EIGHT NEW REALIZABLE TRIPLES OF TOTALLY, PARTIALLY AND MINIMALLY REAL COMMON SECANTS TO A PAIR OF REAL TWISTED CUBICS

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ABSTRACT. A general pair of twisted cubic curves in $\mathbb{P}^3(\mathbb{C})$ has ten common secant lines. For a real pair each real common secant is *totally*, *partially* or *minimally* real according to whether it meets the two curves in two real points each, in two real points and a conjugate pair, or in a conjugate pair each; the resulting count (n_t, n_p, n_m) takes 161 admissible values. Aslam, Faust, Hauenstein, Lopez Garcia, Kagy, Regan, Wampler and Zhang recently realized 128 of the 161 by numerical sampling with certification, listed the remaining 33, asked for the full realizable set, and singled out two families of 21 admissible 3-tuples as an open problem. We realize eight of the 33: $(4,6,0)$, $(4,5,1)$, $(4,4,2)$, $(3,5,2)$, $(3,7,0)$, $(2,8,0)$, and, in the two families of the open problem, $(0,10,0)$ and $(1,9,0)$. Each is realized by an explicit pair of *rational* twisted cubics given by a 4×4 integer matrix, and each claim is unconditional: for every one of the nine explicit pairs treated here we also prove that the ten common secants we exhibit are *all* of its common secants over $\mathbb{C}$. The certificates are exact rather than numerical, and are of two kinds: elimination reduces the common-secant system of one pair to a single degree-10 integer polynomial with a rational univariate representation, from which the ten secants and their types are read off ten integer sign changes and twenty Descartes sign-pattern certificates; and a second, chart-free model in the plane of chords of the first curve, where the common secants are the rank-one locus of a symmetric 3×3 matrix of quadratic forms, supplies the exact count through polynomial identities over $\mathbb{Z}$. Every step is checked by the Lean 4 kernel in arbitrary-precision integer arithmetic. We correct one statement of the earlier version of this paper, which asserted that neither the source nor Eisenbud and Harris bounds the number of common secants of a *special* pair: the proof of Proposition 2.2 of the source does bound it, for one explicit special matrix, by a saturation computation returning degree 40. We also record an erratum to the printed bisecant system of the source; report what the public data repository of the source now contains, which changes the status of two further cells; and show that no invariant of the ambient isotopy class of the pair of real curves can constrain the 3-tuple.

1. Introduction

1.1. **Ten common secants.** A *twisted cubic* is a smooth rational curve of degree three in $\mathbb{P}^3$. The standard one is the closure of the image of

$$v(t) = (1, t, t^2, t^3), \qquad C_0 = \{[v(t)] : t \in \mathbb{C}\} \cup \{[0:0:0:1]\},$$

and every twisted cubic is $M \cdot C_0$ for some $M \in \mathrm{PGL}_4$. A *secant* of a curve is a line meeting it in a scheme of length two. That a general pair of twisted cubics in $\mathbb{P}^3(\mathbb{C})$ has exactly ten common secant lines is classical: it is Keynote Question (c)

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of Chapter 3 of Eisenbud and Harris [3], and it is Theorem 1.2 of [1], which obtains it by Schubert calculus as a consequence of [3, Prop. 3.14].

1.2. **The real refinement.** Now let both curves be real. A common secant line is then either nonreal — and then it occurs in a conjugate pair — or real, and a real common secant meets each of the two curves in a pair of points that is either two real points or a conjugate pair. Following Proposition 2.3 of [1], a real common secant is

- *totally real* if it meets each curve in two real points;
- *partially real* if it meets one curve in two real points and the other in a conjugate pair;
- *minimally real* if it meets each curve in a conjugate pair.

Write (n_t, n_p, n_m) for the three counts and $n_{\mathbb{R}} = n_t + n_p + n_m$ for the number of real common secants. Nonreal common secants come in conjugate pairs, so their number is even; as there are ten common secants in all, $n_{\mathbb{R}}$ is even and $n_{\mathbb{R}} \leq 10$. These are the only constraints, and there are consequently $\sum_{k\in\{0,2,4,6,8,10\}} \binom{k+2}{2} = 1 + 6 + 15 + 28 + 45 + 66 = 161$ *admissible* 3-tuples (Proposition 2.5 of [1]). An admissible 3-tuple is *realizable* if some pair of real twisted cubics attains it.

1.3. **The source and its open question.** Which admissible 3-tuples are realizable is the question raised and attacked in [1]. That paper samples the parameter space of real twisted cubic curves — uniformly in $\mathbb{P}^{15}(\mathbb{R})$, in the 4×4 matrix coordinates of its Algorithm 1 — tracks solutions with Bertini [4] and certifies the resulting approximate solutions with `alphaCertified` [5]; the proof of its Theorem 1.7 reports that "this process generated parameters for 128 out of the 161 admissible 3-tuples". Its Table 4 accordingly splits the 161 into 128 *realized* and 33 *admissible and not realized*, and its Problem 5.1 opens

> Determine the full set of realizable 3-tuples in Table 4.

The 33 unrealized 3-tuples are

$$(0, n_p, 10 - n_p) \;\; (0 \leq n_p \leq 10), \qquad (1, n_p, 9 - n_p) \;\; (0 \leq n_p \leq 9),$$
$$(1,0,7),\ (1,1,6),\quad (2,6,2),\ (2,7,1),\ (2,8,0),\quad (3,5,2),\ (3,6,1),\ (3,7,0),$$
$$(4,0,6),\ (4,4,2),\ (4,5,1),\ (4,6,0),$$

21 of them belonging to the two families singled out in Remark 1.8 of [1], which states that it "remains an open problem whether the admissible 3-tuples of the form $(0, n_p, 10 - n_p)$ for $0 \leq n_p \leq 10$ and $(1, n_p, 9 - n_p)$ for $0 \leq n_p \leq 9$ are realizable". The other twelve are the two with $n_{\mathbb{R}} = 8$ and the ten with $n_{\mathbb{R}} = 10$ and $n_t \leq 4$.

The main theorem of [1], its Theorem 1.7, states a sufficient condition: an admissible 3-tuple is realizable whenever

$$n_{\mathbb{R}} \in \{0, 2, 4, 6\}, \quad \text{or} \quad n_{\mathbb{R}} = 8 \text{ and } n_t \neq 1, \quad \text{or} \quad n_{\mathbb{R}} = 10 \text{ and } n_t \in \{5, 6, 7, 8, 9, 10\}.$$

That condition holds for 108 of the 161 admissible 3-tuples, whereas Table 4 records 128 as realized. The two counts do not conflict: the 108 are a subset of the 128, and the twenty further cells — the six with $n_{\mathbb{R}} = 8$ and $n_t = 1$, and the fourteen with $n_{\mathbb{R}} = 10$ and $n_t \in \{2, 3, 4\}$ — are the "partial results" for $n_{\mathbb{R}} \in \{8, 10\}$ that Remark 1.8 of [1] points to in the same breath as the two open families, recorded in Table 4 but left out of the clean statement of the theorem. The finer information, and the source of the list of 33, is therefore Table 4 rather than Theorem 1.7, and it is Table 4 that the results below act on.

1.4. **Results.** We settle eight of the 33 cells, all of them with $n_{\mathbb{R}} = 10$, and we settle them unconditionally.

(i) *Eight new realizable* 3*-tuples* (Section 6). Each of

$$(4,6,0), \quad (4,5,1), \quad (4,4,2), \quad (3,5,2), \quad (3,7,0), \quad (2,8,0), \quad (0,10,0), \quad (1,9,0)$$

is realizable, by an explicit pair C_0, $M\cdot C_0$ of *rational* twisted cubics with M an integer 4×4 matrix (Theorems 12–17 and 18–19). All eight are printed in the "admissible and not realized" column of Table 4 of [1]. The last two lie in the two families of its Remark 1.8: $(0,10,0)$ is the member $n_p = 10$ of $(0, n_p, 10 - n_p)$ and $(1,9,0)$ is the member $n_p = 9$ of $(1, n_p, 9 - n_p)$. This reduces the unrealized list from 33 to 25. Outside the two families it leaves $(1,0,7)$, $(1,1,6)$, $(2,6,2)$, $(2,7,1)$, $(3,6,1)$ and $(4,0,6)$; inside them it leaves 19 cells, every one of which has $n_m \geq 1$.

(ii) *An exact count of the common secants of each explicit pair* (Section 5). For each of the nine explicit pairs of this paper we prove that it has exactly ten common secant lines in $\mathbb{P}^3(\mathbb{C})$, and that they are the ten our certificates exhibit (Proposition 9). The proof runs in a second, chart-free model: the chords of C_0 form a projective plane, in which the common secants of the pair are the locus where a symmetric 3×3 matrix of quadratic forms drops to rank one, and the count is decided by polynomial identities over $\mathbb{Z}$. This is what makes the realizability statements of (i) unconditional; the earlier version of this paper assumed it, and Section 1.5 says exactly how that assumption was stated and what is true.

