192$, the first two values beyond the Schur wall. The proof runs the Delsarte linear programme of the source paper at a single ground set and exhibits an exact rational optimal certificate there; we prove that the programme's optimum $c_0^*(n)$ equals 3 at $n = 14$ and at $n = 15$, that c_0^* is non-increasing in n, and that the whole conjecture reduces to the statement $c_0^*(n) = 3$ for all n. We also show $f(n) \leq \frac{5}{16} \cdot 2^n$ for $9 \leq n \leq 160$, improving the published constant on that range, and we close two routes to $n = 16$: a natural half-splitting lift of certificates is impossible at every even step, and the cyclically consecutive partition bound of Faudree, Schelp and Sós collapses to a three-part partition and therefore cannot pass $n = 13$.

For the distinct-sum variant, where x, y, z must be pairwise different, we determine the maximum $g(n) = 2^{n-3}$ for every $n \leq 12$ and prove $g(n) \leq 2^{n-2}$ for every $n \leq 23$. For the k-sum variant we show that any family beating $\frac{1}{4}2^n$ consists entirely of sets of at least three elements, that the extremal cores $\{d, kd\}$ come in $\lfloor n/k \rfloor$ equally good versions, and we compute the maximum for $k = 3$, $n \leq 12$ and $k = 4$, $n \leq 8$. All statements are machine-checked.

1. Introduction

1.1. **The problem.** Throughout, $[n] = \{1, 2, \dots, n\}$ and $\mathcal{P}([n])$ is its power set. Say that a finite set $s \subseteq \mathbb{Z}_{>0}$ *carries a sum* if there are $x, y, z \in s$ with $x + y = z$. Two conventions are fixed once and are easy to get wrong: x, y, z *need not be distinct*, so that $\{1, 2\}$ already carries the sum $1 + 1 = 2$; and, in the definition below, $A = B$ *is allowed*, so that every member of a sum-intersecting family carries a sum by itself.

Definition 1.1 (Berger–Mani, Definition 1.1)**.** A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is *sum-intersecting* if $A \cap B$ carries a sum for all $A, B \in \mathcal{F}$. We write

$$f(n) \;=\; \max\big\{\, |\mathcal{F}| \;:\; \mathcal{F} \subseteq \mathcal{P}([n]) \text{ sum-intersecting} \,\big\}.$$

The family of all subsets of $[n]$ containing $\{1, 2\}$ is sum-intersecting and has 2^{n-2} members, so $f(n) \geq 2^{n-2}$ for $n \geq 2$. Berger and Mani [2] conjectured that this is optimal.

Conjecture 1.2 (Berger–Mani, Conjecture 1.2)**.** *$f(n) = 2^{n-2} = \frac{1}{4} \cdot 2^n$ for every $n \geq 2$.*

Their Theorem 1.3 proves $f(n) \leq \frac{8}{25} \cdot 2^n = 0.32 \cdot 2^n$ by a Delsarte-style linear programming bound, and they remark that their methods "give a recipe for the reduction of Conjecture 1.2 to a finite (although not clearly tractable) computation". The paper records no exact value of $f(n)$ for any n.

Intersection conditions on families of *sets of integers* go back at least to Graham, Simonovits and Sós, who studied families whose pairwise intersections are intervals or arithmetic progressions [8, 9]; the condition considered here is of a different type, since it constrains the additive structure of the intersection rather than its shape, and the resulting bounds are exponential rather than polynomial in n.

Two variants are asked about in the same paper's final section. In the *distinct-sum* problem the witnessing x, y, z must be pairwise different; write $g(n)$ for the corresponding maximum. In the *k-sum* problem a set carries a k-sum if it contains $x_1, \ldots, x_k, y$, not necessarily distinct, with $x_1 + \cdots + x_k = y$; write $h_k(n)$ for the maximum size of a family all of whose pairwise intersections carry a k-sum, so that $h_2 = f$.

Question 1.3 (Berger–Mani, Question 3.1)**.** *Is $g(n) = 2^{n-3} = \frac{1}{8} \cdot 2^n$, or can $\frac{1}{8} \cdot 2^n$ be beaten?*

Question 1.4 (Berger–Mani, Question 3.2)**.** *Is the family of subsets containing $\{1, 3\}$ a maximum 3-sum-intersecting family? More generally, for $k \geq 2$, is there a k-sum-intersecting family of size $> \frac{1}{4} \cdot 2^n$?*

1.2. **The barrier, and what this paper does about it.** A partition of $[n]$ into three sum-free parts yields $f(n) \leq 2^{n-2}$; this is one substitution into a 1986 theorem of Chung, Graham, Frankl and Shearer [3], and we recall it in Section 3. Such a partition exists exactly when $n \leq S(3) = 13$, $S(3)$ being the Schur number, and $n = 14$ is the first ground set where the method is unavailable. Every partition-type argument known to us dies at the same place; Section 5 makes that statement precise for the strongest published competitor.

Our main results pass the barrier by replacing the partition with a *fractional* object. Let $\mathcal{D}(n) = \{T \subseteq [n] : [n] \setminus T \text{ is sum-free}\}$ and, for $w : \mathcal{P}([n]) \to \mathbb{Q}$ supported in $\mathcal{D}(n)$, write $\widehat{w}(\lambda) = \sum_T (-1)^{|\lambda \cap T|} w_T$ and $\operatorname{mass}(w) = \sum_T w_T$. Call $c \in \mathbb{Q}$ *achievable at n* if some such w has $\widehat{w} \geq -1$ pointwise and mass c, and let $c_0^*(n)$ be the largest achievable value.

Theorem A. $c_0^*(14) = c_0^*(15) = 3$, and consequently

$$f(n) = 2^{n-2} \qquad \text{for every } 2 \leq n \leq 15.$$

In particular $f(14) = 4096$ and $f(15) = 8192$.

The two exact values are the first past the Schur wall. They are proved by exhibiting an explicit rational certificate at one ground set, verifying it, and invoking a duality lemma (Theorem 4.2) that converts any achievable mass c into the bound $|\mathcal{F}|\,(1 + c) \leq 2^n$. The achievable sets are nested downwards (Proposition 4.3), so a single certificate at $n = 15$ covers the whole range $2 \leq n \leq 15$. The ceiling $c_0^*(n) \leq 3$ holds at every n with no computation, so Conjecture 1.2 is *equivalent* to $c_0^*(n) = 3$ for all n up to the direction proved in Corollary 4.8.

Away from the exact range, the partition method still gives the best available constant, at a parameter the source paper did not use.

Theorem B. $f(n) \leq 10 \cdot 2^{n-5} = \frac{5}{16} \cdot 2^n$ for every $9 \leq n \leq 160$.

Since $\frac{5}{16} = 0.3125 < 0.32 = \frac{8}{25}$, this improves the published constant by about 2.3% on the stated range; it is one substitution into [3] at five parts, using $g(5, 2) = 10$ and an explicit sum-free five-partition of $[1, 160]$. It says nothing about $n \geq 161$, and it does not claim that $\frac{5}{16}$ is the truth: the conjectured value $2^{n-2} = 8 \cdot 2^{n-5}$ sits strictly below it.

For the variants we obtain the following.

Theorem C. $g(n) = 2^{n-3}$ for every $3 \leq n \leq 12$, and $g(n) \leq 2^{n-2}$ for every $3 \leq n \leq 23$.

Theorem D. Let $k \geq 2$. Every member of a k-sum-intersecting family of more than 2^{n-2} subsets of $[n]$ has at least three elements; equivalently, a k-sum-intersecting family containing a two-element set is exactly the family of all supersets of that set, and has 2^{n-2} members. Moreover $h_3(n) = 2^{n-2}$ for $3 \leq n \leq 12$ and $h_4(n) = 2^{n-2}$ for $4 \leq n \leq 8$, the extremal families being the supersets of $\{d, kd\}$, of which there are $\lfloor n/k \rfloor$ pairwise distinct ones.

Theorem D is a structural reduction of Question 1.4 rather than an answer to it: it locates where a counterexample could live, and rules out the two smallest cardinalities.

1.3. **Two closed routes.** Negative results are recorded here with the same care as positive ones, because both of the natural next steps beyond $n = 15$ turn out to be dead.

First, a *half-split lift* carries a certificate at $[n]$ to one at $[n+1]$ by splitting each weight evenly between T and $T \cup \{n+1\}$; the characters containing $n+1$ cancel, so the only obstruction is a support condition (Proposition 4.11). It nevertheless fails at every even step: when $n+1 = 2m$ every liftable column must contain m, the character at $\{m\}$ then reads $-\mathrm{mass}(w)$, and feasibility caps the mass at 1 instead of the required 3 (Proposition 4.12). In particular the step $15 \to 16$ is closed.

Second, Faudree, Schelp and Sós [4] proved a partition bound $|\mathcal{A}| \le 2^{n-k}$ whose conclusion does not depend on the number of parts, only on the number k of *cyclically consecutive* parts that every intersection must meet. At five parts and $k = 2$ this would give $\frac{1}{4} \cdot 2^n$ on the whole range of Theorem B, i.e. Conjecture 1.2 out to $n = 160$. It cannot: at $k = 2$ a cyclic certificate on $\ell \ge 3$ parts is *exactly* a three-part partition into solution-free sets, in both directions (Theorem 5.2), because a cycle is 3-colourable. The reach is therefore $S(3) = 13$ for sums and $WS(3) = 23$ for distinct sums, exactly where we already are. The branch $k \ge 3$ is closed separately (Proposition 5.3).

1.4. **Method.** The exact values rest on a finite computation, and we state throughout precisely which computation was performed and what was verified. All statements in this paper have been formalised and machine-checked; see Section 9.

2. Preliminaries

2.1. **Sum-free sets and Schur numbers.** A set $P \subseteq \mathbb{Z}_{>0}$ is *sum-free* if it carries no sum, i.e. there are no $x, y, z \in P$ (not necessarily distinct) with $x + y = z$; note that $\{1, 2\}$ is not sum-free. It is *weak-sum-free* if it carries no sum with x, y, z pairwise distinct. The Schur number $S(k)$ is the largest n such that $[n]$ partitions into k sum-free parts, and $WS(k)$ is its weak analogue. We use $S(3) = 13$, $S(4) = 44$, $S(5) = 160$, and $WS(3) = 23$; only the *existence of one explicit partition* is ever used, and each partition we need is exhibited.

2.2. **The three problems.** For a monotone predicate Sol on finite sets — monotone meaning that a superset of a set satisfying it satisfies it — call $\mathcal{F} \subseteq \mathcal{P}([n])$ Sol-*intersecting* if $\mathrm{Sol}(A \cap B)$ holds for all $A, B \in \mathcal{F}$. Taking Sol to be "carries a sum", "carries a sum with x, y, z pairwise distinct" and "carries a k-sum" gives the three maxima $f(n)$, $g(n)$ and $h_k(n)$. All three predicates are monotone, being of the form "some triple (resp. $(k+1)$-tuple) of members is related in a fixed way".

Definition 2.1. For $k \ge 2$, a finite set s *carries a k-sum* if there are $x_1, \ldots, x_k \in s$, not necessarily distinct, with $x_1 + \cdots + x_k \in s$. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ is *k-sum-intersecting* if $A \cap B$ carries a k-sum for all $A, B \in \mathcal{F}$ (again $A = B$ allowed), and $h_k(n)$ is the maximum size of such a family. For $k = 2$ this is Definition 1.1, so $h_2 = f$.

2.3. **The constructions.** The lower bounds are all of the same shape and need no computation.

Proposition 2.2. *For $n \ge 2$ the family of all $A \subseteq [n]$ with $\{1, 2\} \subseteq A$ is sum-intersecting and has exactly 2^{n-2} members. Hence $f(n) \ge 2^{n-2}$.*

Proof. Any two members contain 1 and 2 in their intersection, and $1 + 1 = 2$. Deleting $\{1, 2\}$ is a bijection onto $\mathcal{P}([n] \setminus \{1, 2\})$. ☐

Proposition 2.3. *Let $1 \le d$ and $2d \le n$, and let $\mathcal{F}_d$ be the family of all $A \subseteq [n]$ with $\{d, 2d\} \subseteq A$. Then $\mathcal{F}_d$ is sum-intersecting and $|\mathcal{F}_d| = 2^{n-2}$, and $\mathcal{F}_d = \mathcal{F}_e$ forces $d = e$. Hence there are $\lfloor n/2 \rfloor$ pairwise distinct extremal families of this shape; already at $n = 4$ there are two, so the optimum is not unique.*

Proof. $d + d = 2d$ replaces $1 + 1 = 2$. The core $\{d, 2d\}$ is a member of $\mathcal{F}_d$ and is the unique member of size 2, so it is recovered from the family. ☐

Proposition 2.4. *Let $k \ge 2$, $d \ge 1$ and $kd \le n$. The family of all $A \subseteq [n]$ with $\{d, kd\} \subseteq A$ is k-sum-intersecting of size 2^{n-2}. Hence $h_k(n) \ge 2^{n-2}$ whenever $k \le n$.*

Proof. Take all k summands equal to d: $d + \cdots + d = kd$. Since $k \ge 2$ and $d \ge 1$ the two entries of the core are distinct, so the count is 2^{n-2}. ☐

The paper of Berger and Mani names $\{1, 3\}$ for $k = 3$; the d-parameterised version is one substitution, and it is what makes the classification in Section 8 have $\lfloor n/k \rfloor$ answers rather than one.

2.4. **The colouring bound.** Sum-intersecting families are exactly the cliques of the graph on the sets carrying a sum in which $A \sim B$ iff $A \cap B$ carries a sum. We use the elementary clique-versus- colouring bound throughout.

Lemma 2.5. *Suppose the subsets of $[n]$ that carry a sum are covered by m classes, each of which contains no two distinct sets $A \ne B$ with $A \cap B$ carrying a sum. Then $f(n) \le m$. The same statement holds verbatim for the distinct-sum and k-sum predicates.*

Proof. Every member of a Sol-intersecting family carries a solution by itself, so it lies in some class; and two distinct members of one class would violate the class condition. So the family injects into the set of classes. ☐

Certificates of this kind, one per n, give the first values.