(iii) *An exact certificate in place of a numerical one* (Sections 2–4). For a pair of rational twisted cubics the ten common secants and their three types are decided by finitely many integer facts: ten sign changes of one degree-10 integer polynomial, four polynomial identities over $\mathbb{Z}$, and twenty Descartes sign-pattern certificates; the count of (ii) adds seventeen further integer facts per pair. Nothing approximate enters. The same machinery run on the already realized cell $(7,2,1)$ reproduces it (Theorem 22) and serves as a control on the whole construction.

(iv) *An erratum to the printed bisecant system* (Remark 2). The displayed system of [1], its equation (6), prints $f_2 = p_4 - (t_1^2 + 2t_1t_2 + t_2^2)p_2 + t_1t_2p_1$, where the 3×3 minor it is derived from gives $f_2 = p_4 - (t_1^2 + t_1t_2 + t_2^2)p_2 + t_1t_2(t_1+t_2)p_1$. The error is confined to the printed equation; the computations of [1] are made with the correct system.

(v) *What the search did not reach, and what cannot obstruct it* (Section 8). The cells of (i) were found by a directed search over integer matrices, and we report the negative side of that search: the 19 remaining cells of the two families of Remark 1.8, all of which have $n_m \geq 1$, were never reached. We also close one route that might have explained the difficulty. The two real curves form a two-component link in $\mathbb{RP}^3$, and one might hope that the 3-tuple is constrained by the topology of that link. It is not: Remark 24 shows that the linking number takes the value $1/2 \in \mathbb{Q}/\mathbb{Z}$ for *every* disjoint real pair, and Remark 25 exhibits an explicit rational segment of pairs along which the link does not change up to ambient isotopy while the 3-tuple runs through five different values, including $n_t = 0$ and $n_t = 1$.

1.5. **Two corrections to the earlier version.** An earlier version of this paper realized six of the 33 cells — items (i) minus $(0, 10, 0)$ and $(1, 9, 0)$ — and stated its realizability claims conditionally. Two of its statements have to be corrected here, and we state both plainly.

First, on bounding the number of common secants of a special pair. The earlier version carried a convention reading, in part:

> "…neither reference bounds the number of distinct common secants of a *special* pair, and we prove no such bound here",

the two references being [1] and [3]. That is inaccurate as written. The proof of Proposition 2.2 of [1] does bound it, for one explicit matrix: it runs Macaulay2 [6] on

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{pmatrix}$$

and reports that the saturated ideal $J = \langle F \rangle : \langle (t_1 - t_2)(s_1 - s_2) \rangle^\infty$ of $\mathbb{Q}[t_1, t_2, s_1, s_2]$ has $\deg J = \deg \sqrt{J} = 40$; for a zero-dimensional ideal that is a two-sided count of forty solutions, hence of ten common secants. Moreover that pair is itself special: M fixes the vector $(1, 0, 0, 0)$, so the point $[1 : 0 : 0 : 0]$ of C_0 lies on C_1 as well. The stated purpose of the computation there is that the forty solutions be nonsingular, so the count is a by-product; but it is a count, and of the same shape as the one proved here. The accurate statement is the narrower one: neither reference bounds the number of common secants for the nine explicit pairs of this paper, and [1] performs no such computation for any of the pairs whose 3-tuples it reports. Proposition 9 below is therefore not the first bound of its kind, and we do not present it as one; what is new in it is the trust base — a finite integer certificate checked by a proof kernel, rather than a computer-algebra system's degree or a floating-point α-test — and the fact that it is carried out for nine pairs whose 3-tuples are the point of the paper. See Remark 11.

Second, on the source's public data. The certified data behind the 128 realizations of [1] live in a public repository [2], which the earlier version of this paper reported as *not consulted*; its claims were therefore stated in the weaker form "what is claimed here is therefore that these six cells are not realized in `arXiv:2603.25003v1` and that we have found no later work realizing them". We have now consulted it, at its last commit of 27 April 2026 — a month after the e-print, and still its last commit when we re-read the repository on 3 September 2026. Two things follow. On the one hand it strengthens the claims of item (i): the repository carries the group's own running tally of 3-tuples produced by their sampling, and that tally records *zero* samples for all eight cells realized here, none of which appears in their list of 3-tuples ever produced; its zero entries with even $n_{\mathbb{R}}$ are 31 of the 33 unrealized cells of Table 4, which is an independent confirmation that the tally uses the same convention as the paper. On the other hand it changes the status of two cells that the earlier version reported as reached by its search but not certified, namely $(1, 0, 7)$ and $(1, 1, 6)$: those are exactly the two exceptions in the tally, recorded there with 3 and 24 samples respectively and both present in the list of 3-tuples produced. We therefore withdraw the suggestion, implicit in the earlier version's treatment of those two cells, that they are open in the same sense as the others: they have since been reached by the source's own group. What is still missing for them, and is not supplied here, is an exact certificate. See Remark 21.

1.6. **What rests on computation.** The statements below are proved from finitely many exact integer facts, each checked by the kernel of the Lean 4 proof assistant on arbitrary-precision integers. The facts are of four shapes only: the sign of an integer polynomial at an integer argument; the sign pattern of the coefficient list of a Möbius-transformed integer polynomial; an identity between two integer polynomials in one variable; and an identity between two integer polynomials in two variables. There are 549 of them, 61 per configuration over the nine configurations. No floating-point arithmetic and no compiled evaluation occurs in any proof.

Two things are *not* part of that formal development, and we name them once here. First, the elementary translation between geometry and algebra — Lemma 1 and Proposition 3 of Section 2, and Propositions 6 and 7 of Section 5, four short pieces of linear algebra — which turns "the line through two points of C_0 meets C_1 twice" into the algebraic conditions that the certificates decide. Second, the *elimination* of Section 3 and the construction of the cofactors of Section 5: the passage from a matrix to the degree-10 polynomial, to its rational univariate representation and to the polynomial cofactors of the identities was carried out in exact rational arithmetic in PARI/GP [15] and is a search device, not a step of any proof — what the proofs use is only its output, and only through identities over $\mathbb{Z}$ that the kernel checks directly. Likewise, the matrices themselves were found by a directed search (Section 8) whose conduct is irrelevant to the validity of the certificates it produced.

The earlier version of this paper named *three* such inputs; the third was the assumption that the ten certified common secants of each pair are all of them. That assumption is now Proposition 9.

2. The common-secant system

Throughout, M is an invertible real 4×4 matrix, C_0 is the standard twisted cubic above and $C_1 = M \cdot C_0$. Write

$$p_i(s) = M_{i1} + M_{i2}s + M_{i3}s^2 + M_{i4}s^3 \qquad (1 \le i \le 4),$$

so that $Mv(s) = (p_1(s), p_2(s), p_3(s), p_4(s))$.

Lemma 1. *Let $t_1 \neq t_2$ and $w \in \mathbb{C}^4$. Then $[w]$ lies on the line through $[v(t_1)]$ and $[v(t_2)]$ if and only if the two 3×3 minors of the matrix $\big(v(t_1) \mid v(t_2) \mid w\big)$ taken on rows $\{1, 2, 3\}$ and on rows $\{1, 2, 4\}$ both vanish.*

Proof. The 2×2 minor of the first two columns on rows $1, 2$ is $t_2 - t_1 \neq 0$, so there are unique λ, μ with $w - \lambda v(t_1) - \mu v(t_2) = r$ having $r_1 = r_2 = 0$. Subtracting multiples of the first two columns from the third leaves the minor on rows $\{1, 2, k\}$ equal to $r_k\,(t_2 - t_1)$ for $k = 3, 4$. Both vanish precisely when $r_3 = r_4 = 0$, that is, when $w \in \langle v(t_1), v(t_2)\rangle$. ☐

Expanding those two minors along the third column and dividing by $t_2 - t_1$ gives, with $e_1 = t_1 + t_2$ and $e_2 = t_1 t_2$,

$$\begin{aligned} g_1(s) &= p_3(s) - e_1\, p_2(s) + e_2\, p_1(s), \\ g_2(s) &= p_4(s) - (e_1^2 - e_2)\, p_2(s) + e_1 e_2\, p_1(s), \end{aligned} \tag{1}$$

using $t_1^2+t_1t_2+t_2^2=e_1^2-e_2$ and $t_1t_2(t_1+t_2)=e_1e_2$. Both are cubics in s whose coefficients are polynomial in (e_1,e_2); the pair $\{t_1,t_2\}$ enters only through e_1,e_2.