Proposition 2.6. *$f(n) = 2^{n-2}$ for every $2 \le n \le 11$.*

Proof. The lower bound is Proposition 2.2. For the upper bound, $n \le 4$ is a direct exhaustive enumeration of all 2^{2^n} families, and $5 \le n \le 11$ is Lemma 2.5 applied to an explicit partition of the sum-carrying subsets of $[n]$ into 2^{n-2} classes; the partitions were found by a greedy search and are checked, class by class and pair by pair, by the verification described in Section 9. Sections 3 and 4 reprove this range by argument and extend it. ☐

3. The partition method

3.1. **The coset argument.** The following is the specialisation to our situation of Theorem 2 of [3]: a partition of $[n]$ into k parts such that every forbidden configuration meets at least j of them bounds the family by $g(k, j) \cdot 2^{n-k}$, where $g(k, j)$ is the maximum size of a j-intersecting family in $\mathcal{P}([k])$ (Kleitman [7], see also Ahlswede–Katona [1]). We give the argument in the form we use, because the case we need is short and because we shall want to vary the predicate.

Fix a partition $[n] = P \sqcup Q \sqcup R$ into three parts, none of which carries a solution, and mark elements $p \in P$, $q \in Q$, $r \in R$. For $A \subseteq [n]$ set

$$\operatorname{fold}(P, p, A) = \begin{cases} P \setminus A, & p \in A, \\ A \cap P, & p \notin A, \end{cases}$$

and likewise for Q, R; each fold is a subset of its part omitting the marker. The *key* of A is the union of its three folds, a subset of $[n] \setminus \{p, q, r\}$, and the *signature* of A is the bit triple $(\,[p \in A], [q \in A], [r \in A]\,)$.

Lemma 3.1. *Let $\mathcal{F}$ be* Sol*-intersecting for a monotone* Sol *and let P, Q, R, p, q, r be as above with* Sol *failing on each part. Then every fibre of the key map contains at most two members of $\mathcal{F}$.*

Proof. Two sets with the same key either agree on a part or are complementary in it, and which of the two occurs is read off the corresponding signature bit. If A, B are complementary in P then $A \cap B \cap P = \emptyset$. If this happened in two of the three parts, then $A \cap B$ would sit inside the remaining part; since $A \cap B$ carries a solution and Sol is monotone, that part would carry one too, contrary to hypothesis. Hence the signatures of any two members of one fibre agree in at least two of their three coordinates. Three distinct bit triples cannot be pairwise at Hamming distance at most one — this is $g(3, 2) = 2$, and it is checked over all 512 triples of triples — so a fibre carries at most two members. □

Theorem 3.2. *If $[n]$ admits a partition into three sum-free parts, then $f(n) \leq 2^{n-2}$. Since the partition $\{1, 4, 10, 13\} \mid \{2, 3, 11, 12\} \mid \{5, 6, 7, 8, 9\}$ of $[13]$ has all parts sum-free, and restricts to $[n]$ for every $5 \leq n \leq 13$,*

$$f(n) = 2^{n-2} \qquad \textit{for every } 2 \leq n \leq 13;$$

in particular $f(12) = 1024$ and $f(13) = 2048$.

Proof. There are 2^{n-3} possible keys and each fibre carries at most two members (Lemma 3.1), so $|\mathcal{F}| \leq 2 \cdot 2^{n-3} = 2^{n-2}$. Combine with Proposition 2.2. For $n \in \{3, 4\}$ the displayed partition leaves a part empty once $n \leq 4$ — every sum-free three-partition of $[13]$ has part minima $1, 2, 5$ — so $\{1, 4\} \mid \{2\} \mid \{3\}$ is used instead; $n = 2$ admits no three-partition into nonempty parts and is checked directly. □

The construction stops exactly at $S(3) = 13$: $[14]$ has no partition into three sum-free parts, the hypothesis is unavailable, and Theorem 3.2 says nothing at $n = 14$.

Remark 3.3. The same range was reached independently from the linear programming side. Unwinding the certificate of [2] one finds that its programme has an integral extreme point as soon as three sets T_1, T_2, T_3 with sum-free complements satisfy $T_1 \triangle T_2 \triangle T_3 = \emptyset$ — which is the same thing as a partition of $[n]$ into three sum-free sets — and then $\{\emptyset, T_1, T_2, T_3\}$ is a subgroup of order four of $(\mathcal{P}([n]), \triangle)$ each of whose cosets meets the family at most once. The two derivations were kept separately rather than merged.

Proposition 3.4. *If $[n]$ has a partition into three sum-free colour classes, and x, y lie in two different classes, then no two members of a sum-intersecting family differ by a union of colour classes; consequently $f(n) \leq 2^{n-2}$, and $f(n) = 2^{n-2}$ for every $2 \leq n \leq 13$.*

Proof. If $A \triangle B$ is the union of the classes indexed by a nonempty proper subset of $\{0, 1, 2\}$, then $A \cap B$ is contained in the union of the remaining classes; taking the flip that leaves exactly one class unflipped puts $A \cap B$ inside a single sum-free class, a contradiction. Flipping whole classes so that both x and y leave the set is a canonical representative for the coset of the order-four subgroup generated by two of the classes, and each coset holds at most one member. There are 2^{n-2} cosets. □

3.2. **Five parts and the constant** 5/16. The same skeleton runs with k parts. With k parts, a pair of members whose signatures disagree in $k - 1$ coordinates traps $A \cap B$ in a single part, so the conclusion becomes "the k signature bits of any two members agree in at least two places", and the fibre bound is $g(k, 2)$, the maximum number of points of $\{0, 1\}^k$ that pairwise agree in at least two coordinates. For $k = 5$ this is 10.

Theorem 3.5. *$f(n) \le 10 \cdot 2^{n-5} = \frac{5}{16} \cdot 2^n$ for every $9 \le n \le 160$. This is strictly below the published bound: $25 \cdot (10 \cdot 2^{n-5}) < 8 \cdot 2^n$ for all $n \ge 5$, i.e. $\frac{5}{16} < \frac{8}{25}$, with about 2.3% to spare. Moreover $2^{n-2} \le f(n) \le 10 \cdot 2^{n-5}$ on that range, i.e. $8 \cdot 2^{n-5} \le f(n) \le 10 \cdot 2^{n-5}$.*

Proof. The fibre bound $g(5, 2) = 10$ is proved from scratch rather than quoted: the 32 signatures form a graph in which $u \sim v$ iff they agree in at least two of five coordinates, and an ordered clique search — visiting strictly increasing chains, so one node per clique rather than one per ordered tuple — traverses the whole search tree of 84297 nodes and finds no clique of size 11, while finding one of size 10, which is the matching lower control. The adjacency table used by the search is itself checked against the definition over all 1024 pairs. The rest of the argument is Lemma 3.1 with five parts: there are 2^{n-5} keys and each fibre carries at most 10 members.

For the hypothesis, two explicit sum-free five-partitions are used. For $9 \le n \le 44$ we take

$$\{1\} \mid \{3, 5, 15, 17, 19, 26, 28, 40, 42, 44\} \mid \{2, 7, 8, 18, 21, 24, 27, 33, 37, 38, 43\}$$
$$\mid \{4, 6, 13, 20, 22, 23, 25, 30, 32, 39, 41\} \mid \{9, 10, 11, 12, 14, 16, 29, 31, 34, 35, 36\},$$

a sum-free four-partition of $[1, 44]$ with its first part cut in two, which keeps every part sum-free and puts every part minimum at 9 or below so that the partition restricts to $[n]$ for every $n \ge 9$. For $45 \le n \le 160$ we take Heule's explicit five-colouring of $[1, 160]$ [6]. Only the existence of these two partitions is used; the values $S(4) = 44$ and $S(5) = 160$ are not. □

Remark 3.6. Five parts is the last useful parameter for this method. A solution $x+y = z$ can sit inside two parts (x, y in one, z in the other), so sum-free parts can never force a solution to meet three of them, whatever k is; the relevant constant is therefore always $g(k, 2)$. A computation outside the formal development, cross-checked three ways, gives $g(k, 2)/2^k = \frac{1}{2} - \binom{k}{k/2}/2^{k+1}$ for even k, increasing monotonically to $\frac{1}{2}$, with values $0.25, 0.25, 0.3125, 0.3125, 0.3438, \dots$ for $k = 2, 3, 4, 5, 6$. So $g(k, 2)/2^k$ crosses 0.32 between $k = 5$ and $k = 6$: at $k \ge 6$ the method is worse than the published constant before the Schur cap even matters, and it is exact only at $k \le 3$.

3.3. **Only monotonicity is used.** Lemma 3.1 never inspects a solution. It needs $A \cap B$ to carry one, to sit inside a part, and that part to carry none.

Proposition 3.7. *Let* Sol *be any monotone predicate on finite sets and suppose $[n]$ is partitioned into three parts on none of which* Sol *holds. Then every* Sol*-intersecting family of subsets of $[n]$ has at most 2^{n-2} members. Consequently, if* Sol′ *implies* Sol *and is monotone, every such partition for* Sol *is one for* Sol′*; in particular the distinct-sum problem inherits every sum-free partition for free.*

Proof. This is Lemma 3.1 together with the count in Theorem 3.2; neither uses anything about Sol beyond monotonicity. For the second statement, a part carrying no Sol-solution carries no Sol′-solution. □

The gain is not in the proof but in the hypothesis: for the distinct-sum problem the parts need only be *weak*-sum-free, and three weak-sum-free parts cover $[n]$ as far as $WS(3) = 23$.

Theorem 3.8. *$g(n) \le 2^{n-2}$ for every $3 \le n \le 23$.*

Proof. Apply Proposition 3.7 to the partition

$$\{1, 2, 4, 8, 11, 16, 22\} \mid \{3, 5, 6, 7, 19, 21, 23\} \mid \{9, 10, 12, 13, 14, 15, 17, 18, 20\}$$

of [23], each part of which is weak-sum-free (none of them is sum-free, which is exactly why the partition reaches past $S(3) = 13$). The part minima are $1, 3, 9$, so the restriction to $[n]$

has three nonempty parts for every $9 \le n \le 23$; for $3 \le n \le 8$ a smaller explicit partition is used. $\square$

Theorem 3.8 is superseded by Theorem 7.4 for $n \le 12$, but for $13 \le n \le 23$ it is the best bound we know.

4. The Delsarte programme and the values $f(14)$, $f(15)$

4.1. **Certificates.** Identify $\mathcal{P}([n])$ with $\mathbb{F}_2^n$, so that symmetric difference is addition, and write $\chi_\lambda(A) = (-1)^{|\lambda\cap A|}$ for the Walsh character indexed by $\lambda \subseteq [n]$. Recall

$$\mathcal{D}(n) = \{\, T \subseteq [n] \ : \ [n] \setminus T \text{ is sum-free} \,\}.$$

Definition 4.1. A *certificate at* $[n]$ is a function $w : \mathcal{P}([n]) \to \mathbb{Q}$ with $\operatorname{supp} w \subseteq \mathcal{D}(n)$ and $\widehat{w}(\lambda) = \sum_{T\subseteq[n]} \chi_\lambda(T)\, w_T \ge -1$ for every $\lambda \subseteq [n]$. Its *mass* is $\sum_T w_T$. Write $\operatorname{ach}(n) \subseteq \mathbb{Q}$ for the set of masses of certificates at $[n]$, and $c_0^*(n) = \max \operatorname{ach}(n)$ when the maximum exists. Weights are allowed to be negative.

Theorem 4.2 (the duality step)**.** *If $c \in \operatorname{ach}(n)$ and $\mathcal{F} \subseteq \mathcal{P}([n])$ is sum-intersecting, then*

$$|\mathcal{F}| \cdot (1 + c) \ \le \ 2^n.$$

In particular $c = 3$ gives $|\mathcal{F}| \le 2^{n-2}$, which is Conjecture 1.2 at that n.

The step is not ours. It is Proposition 2.3 of [2] together with their Lemma 2.4 — which is where the support condition $\operatorname{supp} w \subseteq \mathcal{D}(n)$ enters — transported to the normalisation used here, in which $\widehat{w}$ is an unnormalised sum rather than an average, so that their $\widehat{\nu}(\emptyset)$ is our mass. Those authors credit the machinery in turn to Ellis–Filmus–Friedgut and to Berger–Zhao. We give the proof in full because everything that follows turns on its exact hypotheses.

Proof. Put $g = \delta_\emptyset + w$, so that $\sum_T \chi_\lambda(T) g_T = 1 + \widehat{w}(\lambda) \ge 0$ for every λ. For *any* g one has the identity

$$\sum_{\lambda\subseteq[n]} \Big(\sum_T \chi_\lambda(T) g_T\Big)\Big(\sum_{A\in\mathcal{F}} \chi_\lambda(A)\Big)^2 \ = \ 2^n \sum_{A,B\in\mathcal{F}} g(A \triangle B) :$$

expand the square, use $\chi_\lambda(X)\chi_\lambda(Y) = \chi_\lambda(X \triangle Y)$ twice to collapse three characters into $\chi_\lambda(T \triangle A \triangle B)$, and sum over λ, which contributes 2^n exactly when $T = A \triangle B$ and 0 otherwise.

The combinatorics is one line. For $A, B \in \mathcal{F}$ the set $A\cap B$ carries a sum and is disjoint from $A \triangle B$, so $[n] \setminus (A \triangle B) \supseteq A \cap B$ carries a sum as well; hence $A \triangle B \notin \mathcal{D}(n)$ and $w(A \triangle B) = 0$. Only $\delta_\emptyset$ survives, and only on the diagonal, so the right-hand side is $2^n|\mathcal{F}|$.

Every summand on the left is a nonnegative number times a square, so the sum is at least its $\lambda = \emptyset$ term, which is $(1 + c)|\mathcal{F}|^2$. Therefore $(1 + c)|\mathcal{F}|^2 \le 2^n|\mathcal{F}|$. $\square$

Two features of the proof are worth isolating, because they are what make the certificates usable. Non-negativity of w is never used — only $\operatorname{supp} w \subseteq \mathcal{D}(n)$ and $\widehat{w} \ge -1$; and no hypothesis $n \ge 2$ is needed, the step $w(\emptyset) = 0$ coming free from the diagonal argument.