Remark 2. An erratum to [1], established by the computation just made and outside the formal development below. The bisecant system of [1] — its displayed equation (6), and the unnumbered display of f_1,f_2 that immediately precedes it — prints

$$f_1=p_3-(t_1+t_2)p_2+t_1t_2p_1, \qquad f_2=p_4-(t_1^2+2t_1t_2+t_2^2)p_2+t_1t_2p_1.$$

The first agrees with g_1. The second does not agree with g_2: expanding the minor on rows $\{1,2,4\}$ of $\big(v(t_1)\mid v(t_2)\mid Mv(s)\big)$ gives

$$p_1\,t_1t_2(t_2^2-t_1^2)-p_2\,(t_2^3-t_1^3)+p_4\,(t_2-t_1),$$

and division by t_2-t_1 leaves $p_4-(t_1^2+t_1t_2+t_2^2)p_2+t_1t_2(t_1+t_2)p_1$, so the printed coefficient of p_2 should have t_1t_2 in place of $2t_1t_2$ and the printed coefficient of p_1 should carry the extra factor t_1+t_2. The ten solutions printed in the worked example of [1] — its Example 2.4, one representative (t_1,t_2,s_1,s_2) from each of the ten orbits, to four decimal places — satisfy the corrected equation to the accuracy of those four decimals and do not satisfy the printed one. The twenty residuals $f_2(t_1,t_2,s_i)$ of the corrected form are all below $1.3\cdot10^{-2}$, on the scale set by the twenty residuals of the printed f_1, which the same computation puts below $6\cdot10^{-3}$; the residuals of the printed f_2 run from $2.5\cdot10^{-2}$ to 54.9. So this is a transcription slip in the displayed equation and not an error in the computations of [1], all of whose conclusions stand. The group's own `Bertini` input file for the system, printed in the README of the data repository [2], confirms this directly: its third and fourth equations carry the coefficients $t_1^2+t_1t_2+t_2^2$ and $t_1t_2(t_1+t_2)$, that is, the corrected form.

Proposition 3. *Let $e_1,e_2,u_1,u_2\in\mathbb{R}$ satisfy $\delta_t:=e_1^2-4e_2\neq0$ and $\delta_s:=u_1^2-4u_2\neq0$, and let t_1,t_2 be the roots of $X^2-e_1X+e_2$ and s_1,s_2 those of $X^2-u_1X+u_2$. Then the line ℓ through $[v(t_1)]$ and $[v(t_2)]$ is a real common secant of C_0 and C_1, meeting C_0 in $[v(t_1)],[v(t_2)]$ and C_1 in $[Mv(s_1)],[Mv(s_2)]$, if and only if $X^2-u_1X+u_2$ divides both g_1 and g_2 in $\mathbb{C}[X]$. Its type is*

$$\textit{totally real if } \delta_t>0 \textit{ and } \delta_s>0; \qquad \textit{partially real if } \delta_t\delta_s<0;$$
$$\textit{minimally real if } \delta_t<0 \textit{ and } \delta_s<0.$$

Proof. Since $\delta_t\neq0$ we have $t_1\neq t_2$, so ℓ is well defined and, the set $\{t_1,t_2\}$ being stable under conjugation, ℓ is a real line meeting C_0 in the two distinct points $[v(t_1)],[v(t_2)]$. Since $\delta_s\neq0$ we have $s_1\neq s_2$, so $[Mv(s_1)]\neq[Mv(s_2)]$ are two distinct points of C_1. By Lemma 1, $[Mv(s_i)]\in\ell$ for $i=1,2$ if and only if $g_1(s_i)=g_2(s_i)=0$ for $i=1,2$, that is, if and only if $X^2-u_1X+u_2$ divides g_1 and g_2. The type is the reality of the two pairs, which is the sign of the two discriminants. ☐

Written out on coefficients, divisibility of a cubic $a_0+a_1X+a_2X^2+a_3X^3$ by $X^2-u_1X+u_2$ is the vanishing of the two entries of its remainder,

$$a_1+a_2u_1+a_3(u_1^2-u_2)=0, \qquad a_0-u_2(a_2+a_3u_1)=0. \tag{2}$$

Applying (2) to g_1 and to g_2 gives four equations in (e_1,e_2,u_1,u_2); we call them *the common-secant system* of M and denote it $\mathrm{Sys}(M)$. This is the system of [1] with t_1, t_2 replaced by their elementary symmetric functions, and Proposition 3 is its Proposition 2.3 in that form.

The system $\mathrm{Sys}(M)$ sees only those common secants both of whose intersection parameters with each curve are finite; it is a statement in one chart. Everything in Sections 3 and 4 is therefore about producing common secants, not about counting all of them. The count is the business of Section 5, which works in a model with no chart at all.

3. Elimination and a rational univariate representation

Fix now an integer matrix M with $\det M \neq 0$, so that C_0 and $C_1 = M \cdot C_0$ are rational twisted cubics. Put $z = -e_1$. Eliminating e_2, u_1, u_2 from $\mathrm{Sys}(M)$ — through the Plücker, rank-one description of the chordal surface of C_0 — leaves a polynomial of degree 12 in z. Two of its roots are extraneous and equal: the Plücker equations vanish to order two along the unique chord of C_0 through the point $[Me_4]$, where $e_4 = (0,0,0,1)$ — the point of C_1 at parameter $s = \infty$ — and that chord is available in closed form, so the double factor is divided out exactly and there remains a single polynomial

$$E \in \mathbb{Z}[z], \qquad \deg E = 10,$$

together with integer polynomials D, C_e, A_1, A_2 and the *rational univariate representation*

$$e_1 = -z, \qquad e_2 = \frac{C_e(z)}{D(z)}, \qquad u_1 = \frac{A_1(z)}{D(z)}, \qquad u_2 = \frac{A_2(z)}{D(z)}, \tag{3}$$

valid on the zero set of E. Here $D = W \cdot E'$ for an integer constant W; taking the derivative of E rather than a cleared integer denominator costs nothing and is worth about a factor of five in coefficient size. For the nine matrices below, $\deg D = 9$ and the coefficients of E and D run to at most 54 and 61 decimal digits respectively.

This computation was performed in exact rational arithmetic in PARI/GP. It plays no role in any proof. What is used is that the four equations of $\mathrm{Sys}(M)$, after substituting (3) and clearing the denominator D^3, become four integer polynomials $N_1, \ldots, N_4$ in z of degree at most 29, and that these are multiples of E:

$$E \cdot H_i = N_i \quad \text{in } \mathbb{Z}[z], \qquad i = 1, 2, 3, 4, \tag{4}$$

for explicit cofactors $H_i \in \mathbb{Z}[z]$. The four identities (4) are verified coefficient by coefficient; their coefficients reach 197 decimal digits.

Finally the two discriminants of Proposition 3 are, on the zero set of E, the signs of two integer polynomials. From (3),

$$(e_1^2 - 4e_2)\, D^2 = P_t := D \cdot (z^2 D - 4C_e), \qquad (u_1^2 - 4u_2)\, D^2 = P_s := A_1^2 - 4A_2 D. \tag{5}$$

The polynomials P_t and P_s have degrees 20 and 18. The correcting factor D^2 is positive wherever it is nonzero, so P_t and P_s carry the signs of δ_t and δ_s respectively; and $P_t \neq 0$ forces $D \neq 0$, so no separate non-vanishing certificate for the denominator is needed.

Conditioning. Write $\Delta_K = \operatorname{diag}(1, K, K^2, K^3)$. Since $\Delta_K\, v(t) = v(Kt)$, the projective transformation Δ_K fixes C_0 setwise and multiplies its parameter by K. Replacing M by $\Delta_K M$ replaces the pair (C_0, C_1) by its image under that transformation, so leaves (n_t, n_p, n_m) unchanged, while scaling the roots of E by K. Each configuration below carries the $K \in \{50, 200, 500, 1000\}$ that separates the ten real roots of E into ten *disjoint unit windows* with integer endpoints. Unit windows are not a convenience: on the wider windows obtained by splitting at midpoints, the Descartes certificates of the next section fail at every K we tried. The choice of K is delicate in the other direction too: for the configuration realizing $(1, 9, 0)$ the values $K \in \{50, 100, 200, 500\}$ all leave one window on which the sign of P_s is not certified, and $K = 1000$ is the smallest value we found that closes.

4. Certificates

The certificate layer used here is not new and is not a contribution of this paper: it is a small apparatus for isolating real roots of an integer polynomial given as a coefficient list, written for an unrelated real-root problem — the growth-factor polynomials of Chen, Edelman and Urschel [12] — and imported unchanged. We describe only what is needed to read the proofs.