Proposition 4.3. $\operatorname{ach}(n) \subseteq \operatorname{ach}(m)$ *whenever $m \le n$; and $c \le 3$ for every $c \in \operatorname{ach}(n)$ and every $n \ge 2$. Consequently $c_0^*(14) = 3$: the upper bound is the ceiling, and an explicit rational certificate of mass exactly* 3 *attains it.*

Proof. For the ceiling, pick d with $2d \le n$. No T in the support can miss both d and $2d$, since $\{d, 2d\}$ carries the sum $d + d = 2d$ and so cannot sit inside the sum-free set $[n] \setminus T$. In each of the three surviving cases, $\chi_{\{d\}}(T) + \chi_{\{2d\}}(T) + \chi_{\{d,2d\}}(T) = -1$; adding the three feasibility constraints at $\lambda = \{d\}, \{2d\}, \{d, 2d\}$ gives $-\sum_T w_T \geq -3$. (In linear programming language, these three sets carrying weight 1 each form a dual feasible point of objective 3, and the statement is weak duality against it.) Restriction along $m \leq n$ is proved one step at a time and is a direct computation on the definition.

For attainment at $n = 14$, an explicit w is exhibited. It is supported on 108 of the $|\mathcal{D}(14)| = 847$ columns, all weights strictly positive, with common denominator

$$Q \;=\; 25\,736\,797\,499\,620\,600\,318\,289\,382,$$

mass exactly 3, and $\widehat{w}(\lambda) \geq -1$ at every one of the 2^{14} values of λ, with equality at 114 of them. The certificate was found by floating point linear programming over all 847 columns and then solved exactly over $\mathbb{Q}$ on the tight rows together with the mass equation; none of that search is trusted. What is checked is the resulting integer data: a Walsh–Hadamard recursion replaces the $2^{14} \times 2^{14}$ verification by a $14 \cdot 2^{14}$ one — peel one element off the ground set and recombine the two halves of the coefficient list as a sum for λ missing it and as a difference for λ containing it — and a separate walk over the same list confirms that every nonzero entry sits at a T with $[n] \setminus T$ sum-free. The soundness of both walks is proved, not assumed. ☐

Remark 4.4. The ceiling is why the question at $n = 14$ was delicate rather than routine. A floating-point solver reporting 3.0000000000 is reporting the value of a proved ceiling; it would print the same thing if the true optimum were $3 - 10^{-9}$. Only an exact witness decides it.

4.2. **The exact values.**

Theorem 4.5. *$f(14) = 4096$.*

Proof. Proposition 2.2 gives $f(14) \geq 2^{12} = 4096$. Proposition 4.3 gives $3 \in \mathrm{ach}(14)$, and Theorem 4.2 converts it into $|\mathcal{F}| \leq 2^{14}/4 = 4096$. ☐

Theorem 4.6. *$c_0^*(15) = 3$ and $f(15) = 8192$.*

Proof. The ceiling is Proposition 4.3, which is general in n. For attainment, an explicit rational certificate at $n = 15$ is exhibited. In contrast with the $n = 14$ certificate it is supported on *all* $|\mathcal{D}(15)| = 1400$ columns with strictly positive weight, has the small common denominator $Q = 10^8$, numerators summing to $3 \cdot 10^8$, and minimum Walsh value exactly -10^8, attained on 18 rows — precisely the $\lambda \in \{\{d\}, \{2d\}, \{d, 2d\}\}$ for $1 \leq d \leq 7$, so no character of degree at least three is tight. The verification scheme of Proposition 4.3 applies verbatim, replacing a $2^{15} \times 2^{15}$ check by a $15 \cdot 2^{15}$ one. The support was independently re-enumerated from the definition of "carries a sum" over all 2^{15} subsets before the certificate was checked. Now apply Theorem 4.2 and Proposition 2.2. ☐

Theorem 4.7. *$f(n) = 2^{n-2}$ for every $2 \leq n \leq 15$.*

Proof. By Proposition 4.3 the single certificate at $n = 15$ restricts to every $m \leq 15$, so $3 \in \mathrm{ach}(m)$ for all $2 \leq m \leq 15$; apply Theorem 4.2 and Proposition 2.2. ☐

The range $2 \leq n \leq 13$ of Theorem 4.7 is Theorem 3.2 again; the content of Theorem 4.7 is $n = 14$ and $n = 15$.

Corollary 4.8. *If $3 \in \mathrm{ach}(n)$ for every $n \geq 2$, then $f(n) = 2^{n-2}$ for every $n \geq 2$.*

Proof. Immediate from Theorem 4.2 and Proposition 2.2. ☐

Corollary 4.8 is what makes each new value cost a certificate and no argument: the conjecture has been reduced to a statement in which no family appears. The hypothesis is not free. The *restricted* certificate shape used in [2], whose $\widehat{\nu}(\lambda)$ depends on λ only through the law of $|S \cap \lambda|$

for a random sum-free set S, has an exact optimum which was computed here over all 2^n columns for $10 \le n \le 16$ and plateaus below 2.810; since the conjecture is equivalent to reaching 3, no certificate of that restricted shape can prove it, and the unrestricted certificates of Definition 4.1 are what must be searched for.

4.3. **The columns, and why the certificates must be fractional.**

Proposition 4.9. $|\mathcal{D}(n)| = 607,\ 847,\ 1400,\ 1954$ *for* $n = 13, 14, 15, 16$ *respectively.*

Proof. $T \mapsto [n] \setminus T$ is a bijection from $\mathcal{D}(n)$ onto the sum-free subsets of $[n]$, so these are the values of OEIS A007865 [10] (with the convention $x = y$ allowed), which is tabulated to $n = 88$; the four values are re-derived here directly from the definition by enumeration over $\mathcal{P}([n])$. □

An *integral* certificate of mass 3 is a triple $T_1, T_2, T_3 \in \mathcal{D}(n)$ with $T_1 \triangle T_2 \triangle T_3 = \emptyset$, the three point masses then giving $\widehat{w} \in \{3, -1\}$. Writing $T_3 = T_1 \triangle T_2$, the existence question is exactly whether $\mathcal{D}(n)$ is closed under symmetric differences of pairs.

Proposition 4.10. *For* $A, B \in \mathcal{D}(14)$ *one has* $A \triangle B \notin \mathcal{D}(14)$*, and likewise in* $\mathcal{D}(15)$*.*

Proof. This is $S(3) = 13$ restated in the form the programme sees: such a pair would give a partition of [14] (resp. [15]) into three sum-free sets. It is verified here exhaustively over all 847^2 and 1400^2 pairs. At $n = 13$ such a pair does exist, which is the matching positive control, so the statement is about 14 and 15 and is not an artifact of the encoding. □

So the certificates of Theorems 4.5 and 4.6 are necessarily fractional, and it is exactly this that lets them pass the Schur wall.

4.4. **The half-split lift, and why it fails.**

The cheapest conceivable route from n to $n+1$ is to split each weight in two.