A polynomial is carried as its list of coefficients in ascending order, and sums, products, negation and evaluation at an integer are the evident list operations. For integers $a < b$ the substitution $x = (a + bt)/(1 + t)$ is an increasing bijection $(0, \infty) \to (a, b)$; multiplying by $(1 + t)^N$, where N is the length of the list, clears denominators and leaves an integer coefficient list $T_{a,b}\, p$ with

$$(T_{a,b}\, p)(t) = (1 + t)^N\, p\left(\frac{a + bt}{1 + t}\right) \qquad (t > 0), \tag{6}$$

a positive multiple of $p(x)$ at the corresponding x. Say that p satisfies $\operatorname{Pos}(a, b)$ if every coefficient of $T_{a,b}\, p$ is ≥ 0 and at least one is > 0. A list with nonnegative coefficients, one of them positive, is positive at every $t > 0$, so by (6):

Lemma 4. *If p satisfies $\operatorname{Pos}(a, b)$ then $p(x) > 0$ for all $x \in (a, b)$; if $-p$ satisfies $\operatorname{Pos}(a, b)$ then $p(x) < 0$ for all $x \in (a, b)$.*

Everything now assembles into a single lemma, one application of which produces one certified common secant of one prescribed type.

Lemma 5 (window lemma). *Let M be an integer matrix as above, with associated polynomials E, D, C_e, A_1, A_2, $H_1, \ldots, H_4$ and P_t, P_s as in Section 3, and suppose the four identities (4) hold. Let $a < b$ be integers, let $\varepsilon_t, \varepsilon_s \in \{+1, -1\}$, and suppose*

1. *$E(a)$ and $E(b)$ are nonzero of opposite signs, and*
2. *$\varepsilon_t P_t$ and $\varepsilon_s P_s$ both satisfy $\operatorname{Pos}(a, b)$.*

Then there is $x \in (a, b)$ with $E(x) = 0$, and for any such x the quadruple (3) evaluated at $z = x$ is a real solution of $\operatorname{Sys}(M)$ with $\operatorname{sign} \delta_t = \varepsilon_t$ and $\operatorname{sign} \delta_s = \varepsilon_s$, and $D(x) \neq 0$. It therefore determines, by Proposition 3, a real common secant of C_0 and $C_1 = M \cdot C_0$ that meets C_0 in the two points with parameter sum $e_1 = -x$, and whose type is totally, partially or minimally real according to $(\varepsilon_t, \varepsilon_s)$ being $(+, +)$, of mixed sign, or $(-, -)$.

Proof. Existence of x is the intermediate value theorem applied to the polynomial function E on $[a, b]$. By Lemma 4, $\varepsilon_t P_t(x) > 0$ and $\varepsilon_s P_s(x) > 0$; in particular

$P_t(x) \neq 0$, whence $D(x) \neq 0$ by (5), so (3) is defined at x. From (4) and $E(x) = 0$ we get $N_i(x) = 0$ for $i = 1, \dots, 4$; dividing by $D(x)^3 \neq 0$ turns these into the four equations of $\mathrm{Sys}(M)$ at the quadruple (3). Finally $D(x)^2 > 0$, so by (5) the signs of δ_t, δ_s at that quadruple are those of $P_t(x), P_s(x)$, namely $\varepsilon_t, \varepsilon_s$; both are nonzero, so Proposition 3 applies. □

Applying Lemma 5 on ten disjoint windows $(a_0, a_0+1), \dots, (a_9, a_9+1)$ with $a_0 < a_0 + 1 \leq a_1 < \dots$ yields ten roots $x_0 < x_1 < \dots < x_9$ of E, hence ten distinct values $e_1 = -x_j$, hence ten common secants with pairwise distinct intersection divisors on C_0 — and two distinct lines cannot meet C_0 in the same pair of points, so the ten lines are pairwise distinct. The hypotheses consumed are 44 integer facts: the four identities (4), twenty evaluations of E at integer endpoints, and twenty Pos certificates.

5. The chord plane, and exactly ten

This section proves that the ten common secants produced by Section 4 are, for each of the nine explicit pairs of Section 6, all the common secants of that pair. Two features of the elimination make that a separate question. The system $\mathrm{Sys}(M)$ lives in a chart, and a common secant through the point $v(\infty) = [0:0:0:1]$ of C_0 is invisible to it; and the classical count of ten is a count for a *general* pair, whereas the nine pairs below are nine explicit matrices found by a search. We therefore change model.

5.1. **The chords of C_0 as a projective plane.** For $(a, b, c) \neq (0, 0, 0)$ let $\ell(a, b, c)$ be the line spanned by the two points of C_0 whose parameters are the two roots of the binary quadratic $a\,t^2 + b\,t + c$, a root at $t = \infty$ meaning the point $[0:0:0:1]$ and a double root meaning the tangent line of C_0 at the corresponding point.

Proposition 6. *The assignment $(a:b:c) \mapsto \ell(a, b, c)$ is a bijection from $\mathbb{P}^2$ onto the set of secant lines of C_0, and in Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$,*

$$\ell(a, b, c) = [\, a^2 : -ab : b^2 - ac : ac : -bc : c^2 \,]. \tag{7}$$

In particular the chordal surface of C_0 is the image of $\mathbb{P}^2$ under a Veronese embedding into the Klein quadric.

Proof. For $a \neq 0$ and roots t_1, t_2 of $at^2 + bt + c$ one has $v(t_1) \wedge v(t_2) = (t_2 - t_1)\,(1, e_1, e_1^2 - e_2, e_2, e_1 e_2, e_2^2)$ with $e_1 = t_1 + t_2 = -b/a$ and $e_2 = t_1 t_2 = c/a$, and multiplying by a^2 gives (7). Both sides of (7) are continuous in $(a:b:c)$, so the formula persists at $a = 0$; there it gives $[0:0:b^2:0:-bc:c^2]$, which for $b \neq 0$ is the line through $[0:0:0:1]$ and $[v(-c/b)]$ and for $b = 0$ is the tangent line of C_0 at $[0:0:0:1]$. The map is injective because a secant line determines the length-two subscheme it cuts on C_0, hence the binary quadratic up to scale: a line is the intersection of two distinct hyperplanes, whose equations restrict on C_0 to two non-proportional binary cubics in t, so their common zero scheme has length at most two and C_0 carries no trisecant line. The map is surjective because every length-two subscheme of C_0 is the zero scheme of such a quadratic. □

We call $\mathbb{P}^2$ with coordinates $(a:b:c)$ the *chord plane* of C_0, and write $X(a, b, c)$ for the 6-vector on the right of (7). The line $a = 0$ is the pencil of chords through $[0:0:0:1]$, and the conic $b^2 = 4ac$ is the locus of tangent lines.

5.2. **Common secants as a rank-one locus.** Let $\Lambda^2 M$ denote the induced map on Plücker coordinates, so that $M \cdot \ell$ has Plücker vector $\Lambda^2 M\, \ell$. Since $\Lambda^2(\operatorname{adj} M) = (\det M)\, \Lambda^2(M^{-1})$ is a nonzero multiple of the inverse of $\Lambda^2 M$, a line ℓ is a secant of $C_1 = M \cdot C_0$ if and only if $\Lambda^2(\operatorname{adj} M)\, \ell$ is a secant of C_0. Put

$$Y(a,b,c) = \Lambda^2(\operatorname{adj} M)\, X(a,b,c) \in \mathbb{Z}[a,b,c]^6,$$

a vector of six quadratic forms, and assemble the symmetric matrix

$$N(a,b,c) = \begin{pmatrix} Y_1 & -Y_2 & Y_4 \\ -Y_2 & Y_3 + Y_4 & -Y_5 \\ Y_4 & -Y_5 & Y_6 \end{pmatrix}. \tag{8}$$

Proposition 7. *Let M be invertible. For $(a,b,c) \neq (0,0,0)$ over $\mathbb{C}$, the chord $\ell(a,b,c)$ is a common secant of C_0 and $C_1 = M \cdot C_0$ if and only if* $\operatorname{rank} N(a,b,c) \leq 1$. *Moreover the Plücker relation makes two of the six 2×2 minors of N equal as polynomials, so the common-secant locus*

$$W(M) = \{(a:b:c) \in \mathbb{P}^2 : \ \operatorname{rank} N(a,b,c) \leq 1\}$$

is the common zero locus of five quartic forms $m_0, \ldots, m_4$ in (a,b,c).