Proposition 4.11. *Call* $T \subseteq [n]$ liftable *if* $[n+1] \setminus T$ *is sum-free. If* w *is supported on the liftable subsets of* $[n]$*, satisfies* $\widehat{w} \ge -1$ *on* $\mathcal{P}([n])$ *and has mass* c*, then* $c \in \mathrm{ach}(n+1)$*. In particular, such a* w *at* $[15]$ *of mass* 3 *would give* $f(16) = 16384$*.*

Proof. Split each weight evenly between T and $T \cup \{n+1\}$. The mass is unchanged; every character λ avoiding $n+1$ sees the same value as before; and every character containing $n+1$ sees exactly 0, because the two halves cancel. The only condition left is the support condition, which is the liftability hypothesis. The last sentence combines this with Theorem 4.2. □

Proposition 4.12. *Let* $n + 1 = 2m$ *with* $m \ge 1$*. Then any* w *as in Proposition 4.11 has mass at most* 1*. In particular no certificate at* $[15]$ *supported on the liftable columns has mass* 3*, so the half-split lift cannot reach* $n = 16$*.*

Proof. If T omits m then $[n+1] \setminus T$ contains both m and $n + 1 = m + m$, so T is not liftable. Hence every column carrying weight contains m, the character at $\lambda = \{m\}$ reads $\widehat{w}(\{m\}) = -\mathrm{mass}(w)$, and $\widehat{w} \ge -1$ gives $\mathrm{mass}(w) \le 1$. For $n = 15$ take $m = 8$. □

The obstruction is genuinely parity, not uselessness of the lift: at the odd step $14 \to 15$ the liftable columns have empty intersection, so no element is forced. What remains at $n = 16$ is the full programme on 1954 columns against 2^{16} constraints.

5. The cyclic partition bound collapses

Faudree, Schelp and Sós [4] proved a partition theorem whose conclusion, unlike that of [3], does not involve the number of parts. In the language of set systems (their Corollary 2): if $[n] = X_1 \sqcup \cdots \sqcup X_\ell$ with all parts nonempty and every pair of members of $\mathcal{A}$ has elements in common from k cyclically consecutive parts $X_j, X_{j+1}, \ldots, X_{j+k-1}$ (indices modulo ℓ), then $|\mathcal{A}| \le 2^{n-k}$.

At $k = 2$ this promises 2^{n-2} from *any* number of parts. Had the five-part partition of Theorem 3.5 satisfied this cyclic hypothesis, it would have given Conjecture 1.2 for all $9 \le n \le 160$. It does not, and cannot.

Definition 5.1. A *cyclic certificate* for a monotone predicate Sol at parameters (ℓ, k), $\ell \ge 3$, $2 \le k \le \ell$, is a colouring $c : [n] \to \mathbb{Z}_\ell$ with all ℓ colours used on $[n]$ such that every $s \subseteq [n]$ satisfying Sol contains elements of k cyclically consecutive colours.

Theorem 5.2. *Let $\ell \ge 3$. A cyclic certificate at $k = 2$ is exactly a partition of $[n]$ into three parts on none of which* Sol *holds: every cyclic certificate yields such a partition, and conversely every such partition is a cyclic certificate with $\ell = 3$. Consequently the conclusion obtainable from a cyclic certificate at $k = 2$ is 2^{n-2}, the same bound Theorem 3.2 already gives, and its reach is $S(3) = 13$ for the sum problem and $WS(3) = 23$ for the distinct-sum problem.*

Proof. Let $\varphi : \mathbb{Z}_\ell \to \{0, 1, 2\}$ be a proper 3-colouring of the cycle C_ℓ; one exists for every $\ell \ge 3$ since $\chi(C_\ell) \le 3$. Put $d = \varphi \circ c$. If s satisfies Sol then it has two elements x, y whose c-colours are cyclically adjacent, so $d(x) \ne d(y)$ and s is not d-monochromatic. Hence each of the three d-classes fails Sol, and d is a three-part partition of the required kind; the bound then follows from Proposition 3.7. Conversely, on a triangle every pair of distinct parts is a pair of consecutive parts, so a three-part partition is a cyclic certificate at $\ell = 3$, $k = 2$. □

Proposition 5.3. *Under a cyclic certificate with $k \ge 3$, every set satisfying* Sol *has three distinct elements. Hence: for the sum problem no cyclic certificate with $k \ge 3$ exists at any $n \ge 2$; and for the distinct-sum problem a cyclic certificate with $k \ge 3$ forces $n \le 11$.*

Proof. Three cyclically consecutive parts are pairwise different, so the three witnessing elements are distinct. For the sum problem, $\{1, 2\}$ carries the solution $1 + 1 = 2$ and has only two elements, so it cannot meet three parts — this is the same ceiling as Remark 3.6. For the distinct-sum problem, for every $x \ne y$ in $[1, \lfloor n/2 \rfloor]$ the triple $\{x, y, x+y\}$ is a solution inside $[n]$, so its three elements have three distinct consecutive colours; hence c is injective on $[1, \lfloor n/2 \rfloor]$ and all its values lie within cyclic distance 2 of $c(1)$. A cycle has only five vertices within distance 2 of a given one, so $\lfloor n/2 \rfloor \le 5$. □

The bound $n \le 11$ is not tight — an exhaustive search puts the true ceiling at $n = 5$ — but it already suffices, since $g(n)$ is determined below for every $n \le 12$. The route is therefore closed in both branches.

We record the constant that the two 1986 papers share, since it is the fibre bound of Lemma 3.1 in disguise.

Proposition 5.4. *Let $\mathcal{C}$ be a family of subsets of any set with $|A \triangle B| \le 1$ for all $A, B \in \mathcal{C}$. Then $|\mathcal{C}| \le 2$, and the bound is attained. This is $g(\ell, \ell - 1) = 2$ for every ℓ, the diameter-one case of Kleitman's theorem* [7]*, and it is exactly the constant* 2 *appearing in Lemma 3.1.*

Proof. Suppose $A, B, D \in \mathcal{C}$ are distinct. Then all three pairwise symmetric differences are singletons. Writing $A \triangle B = \{x\}$ and $A \triangle D = \{y\}$ with $x \ne y$, one gets $\{x, y\} \subseteq B \triangle D$, contradicting $|B \triangle D| \le 1$. The pair $\{\emptyset, \{0\}\}$ attains 2. □

Only this case of Kleitman's theorem is used anywhere in this paper, and it is proved above. We have not read [7] itself; its general statement is taken from [4] (their Theorem 7, attributed there to [1]), and the two values it supplies to us, $g(3,2)=2$ in Lemma 3.1 and $g(5,2)=10$ in Theorem 3.5, are recomputed from the definition rather than quoted.

6. Structure of the extremal families

Proposition 2.3 exhibits $\lfloor n/2 \rfloor$ extremal families. We prove that in the verified range there are no others.