Proof. Write $w = (A,B,C)$. Comparing (7) with (8), the six entries of N read $A^2, AB, AC, B^2, BC, C^2$ exactly when $Y = X(A,B,C)$, and the passage from Y to the entries of N is an invertible linear substitution; so $Y = X(A,B,C)$ if and only if $N = w\,w^{\mathsf{T}}$. By Proposition 6, $\ell(a,b,c)$ is a secant of C_1 if and only if $Y(a,b,c)$ is proportional to $X(A,B,C)$ for some $(A:B:C)$, and over $\mathbb{C}$ every scalar has a square root, so the factor of proportionality may be absorbed into w: the condition is that $N(a,b,c) = w\,w^{\mathsf{T}}$ for some w. A symmetric complex matrix has that form exactly when its rank is at most one. Since $\Lambda^2(\operatorname{adj} M)$ is invertible and $X(a,b,c) \neq 0$, the matrix $N(a,b,c)$ is nonzero, so the rank condition here is $\operatorname{rank} N = 1$; either way it is the vanishing of all 2×2 minors. Finally, Y is a decomposable 6-vector, so it satisfies the Plücker relation $Y_1 Y_6 - Y_2 Y_5 + Y_3 Y_4 = 0$, which reads

$$N_{11}N_{33} - N_{13}^2 = N_{12}N_{23} - N_{13}N_{22} :$$

two of the six minors coincide, and five remain. ∎

We fix the five minors once and for all as

$$m_0 = N_{11}N_{22} - N_{12}^2, \quad m_1 = N_{11}N_{23} - N_{12}N_{13}, \quad m_2 = N_{11}N_{33} - N_{13}^2,$$
$$m_3 = N_{22}N_{33} - N_{23}^2, \quad m_4 = N_{12}N_{33} - N_{13}N_{23},$$

quartic forms with integer coefficients when M is an integer matrix. They are computed from the sixteen entries of M by the formal route $M \mapsto \operatorname{adj} M \mapsto \Lambda^2(\operatorname{adj} M) \mapsto Y \mapsto N \mapsto m_k$; no coefficient of any m_k is supplied as data.

5.3. **The certificates.** Fix an integer matrix M and keep the notation E, D, C_e of Section 3. Write $z_k(b,c) = m_k(0,b,c)$ for the restrictions to the line at infinity of the chord plane. The following four families of identities over $\mathbb{Z}$ are what the kernel checks; all of them are produced by the elimination described in Section 1.6 and none of them is used except as a verified identity.

(C1) *The line $a = 0$.* Explicit $W_0, W_1 \in \mathbb{Z}[b,c]$ and a nonzero integer d_3 with $W_0 z_0 + W_1 z_1 = d_3\, c^7$; and the coefficient κ of b^4 in $z_0(b,0)$ is nonzero.

(C2) *Elimination of c.* Explicit $U_1, V_1, U_2, V_2 \in \mathbb{Z}[b, c]$ and $q_1, q_2 \in \mathbb{Z}[b]$ with

$$U_1 m_0(1, b, c) + V_1 m_1(1, b, c) = E(b)\, q_1(b),$$
$$U_2 m_0(1, b, c) + V_2 m_3(1, b, c) = E(b)\, q_2(b),$$

the two right-hand sides being $\mathrm{Res}_c(m_0, m_1)$ and $\mathrm{Res}_c(m_0, m_3)$, each of degree 16 in b; together with a Bézout identity $c_1 q_1 + c_2 q_2 = d$ in $\mathbb{Z}[b]$ with d a nonzero integer.

(C3) *Shape.* Explicit $U, V \in \mathbb{Z}[b, c]$ and $A, B \in \mathbb{Z}[b]$ with

$$U\, m_0(1, b, c) + V\, m_1(1, b, c) = A(b)\, c + B(b),$$

together with $\alpha A + \beta E = d'$ in $\mathbb{Z}[b]$, d' a nonzero integer.

(C4) *Bridge.* Explicit $H^{(k)} \in \mathbb{Z}[z]$ with $E \cdot H^{(k)} = D^4 \cdot m_k\big(1, z, C_e/D\big)$ in $\mathbb{Z}[z]$ for $k = 0, \ldots, 4$; the right-hand sides have degree 40 and the cofactors degree 30.

Lemma 8. *Suppose M satisfies* (C1)*. Then no common secant of C_0 and $C_1 = M \cdot C_0$ meets C_0 at its point $[0 : 0 : 0 : 1]$ of parameter ∞; equivalently $W(M)$ misses the line $a = 0$ of the chord plane. Consequently $W(M)$ is contained in the affine chart $a = 1$. This holds for each of the nine configurations of Table* 1*.*

Proof. Let $(0 : b : c) \in W(M)$. Then $z_0(b, c) = z_1(b, c) = 0$, so $d_3 c^7 = 0$ by the first identity of (C1), and $c = 0$ since $d_3 \neq 0$. Then $0 = z_0(b, 0) = \kappa b^4$ with $\kappa \neq 0$, so $b = 0$; but $(0 : 0 : 0)$ is not a point of $\mathbb{P}^2$. □

Proposition 9 (exactly ten)**.** *Let M be one of the nine matrices of Table* 1*, and let $x_0 < \cdots < x_9$ be the ten roots of E produced by the ten windows of Table* 2 *through Lemma* 5*. Then the pair C_0, $C_1 = M \cdot C_0$ has exactly ten common secant lines in $\mathbb{P}^3(\mathbb{C})$, namely the ten chords*

$$\ell\Big(1,\ x_j,\ \tfrac{C_e(x_j)}{D(x_j)}\Big), \qquad j = 0, \ldots, 9.$$

In particular the ten common secants exhibited in Theorems 12–17*,* 18*,* 19 *and* 22 *are all the common secants of their pair, and the count is unconditional.*

Proof. By Lemma 8 every point of $W(M)$ is of the form $(1 : b : c)$. Let $(1 : b : c)$ be such a point. From $m_0 = m_1 = 0$ and the first identity of (C2), $E(b)q_1(b) = 0$; from $m_0 = m_3 = 0$ and the second, $E(b)q_2(b) = 0$. If $E(b) \neq 0$ then $q_1(b) = q_2(b) = 0$, contradicting $c_1(b)q_1(b) + c_2(b)q_2(b) = d \neq 0$. Hence $E(b) = 0$. By (C3), $A(b)c + B(b) = 0$, and $A(b) \neq 0$ because $\alpha(b)A(b) + \beta(b)E(b) = d'$ with $E(b) = 0$ and $d' \neq 0$; so $c = -B(b)/A(b)$ is determined by b. The map $(1 : b : c) \mapsto b$ is therefore injective on $W(M)$ with image inside the zero set of E, and E has degree exactly 10 — its coefficient list has eleven entries and its last entry is nonzero — so $\#W(M) \leq 10$.

Conversely each x_j gives a point of $W(M)$. Lemma 5 supplies $E(x_j) = 0$ and $D(x_j) \neq 0$, so evaluating (C4) at $z = x_j$ and dividing by $D(x_j)^4 \neq 0$ gives $m_k\big(1, x_j, C_e(x_j)/D(x_j)\big) = 0$ for $k = 0, \ldots, 4$. The ten points so obtained are pairwise distinct because their b-coordinates $x_0 < \cdots < x_9$ are. Hence $\#W(M) = 10$ exactly, and $W(M)$ consists of the ten listed chords. The identification of those chords with the secants of Lemma 5 is (3): at $z = x_j$ the pair $(e_1, e_2) = (-x_j, C_e(x_j)/D(x_j))$ is exactly the pair of elementary symmetric functions of the two parameters of the secant on C_0, so the chord $\ell(1, x_j, C_e(x_j)/D(x_j))$ is that secant. □

Everything in the proof that is not a line of algebra is an exhaustive comparison of two integer coefficient lists. Per configuration the identities consume seventeen such facts: two resultant identities, one Bézout identity, one shape identity, one coprimality identity, one identity on the line $a = 0$, five bridge identities and six small non-vanishing and length facts. Over the nine configurations that is 153 facts, on top of the 396 of Section 4.

Corollary 10. *For each of the nine configurations, E is squarefree and all ten of its roots are real.*

Proof. E has degree 10 and the ten windows give ten distinct real roots. □

The earlier version of this paper offered Corollary 10 as *evidence* for the assumption that Proposition 9 now replaces; it is a consequence of the same certificates instead.

Remark 11 (prior art for the count). Proposition 9 strengthens the trust base of a computation of a kind that has been done before, and we set out the precedents. As recorded in Section 1.5, the proof of Proposition 2.2 of [1] carries out exactly this count for one explicit — and itself special — integer matrix, by a saturation in Macaulay2 [6] returning $\deg J = \deg \sqrt{J} = 40$. The model of Propositions 6 and 7 is classical: that the chords of a twisted cubic form a Veronese surface on the Klein quadric is the subject of Cossidente, Hirschfeld and Storme [9], and the description of the common secants of two twisted cubics as a rank locus with answer ten goes back to Reye and to Coble [11]: the correspondence between common secants and rank-two quadrics is [10, Lemma 4.3], which attributes it to Reye, and the count of ten rank-two quadrics in a general web is Porteous' formula, stated there in the paragraph before that lemma. Certifying that an explicit instance of a Schubert problem attains its generic count is established practice in numerical algebraic geometry: the abstract of Duff, Hein and Sottile [7] says of the approaches given there that they "may be used to certify individual solutions, reject nonsolutions, or certify that we have found all solutions"; see also Hauenstein, Hein and Sottile [8], and `alphaCertified` [5], the tool [1] itself uses, which is built for the same purpose. What is new here is that the count is decided by a finite integer certificate checked by a proof kernel rather than by a computer-algebra system's degree computation or a floating-point α-test, and that it is carried out for the nine pairs whose 3-tuples are at issue.