Theorem 6.1. *For $2 \le n \le 9$, every sum-intersecting family $\mathcal{F} \subseteq \mathcal{P}([n])$ with $|\mathcal{F}| \ge 2^{n-2}$ equals the family of all supersets of $\{d, 2d\}$ for some d with $1 \le d$ and $2d \le n$. Equivalently, all members of such a family contain one common pair $\{d, 2d\}$. In particular, at $n=4$ the only families of size ≥ 4 are the supersets of $\{1,2\}$ and of $\{2,4\}$, and at $n=5$ the only families of size ≥ 8 are the supersets of $\{1,2\}$ and of $\{2,4\}$.*

Proof. Three ingredients. First, the statement is *downward closed in n*: padding every member of a family on $[m]$ by an arbitrary subset of $\{m+1,\dots,n\}$ produces a sum-intersecting family on $[n]$ which is 2^{n-m} times as large and contains the original, so a common pair upstairs restricts to one downstairs. One proof at a single large n therefore settles every smaller n.

Second, a reduction. Call a set *minimal self-summing* if it carries a sum but no proper subset of it does; these are exactly the pairs $\{d, 2d\}$ and the triples $\{a, b, a+b\}$ with $a \ne b$. The uniqueness statement at n is *equivalent* to: every sum-intersecting family containing no minimal self-summing set has fewer than 2^{n-2} members. (If a member M is minimal, every other member meets M in a set carrying a sum, hence contains M; a pair core gives the conclusion, a triple core gives at most $2^{n-3} < 2^{n-2}$ members.) The threshold 2^{n-2} cannot be lowered: deleting $\{d, 2d\}$ from its own family leaves a minimal-free sum-intersecting family with $2^{n-2}-1$ members.

Third, a certificate. By the reduction and Lemma 2.5 it suffices to colour the sum-carrying subsets of $[n]$ that are *not* minimal self-summing with $2^{n-2}-1$ classes. Such colourings are exhibited for $n = 6,7,8,9$; by the previous paragraph the number of classes is optimal, so the certificates are exactly optimal colourings and their existence had to be measured rather than assumed. The padding argument then covers $2 \le n \le 5$, and the two smallest cases are also confirmed by direct enumeration over all sum-intersecting families of subsets of [4] and [5]. ☐

Remark 6.2. The same greedy search finds such colourings at every n from 6 to 15 — 4095 classes over 15492 sets at $n = 14$, 8191 over 31317 at $n = 15$ — each re-checked by a verifier sharing no code with the search, and with the control that asking for $2^{n-2}-2$ classes fails everywhere. Only $n \le 9$ is formalised; the palettes grow like 2^{n-2} and stop being affordable to check. Note also that an unconditional uniqueness statement would imply Conjecture 1.2, so it is at least as hard as an open problem.

7. The distinct-sum variant

Here a set carries a solution if it contains pairwise distinct x, y, z with $x + y = z$; the maximum is $g(n)$. The construction implicit in Question 1.3 is the family of all supersets of a fixed triple $\{a, b, a+b\}$ with $a \ne b$, of size 2^{n-3}, so $g(n) \ge 2^{n-3}$.

Theorem 7.1. $g(n) = 2^{n-3}$ *for every* $3 \le n \le 10$.

Theorem 7.2. $g(9) = 64$ *and* $g(10) = 128$.

Proof of Theorems 7.1 and 7.2. The lower bounds are the triple construction. For the upper bounds we use Lemma 2.5 with 2^{n-3} classes. The colourings were found by a DSATUR heuristic; the cost is entirely in checking them, and the encoding matters. A subset $A \subseteq [n]$ travels as the integer $\mathrm{enc}(A) = \sum_{x \in A} 2^x$, which is injective; membership in a class becomes an integer comparison, and "$A \cap B$ carries a distinct sum" becomes "some mask $t = 2^x \mid 2^y \mid 2^{x+y}$ with $0 < x < y$, $x + y \le n$, satisfies $t \wedge \mathrm{enc}(A) = t$ and $t \wedge \mathrm{enc}(B) = t$", so that no intersection is ever formed and at most $\lfloor n^2/4 \rfloor$ masks are tested per pair. The bridge lemma — a distinct sum in $A \cap B$ is a solution mask inside both encodings — is proved, so the search and the mathematics cannot drift apart. $\square$

At $n = 11$ the same approach stalls, and the reason is structural.

Lemma 7.3. *Let T be the set of minimal solutions $\{x, y, x+y\}$ with $0 < x < y$ and $x + y \le n$. Counting incidences two ways, $\sum_{A \subseteq [n]} |\{t \in T : t \subseteq A\}| = |T| \cdot 2^{n-3}$. Hence a colouring with 2^{n-3} classes must be* perfect*: the solution sets of each class must partition T exactly.*

At $n = 11$ this reads $6400 = 25 \cdot 256$ with no slack anywhere, which is a 1638-hyperedge, 256-regular hypergraph 1-factorisation on 25 vertices; local search plateaus one class short. The fix is not a better search but a smaller instance.

Theorem 7.4. *$g(11) = 256$, $g(12) = 512$, and $g(n) = 2^{n-3}$ for every $3 \le n \le 12$.*

Proof. Let $\mathcal{F}$ be distinct-sum-intersecting and suppose some $A \in \mathcal{F}$ carries exactly one solution t. Every $B \in \mathcal{F}$ shares a solution with A, and t is the only candidate, so $t \subseteq B$ for every $B \in \mathcal{F}$ and $\mathcal{F}$ lies inside the star of t, which has exactly 2^{n-3} members. Otherwise every member carries at least two solutions, and only those sets need colouring. So $g(n) \le 2^{n-3}$ follows from a proper 2^{n-3}-colouring of the sets with two or more solutions alone.

That is what removes the rigidity of Lemma 7.3: at $n = 11$ it drops 384 of the 1638 vertices, leaving each solution in only 232 to 246 of them, i.e. ten to twenty-four classes of room where the full instance had none. A tabu search then closes $n = 11$ (1254 vertices, 256 classes) and $n = 12$ (2775 vertices, 512 classes) immediately. Both vertex counts are verified rather than asserted, and both certificates are checked exhaustively. Combining with Theorem 7.1 gives the stated range. $\square$

Remark 7.5. One structured route deserves to be recorded as closed. Reading a subset as a vector of $\mathbb{F}_2^n$ and colouring by $A \mapsto MA$ for a matrix M over $\mathbb{F}_2$ with $\dim\ker M = 3$, the colouring is proper exactly when every nonzero element of $\ker M$ meets every minimal solution; dually, exactly when $[n]$ can be 7-coloured by the points of the Fano plane $PG(2, 2)$ so that the union of the three classes on each of the seven lines is weak-sum-free. An exhaustive depth-first search shows such a colouring exists for $n \le 8$ and for no $n \ge 9$ — precisely where the free certificates stop.

Together with Theorem 3.8 this gives $g(n) = 2^{n-3}$ for $n \le 12$ and $g(n) \le 2^{n-2}$ for $13 \le n \le 23$. Question 1.3, which asks about all n, remains open; what is proved is that $\frac{1}{8} \cdot 2^n$ is not beaten anywhere it has been checked.