6. Eight new realizable triples

The nine configurations are recorded in Table 1. Each is an integer matrix $M = \operatorname{diag}(1, K, K^2, K^3)\, M_0$ with M_0 small, and $\det M_0 \neq 0$ is displayed so that invertibility — and hence the fact that $C_1 = M \cdot C_0$ is a twisted cubic — can be checked by inspection. Table 2 records the corresponding ten windows.

Theorem 12. *Let M be the matrix of configuration B of Table 1. Then C_0 and $M \cdot C_0$ are real twisted cubics having exactly ten common secant lines in $\mathbb{P}^3(\mathbb{C})$; all ten are real, four of them totally real and six partially real. Hence the 3-tuple $(4, 6, 0)$ is realizable.*

Theorem 13. *Let M be the matrix of configuration C of Table 1. Then C_0 and $M \cdot C_0$ have exactly ten common secant lines, all real: four totally real, five partially real and one minimally real. Hence $(4, 5, 1)$ is realizable.*

	(n_t, n_p, n_m)	K	$\det M_0$	M_0
A	$(7, 2, 1)$	50	91	$\begin{pmatrix} 4 & -5 & 0 & 5 \\ -4 & 2 & -1 & -2 \\ 5 & 5 & 3 & -4 \\ -5 & 0 & 1 & 6 \end{pmatrix}$
B	$(4, 6, 0)$	50	-63852	$\begin{pmatrix} 4 & 9 & -7 & -17 \\ 6 & -2 & -11 & 5 \\ -16 & 21 & 1 & -13 \\ -30 & 7 & -6 & 11 \end{pmatrix}$
C	$(4, 5, 1)$	200	153014	$\begin{pmatrix} 2 & -2 & 1 & -21 \\ -7 & 21 & -20 & -14 \\ 10 & -8 & -11 & -7 \\ -9 & -13 & 0 & 5 \end{pmatrix}$
D	$(4, 4, 2)$	1000	105780	$\begin{pmatrix} 3 & -4 & -1 & -19 \\ -7 & 17 & -18 & -15 \\ 10 & -8 & -10 & -6 \\ -11 & -9 & 1 & 5 \end{pmatrix}$
E	$(3, 5, 2)$	1000	108848	$\begin{pmatrix} 2 & -4 & 0 & -19 \\ -7 & 15 & -17 & -15 \\ 10 & -8 & -11 & -6 \\ -11 & -9 & 1 & 5 \end{pmatrix}$
F	$(3, 7, 0)$	1000	-82272	$\begin{pmatrix} 4 & 6 & -7 & -14 \\ 5 & -3 & -10 & 10 \\ -18 & 24 & 2 & -11 \\ -27 & 7 & -5 & 12 \end{pmatrix}$
G	$(2, 8, 0)$	1000	-79699	$\begin{pmatrix} 4 & 8 & -6 & -17 \\ 6 & -2 & -9 & 5 \\ -16 & 23 & 1 & -12 \\ -31 & 8 & -7 & 10 \end{pmatrix}$
H	$(0, 10, 0)$	500	-478	$\begin{pmatrix} 3 & 5 & 0 & 0 \\ 0 & 0 & -4 & 5 \\ -1 & 0 & -3 & -3 \\ 1 & -3 & -4 & 0 \end{pmatrix}$
J	$(1, 9, 0)$	1000	-993	$\begin{pmatrix} 3 & 2 & -4 & 0 \\ 1 & 1 & -3 & 8 \\ 0 & 0 & -7 & -1 \\ 1 & -5 & -7 & 1 \end{pmatrix}$

TABLE 1. The nine configurations. The matrix of the pair is $M = \operatorname{diag}(1, K, K^2, K^3)\, M_0$ and the pair is C_0, $M \cdot C_0$. The last two rows realize the two cells of Remark 1.8 of [1] settled here.

A	$-2155^p, -955^t, -573^t, -268^t, -173^t, -97^p, -6^m, 118^t, 142^t, 161^t$
B	$-365^p, -227^t, -153^p, -138^p, -124^t, -33^p, 15^p, 33^t, 50^p, 85422^t$
C	$-3175^p, -488^p, -76^t, -11^m, 124^p, 147^p, 346^t, 385^t, 515^t, 1193^p$
D	$-20202^p, -1483^p, -200^t, 68^m, 460^m, 1150^t, 1330^p, 1550^t, 2247^t, 5478^p$
E	$-6702^p, -1541^p, -218^t, 35^m, 606^p, 1251^p, 1518^t, 1572^p, 1829^t, 4564^m$
F	$-25740^p, -9333^t, -2571^p, -2471^p, -1764^t, -283^p, 344^p, 1035^t, 1253^p, 1368^p$
G	$-10646^t, -2574^t, -2462^p, -2071^p, -549^p, 563^p, 607^p, 1882^p, 2157^p, 44478^p$
H	$-952^p, -572^p, -233^p, -206^p, -173^p, 219^p, 283^p, 421^p, 629^p, 1054^p$
J	$-4367^p, -1294^p, -834^p, -795^p, -373^p, 198^t, 251^p, 892^p, 1098^p, 17835^p$

TABLE 2. The ten windows of each configuration. An entry a^τ means that the window is $(a, a+1)$ and that the secant it produces is totally ($\tau = t$), partially ($\tau = p$) or minimally ($\tau = m$) real. The windows are listed in increasing order and are pairwise disjoint; the corresponding secant meets C_0 in the two points whose parameters have sum $e_1 \in (-a-1, -a)$.

Theorem 14. *Let M be the matrix of configuration D of Table 1. Then C_0 and $M \cdot C_0$ have exactly ten common secant lines, all real: four totally real, four partially real and two minimally real. Hence $(4, 4, 2)$ is realizable.*

Theorem 15. *Let M be the matrix of configuration E of Table 1. Then C_0 and $M \cdot C_0$ have exactly ten common secant lines, all real: three totally real, five partially real and two minimally real. Hence $(3, 5, 2)$ is realizable.*

Theorem 16. *Let M be the matrix of configuration F of Table 1. Then C_0 and $M \cdot C_0$ have exactly ten common secant lines, all real: three totally real and seven partially real. Hence $(3, 7, 0)$ is realizable.*

Theorem 17. *Let M be the matrix of configuration G of Table 1. Then C_0 and $M \cdot C_0$ have exactly ten common secant lines, all real: two totally real and eight partially real. Hence $(2, 8, 0)$ is realizable.*

The two theorems that follow settle two cells of the open problem of Remark 1.8 of [1].

Theorem 18. *Let M be the matrix of configuration H of Table 1, that is $M = \operatorname{diag}(1, 500, 500^2, 500^3)\, M_0$ with*

$$M_0 = \begin{pmatrix} 3 & 5 & 0 & 0 \\ 0 & 0 & -4 & 5 \\ -1 & 0 & -3 & -3 \\ 1 & -3 & -4 & 0 \end{pmatrix}, \qquad \det M_0 = -478.$$

Then C_0 and $M \cdot C_0$ are real twisted cubics having exactly ten common secant lines in $\mathbb{P}^3(\mathbb{C})$; all ten are real and all ten are partially real. Hence the 3-tuple $(0, 10, 0)$ is realizable.

Theorem 19. *Let M be the matrix of configuration J of Table 1, that is $M = \operatorname{diag}(1, 1000, 1000^2, 1000^3)\, M_0$ with*

$$M_0 = \begin{pmatrix} 3 & 2 & -4 & 0 \\ 1 & 1 & -3 & 8 \\ 0 & 0 & -7 & -1 \\ 1 & -5 & -7 & 1 \end{pmatrix}, \qquad \det M_0 = -993.$$

Then C_0 and $M \cdot C_0$ are real twisted cubics having exactly ten common secant lines in $\mathbb{P}^3(\mathbb{C})$; all ten are real, one of them totally real and nine partially real. Hence the 3-tuple $(1, 9, 0)$ is realizable.

Proof of Theorems 12–17 and 18–19. Identical in each case. Fix the configuration. Its matrix M is integral with $\det M = K^6 \det M_0 \neq 0$, so $C_1 = M \cdot C_0$ is a twisted cubic, real because M is real. The four identities (4) for M are checked coefficient by coefficient over $\mathbb{Z}$; this is one exhaustive comparison of two integer coefficient lists of degree at most 29 per identity, four in all. For each of the ten windows $(a, a+1)$ of Table 2 we then check four integer facts: the signs of $E(a)$ and $E(a+1)$, which are opposite, and the two $\operatorname{Pos}(a, a+1)$ certificates for $\varepsilon_t P_t$ and $\varepsilon_s P_s$, with $(\varepsilon_t, \varepsilon_s)$ determined by the type marked in Table 2: $(+,+)$ for t, $(-,-)$ for m, and $(+,-)$ or $(-,+)$ for p. Each such check is the sign pattern of a single integer coefficient list. Lemma 5 then gives, for each window, a common secant of the marked type meeting C_0 in the two points whose parameters sum to $e_1 \in (-a-1, -a)$. The ten windows are disjoint, so the ten values of e_1 are distinct, so the ten secants are pairwise distinct. Counting the marks in the row of Table 2 gives the stated numbers of totally, partially and minimally real secants. Proposition 9 then says that these ten are all the common secants of the pair, which consumes the seventeen further integer facts of Section 5, and the 3-tuple follows.

There is no case analysis and no search inside the proof: the windows and the signs $\varepsilon_t, \varepsilon_s$ are data, and the whole finite check for one configuration is the 61 integer facts listed above. All 549 of them, over the nine configurations, are checked by the Lean 4 kernel; see Section 9. □

Remark 20. The certificates also show that none of the ninety secants is tangent to either curve and that none of the pairs of intersection points degenerates: on every window both P_t and P_s are certified of one strict sign, so both discriminants are nonzero, and each secant meets each of the two curves in two distinct points.

Remark 21. Eight of the 33 cells of Table 4 of [1] are settled by Theorems 12–17 and 18–19, leaving 25. Of those, six lie outside the two families of Remark 1.8 — $(1,0,7)$, $(1,1,6)$, $(2,6,2)$, $(2,7,1)$, $(3,6,1)$, $(4,0,6)$ — and 19 lie inside them, every one with $n_m \geq 1$. Two of the six, namely $(1,0,7)$ and $(1,1,6)$, have since been reached by the sampling recorded in the public data [2] of [1] (Section 1.5). So 23 of the 33 cells of the printed table are open in the sense of being neither settled here nor reached there.

7. The control configuration, and controls outside the development

Configuration A realizes $(7,2,1)$, which is already in the realized column of Table 4 of [1]. We state it because it is the control on the entire construction: the same code path, run on a cell whose realizability is known, must reproduce it, and does.

Theorem 22. *Let M be the matrix of configuration A of Table 1. Then C_0 and $M \cdot C_0$ have exactly ten common secant lines, all real: seven totally real, two partially real and one minimally real. The pair realizes $(7,2,1)$.*

Proof. Word for word the proof of Theorems 12–17, with the row A of Table 2. □

Six further controls were run outside the certified development and are reported here as computations, not as theorems.

First, three of the eleven examples of [1] that print an exact parameter matrix: its Example 2.4 and its Examples 4.9 and 4.10, whose 3-tuples are stated there as $(10,0,0)$, $(1,0,3)$ and $(0,0,0)$. Taking each printed matrix as an exact rational one and rebuilding from the definitions of Section 2, the elimination returns a degree-10 polynomial and those same three 3-tuples.

Second, the distribution of $n_{\mathbb{R}}$ over 20 000 uniformly sampled integer matrices with entries in $[-12,12]$, exactly classified, is

$$n_{\mathbb{R}} = 0,2,4,6,8,10: \qquad 2185,\ 11367,\ 5378,\ 918,\ 100,\ 9$$
$$(10.9\%,\ 56.8\%,\ 26.9\%,\ 4.6\%,\ 0.50\%,\ 0.045\% \text{ of the } 20\,000),$$

the remaining 43 samples being matrices on which the elimination degenerates and which were discarded. Figure 5 of [1] reports the distribution of the same quantity over 100 000 parameter points sampled uniformly in $\mathbb{P}^{15}(\mathbb{R})$, with bin counts 10 620, 55 741, 27 979, 4 940, 660, 60, that is 10.6%, 55.7%, 28.0%, 4.9%, 0.66%, 0.060%. The two agree bin by bin to about one percentage point. They are not the same experiment — integer matrices with bounded entries are not uniform in $\mathbb{P}^{15}(\mathbb{R})$ — so the agreement is a coarse consistency check on the classifier and not a statistical test.

Third, a geometric check bypassing the whole Plücker route, run for the seven configurations $A, \dots, G$: for each of their seventy secants, the four points $v(t_1)$, $v(t_2)$, $Mv(s_1)$, $Mv(s_2)$ of $\mathbb{P}^3(\mathbb{C})$ were formed in floating point at a working precision of 77 significant decimal digits from (3), and all sixteen 3×3 minors of the 4×4 matrix they span were evaluated after normalising each point by its largest coordinate. Every one of those 1120 minors vanished to the working precision, the largest in modulus being below $3 \cdot 10^{-77}$; and counting the reality of the four points directly returns the 3-tuples of Table 1. The Plücker route and the four-point definition of a common secant therefore agree.

Fourth, a control on the count of Section 5 against the classical count. On 200 random integer 4×4 matrices with entries in $[-9, 9]$ and nonzero determinant, the greatest common divisor of the pairwise resultants of the five minors $m_k(1, b, c)$ with respect to c has degree exactly 10 in all 200 cases, and the corresponding computation on the line $a = 0$ returns a nonzero constant in all 200. So the chord model returns the classical answer of ten for a general pair before it is applied to nine special ones.

Fifth, the same computation run on the explicit matrix of Proposition 2.2 of [1] — a special pair, since it puts a point of C_0 on C_1 — returns a nonzero constant on $a = 0$ and a squarefree resultant greatest common divisor of degree exactly 10; that is the source's own count of ten common secants for its own matrix, recovered from the model of Section 5.

Sixth, two checks internal to the machinery. For each of the nine configurations the primitive greatest common divisor of the ten pairwise resultants of the five minors equals the eliminant E of Section 3 exactly; the elimination of Section 3 and the resultant computation in the chord plane are different computations and agree. And a third, wholly separate instrument agrees with both: writing $\mathbb{P}^3$ as the space of binary cubics, so that C_0 is the locus of perfect cubes and is cut out by the vanishing of the Hessian, the common secants of the pair appear as the ten nodes of a rational sextic curve in the chord plane, with the two types read off the discriminants of the corresponding binary quadratics. That model uses no Plücker coordinate and no rank condition, and it returns the same refined counts as Table 2 for all nine configurations.

8. The search, what it did not reach, and what cannot obstruct it

Nothing in this section is a theorem; it is a report on how the matrices of Table 1 were found, on where the same instrument stopped, and on one route to an obstruction that we can rule out.

8.1. **The searches.** Uniform sampling does not reach these cells. In the 20 000-sample experiment above, $n_{\mathbb{R}} = 10$ occurred nine times, and every uniformly sampled pair with $n_{\mathbb{R}} = 10$ that we recorded had $n_t \geq 7$. What worked is a directed walk across the walls of the classification. Across a codimension-one wall a real intersection pair collides and becomes a conjugate pair, so (n_t, n_p, n_m) can only change by $(-1, +1, 0)$ or $(0, -1, +1)$, or by ± 2 in one coordinate when a pair of real secants becomes nonreal. Each proposal costs one exact classification, 1.5 to 3.5 milliseconds in PARI/GP.

The six configurations $B, \dots, G$ were found by annealing on the distance to a target 3-tuple,

$$d(v,u) = |v_t - u_t| + |v_m - u_m| + 3\,|n_{\mathbb{R}}(v) - n_{\mathbb{R}}(u)|,$$

with perturbations of one to three matrix entries by a few parts per thousand of $\max|M_{ij}|$ per step. The run that produced them consisted of fourteen chases of 20 000 to 60 000 proposals, each started from a random matrix already at the target $n_{\mathbb{R}}$; it returned twelve hits on eight distinct previously unrealized 3-tuples. An earlier round of eight chases started from the parameter point of the $(10,0,0)$ example of [1], with coarser steps, descended only to $n_t = 5$ and produced nothing — a fact about the search, not about the geometry. Six further chases of 200 000 proposals in total, seeded at the new low-n_t hits and aimed at $(0,10,0)$, $(1,9,0)$, $(0,9,1)$, $(1,0,9)$, $(0,0,10)$ and $(4,0,6)$, re-found the same eight cells and no new one. In all, roughly 470 000 pairs were exactly classified and no configuration with $n_{\mathbb{R}} = 10$ and $n_t \leq 1$ was seen.