8. The k-sum variant

Theorem 8.1. *Let $k \ge 2$ and let $\mathcal{F} \subseteq \mathcal{P}([n])$ be k-sum-intersecting.*

(1) *If some $C \in \mathcal{F}$ has $|C| = 2$, then $C \subseteq A$ for every $A \in \mathcal{F}$, so $\mathcal{F}$ is contained in the family of all supersets of C and $|\mathcal{F}| \le 2^{n-2}$.*
(2) *If $|\mathcal{F}| > 2^{n-2}$, then every member of $\mathcal{F}$ has at least three elements.*

(3) *More generally, for every* $C \in \mathcal{F}$, $|\mathcal{F}| \leq 2^{n-|C|} \cdot |\mathrm{deck}_k(C)|$, *where* $\mathrm{deck}_k(C) = \{T \subseteq C : T$ *carries a* k*-sum*$\}$. *Consequently, if* $|\mathcal{F}| > 2^{n-2}$ *then* $|\mathrm{deck}_k(C)| > 2^{|C|-2}$ *for every* $C \in \mathcal{F}$*: more than a quarter of the subsets of every member carry a* k*-sum.*

Proof. A k-sum needs two distinct values, since $kx = x$ forces $x = 0 \notin [n]$; so any set carrying a k-sum has at least two elements. For (1), $A \cap C$ carries a k-sum and is contained in the two-element set C, hence equals C, hence $C \subseteq A$; the count is then the number of supersets of a 2-set. For (2), a member A satisfies $A \cap A = A$, so $|A| \geq 2$, and $|A| = 2$ would contradict (1) via the hypothesis $|\mathcal{F}| > 2^{n-2}$.

For (3), fix $C \in \mathcal{F}$. Every $A \in \mathcal{F}$ has $A \cap C$ carrying a k-sum, so $\mathcal{F}$ is contained in $\{S \subseteq [n] : S \cap C \text{ carries a } k\text{-sum}\}$, and splitting S into $S \cap C$ and $S \setminus C$ counts that set exactly as $2^{n-|C|}|\mathrm{deck}_k(C)|$. The last statement follows by writing $2^{n-2} = 2^{n-|C|} \cdot 2^{|C|-2}$. Statement (1) is the case $|C| = 2$, where the deck is $\{C\}$. □

Part (2) is the sharpest general statement we have about Question 1.4. It does not answer it; it says where the answer cannot be: a family beating $\frac{1}{4} \cdot 2^n$ contains no singleton and no pair. It is also what makes the classification searches feasible, since it removes exactly the region where the known optima live.

Proposition 8.2. *Let* $\mathcal{F}$ *be* k*-sum-intersecting,* $C \in \mathcal{F}$*, and let* $\mathcal{D}$ *be any family of subsets of* C *such that every* k*-sum-carrying* $T \subseteq C$ *contains a member of* $\mathcal{D}$*. Then* $|\mathcal{F}| \leq \sum_{E \in \mathcal{D}} 2^{n-|E|}$*. Taking for* $\mathcal{D}$ *the* $\subseteq$*-minimal* k*-sum-carrying subsets of* C*: if* $|\mathcal{F}| > 2^{n-2}$ *then every* $C \in \mathcal{F}$ *either contains a pair* $\{d, kd\}$ *or has at least three minimal cores. Moreover the families of Proposition 2.4 are pairwise distinct for distinct* d*, so the core* $\{1, k\}$ *named in Question 1.4 is one of* $\lfloor n/k \rfloor$ *equally good ones.*

Proof. The union bound over the sets of supersets of the members of $\mathcal{D}$ gives the displayed inequality. A minimal core of size 2 is exactly a pair $\{d, kd\}$; so if C contains no such pair, all its minimal cores have at least three elements, and two cores of size 3 would give only $2 \cdot 2^{n-3} = 2^{n-2}$, which is not more than 2^{n-2}. Hence at least three cores are needed. Distinctness of the families is Proposition 2.3 with $2d$ replaced by kd. □

Remark 8.3. The hypothesis "contains no pair $\{d, kd\}$" in Proposition 8.2 is not removable. At $k = 3$ the set $C = \{1, 3, 9\}$ has exactly the two minimal cores $\{1, 3\}$ and $\{3, 9\}$ and still satisfies the quarter criterion of Theorem 8.1(3).

Theorem 8.4. $h_3(n) = 2^{n-2}$ *for every* $3 \leq n \leq 12$*, and* $h_4(n) = 2^{n-2}$ *for every* $4 \leq n \leq 8$*.*

Proof. Lower bounds are Proposition 2.4. Upper bounds are Lemma 2.5 with 2^{n-2} classes, one explicit colouring per pair (k, n); for $k = 3$ and $n \geq 10$ the certificate is checked by compiled evaluation rather than by kernel reduction. □

Outside the formal development the same computation, accelerated by Theorem 8.1(2), reaches $k = 3$, $n \leq 14$ and $k = 4$, $n \leq 13$: nothing beats $\frac{1}{4} \cdot 2^n$ anywhere in range, and the maximum families are in every case exactly the supersets of some $\{d, kd\}$. This is evidence for the second paragraph of Question 1.4 and an answer to its first paragraph in the checked range, with the qualification that the extremal core is not unique.

9. Verification

Every numbered statement of this paper has been formalised and machine-checked in Lean 4 against Mathlib, with the definitions given here taken as the definitions in the formalisation; the certificates, colourings and exhaustive searches described above are re-executed by the proof kernel from the raw data rather than trusted from the search that produced them, and the negative controls (a claimed value one too large or one too small, an empty certificate, a mass the ceiling forbids) are checked to fail. Three of the statements — the value $f(4)$ obtained by enumerating all 2^{2^4} families, the column counts of Proposition 4.9 and Proposition 4.10, the colouring certificates for $g(12)$ and for $h_3(n)$ at $n = 10, 11, 12$ — are checked by compiled evaluation and so additionally trust the compiler; every other computational statement, including both Delsarte certificates and all of Section 4, reduces in the kernel and depends on no axioms beyond propositional extensionality, choice and quotient soundness. The results quoted as computed *outside* the formal development are marked as such where they occur (Remark 3.6, the restricted-shape plateau below 2.810 in Section 4, the wider ranges in Sections 6 and 8, and the Fano criterion in Section 7) and are not used in the proof of any numbered statement. The repository is currently private; repository reference to be added at submission.

Concluding remarks

The first open value is $n = 16$, and what is missing is one object: a rational w on the 1954 columns of $\mathcal{D}(16)$ with mass 3 and $\widehat{w} \geq -1$. Everything else is general in n — the ceiling, the duality step, the construction, and downward restriction. The two cheap routes to it are closed (Proposition 4.12 and Theorem 5.2), so what remains is the 65536×1954 programme itself. Its feasible set is the optimal face of a concave problem, hence has empty interior, so an approximate solution must still be rounded exactly onto it; at $n = 15$ the forced-tight rows together with the mass equation had rational rank 12 and the rounding corrected 12 coordinates.

For the variants, the natural next target is the remaining conjecture of [4], on agreement along a cyclic translate of a fixed pattern $a_1 < \cdots < a_k$, which is Conjecture 1 of [3] and was settled for $k = 3$ by Füredi, Griggs, Holzman and Kleitman [5]. This is the one place in the present circle of ideas where the ceiling of Remark 3.6 does not immediately apply, because for the k-sum problem a solution set has $k + 1$ elements.

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