That negative result was an artefact of the objective, not of the geometry, as configurations H and J show. What reaches them is a *constrained* objective: maximise the number of real common secants subject to a penalty on the totally real ones,

$$\operatorname{score}(M) = n_{\mathbb{R}}(M) - 3\max\bigl(0,\ n_t(M) - c\bigr), \qquad c \in \{0,1\}.$$

Such a walk never has to travel through intermediate 3-tuples; it climbs $n_{\mathbb{R}}$ while the penalty holds n_t down, and so spends its whole life in the thin region rather than trying to enter it from outside. Ten independent annealed walks of 6 000 proposals each — about 60 000 proposals, some three CPU-minutes — produced a pair with 3-tuple $(0,10,0)$ on the sixth walk. A greedy coordinate descent that decreases $\max|M_{ij}|$ subject to preserving the refined counts then took that witness from largest entry 19 to largest entry 5, which is the matrix M_0 of configuration H; the same step improved the worst discriminant margin of its ten secants from 0.42 to 0.69 and brought the largest coefficient of the cleared equations N_i down from 154 to 142 decimal digits. Configuration J was found the same way, and two further integer matrices with 3-tuple $(1,9,0)$ were found and cross-checked but are not reported here.

8.2. **What the search did not reach.** The two families of Remark 1.8 of [1] contain 21 cells, of which Theorems 18 and 19 settle the two with $n_m = 0$. Every one of the 19 that remain has $n_m \geq 1$, and that is exactly where the instrument stops. About $1.5 \cdot 10^6$ further pairs were exactly classified without reaching any of them. The main searches were: 21 target chases, one per cell, of six walks of 9 000 proposals each, all of which converged to the $n_m = 0$ end of their family and stopped there; two "ratchet" searches from the witness of Theorem 18, holding $n_{\mathbb{R}} = 10$ and $n_t \leq c$ for $c = 0$ and $c = 1$ while maximising n_m, of 25 rounds of 3 000 proposals each, both of which returned $\max n_m = 0$; and an approach from the other side, fixing n_m first with n_t free, which reaches $n_{\mathbb{R}} = 10$ with $n_m \in \{1,2,3,4,7\}$ easily but then stalls at $n_t = 7,6,6,6,2$ respectively. As before, "not found" is not "not realizable", and we draw no conclusion from it; we record it because the next attempt on those cells should know where this instrument stops. The sharpened form of the question left open by Table 4 is therefore: *is there a pair of real twisted cubics with $n_{\mathbb{R}} = 10$, $n_t \leq 1$ and $n_m \geq 1$?*

Remark 23 (two cells measured, not proved). Two further cells, $(1,1,6)$ and $(1,0,7)$, both with $n_{\mathbb{R}} = 8$, were reached by the search of the earlier version of this paper, by the matrices

$$\begin{pmatrix} 5 & 0 & -11 & 6 \\ 10 & 8 & -1 & -11 \\ 6 & 13 & 11 & 10 \\ -17 & -5 & -11 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 12 & 2 & 10 & -6 \\ 7 & -10 & 2 & -12 \\ -2 & -9 & 7 & 4 \\ -8 & -2 & 3 & -10 \end{pmatrix}$$

respectively. For these two the count of real common secants is exact — it is a real root count of the integer eliminant — but the three-way split into types was made from floating-point discriminant signs without Descartes certificates, and an $n_{\mathbb{R}} = 8$ claim additionally needs a certified bound on the number of real roots of a degree-10 polynomial, which the certificates used here do not provide (they use only the free bound that ten sign changes of a degree-10 polynomial account for all of its real roots). We therefore record $(1,1,6)$ and $(1,0,7)$ as measured, not proved. As Section 1.5 records, these are also the two cells that the source's own public data [2] has since reached by sampling, so what is missing for them is not a witness but an exact certificate.

8.3. **No topological obstruction can explain the difficulty.** The two real curves $C_0(\mathbb{R})$ and $C_1(\mathbb{R})$, when disjoint, form an ordered two-component link of smooth circles in $\mathbb{RP}^3$, and it is natural to ask whether the topology of that link constrains (n_t, n_p, n_m) — which would be an explanation for the resistance of the cells with small n_t. It does not. The next two remarks record why; both are elementary, are stated outside the formal development, and are included because they close a route rather than open one.

Remark 24 (the linking number is a constant). A real curve of odd degree meets a real plane in an odd number of real points, nonreal intersections coming in conjugate pairs, so a real twisted cubic represents the nonzero class g of $H_1(\mathbb{RP}^3; \mathbb{Z}/2) = \mathbb{Z}/2$. Neither curve is null-homologous, so no integral linking number exists; and $\mathbb{RP}^3 = L(2,1)$ is a rational homology sphere, where the linking number of two disjoint curves in classes x, y is the value $\lambda(x,y) \in \mathbb{Q}/\mathbb{Z}$ of the linking form, a purely homological quantity. For $L(2,1)$ one has $\lambda(g,g) = 1/2$. Hence the linking number of $C_0(\mathbb{R})$ and $C_1(\mathbb{R})$ equals $1/2$ for *every* disjoint pair of real twisted cubics, and carries no information about the 3-tuple. The signed intersection count in the Grassmannian dies with it: the common secants are the intersection of the two chordal surfaces, embedded copies of $\mathbb{RP}^2$, inside $X = G(2,4)(\mathbb{R}) \cong (S^2 \times S^2)/\pm$, whose deck transformation $(x,y) \mapsto (-x,-y)$ is orientation-preserving but acts by -1 on the class of each factor, so $H_2(X;\mathbb{Q}) = H_2(S^2 \times S^2; \mathbb{Q})^{\mathbb{Z}/2} = 0$ and no $\mathbb{Z}$- or $\mathbb{Q}$-valued signed count is available at all. The mod-2 intersection number returns only the constraint already known, that $n_{\mathbb{R}}$ is even. None of this is new mathematics; it is recorded because it removes a candidate mechanism for all 161 cells at once.

Remark 25 (an explicit segment on which the link is constant and the 3-tuple is not). Sharper, and certified over $\mathbb{Q}$. Let M_A be the matrix M_0 of configuration H, let M_B be M_A with the entry in position $(2,1)$ changed from 0 to 3, and put $M(\theta) = (1-\theta)M_A + \theta M_B$ for $\theta \in [0,1]$. All of the following was computed exactly over $\mathbb{Q}$ in PARI/GP. The determinant is linear in θ with $\det M(0) = -478$ and $\det M(1) = -298$ and no root in $[0,1]$, so every $M(\theta)$ is invertible and $C_1(\theta) = M(\theta) \cdot C_0$ is a smooth rational normal cubic. The two curves stay disjoint: identifying $\mathbb{P}^3$ with the space of binary cubics by $x \mapsto x_0u^3 + 3x_1u^2w + 3x_2uw^2 + x_3w^3$, the curve C_0 is the locus of perfect cubes, that is, exactly the locus where the Hessian vanishes identically; so a value of θ at which the curves meet at a finite parameter is a root of $\mathrm{Res}_s(h_1, h_2)$ for every pair of linear combinations of the three binary sextics $\gamma_i(s) = \mathrm{Hess}\big(M(\theta)v(s)\big)_i$. The greatest common divisor G of those resultants over twelve random integer combinations has degree 9, is squarefree, has $G(0) = -5922380792837$ and $G(1) = -5123411311766$, and a Sturm count returns no root in $[0, 1]$ — indeed none in the widened interval $[-1/10, 11/10]$ — while the greatest common divisor of the three leading coefficients has no root in $[0, 1]$, which rules out a meeting at $s = \infty$. Hence $\theta \mapsto \big(C_0(\mathbb{R}), C_1(\theta)(\mathbb{R})\big)$ is a smooth family of ordered two-component links of disjoint embedded circles in $\mathbb{RP}^3$, so by isotopy extension its ambient isotopy class is constant on $[0, 1]$. The 3-tuple is not: computed exactly at rational values of θ, it runs

$$(0, 10, 0), \quad (0, 8, 0), \quad (0, 6, 0), \quad (0, 4, 0), \quad (1, 3, 0)$$

as θ goes from 0 to 1. One isotopy class of links therefore carries five different cells, including one with $n_\mathbb{R} = 10$ and $n_t = 0$ and one with $n_t = 1$, so the 3-tuple is not a function of the isotopy class and no invariant of that class — linking numbers of lifts, knot types of the components, anything — can be read off the 3-tuple or used to constrain it. A second segment, obtained by changing instead the entry in position $(4, 1)$ from 1 to -2, does the same with a degree-6 greatest common divisor and endpoint 3-tuple $(1, 0, 1)$. As a check on the instrument, the same greatest common divisor has degree exactly 18 on three random integer segments, which is the Thom–Porteous degree of the hypersurface $\{M : C_0 \cap M \cdot C_0 \neq \emptyset\}$ in $\mathbb{P}^{15}$.

What this refutes is precise: any obstruction argument that factors through the 3-tuple and an isotopy invariant. It does not prove the stronger sentence "no isotopy invariant can rule out any cell", which would additionally require knowing which link types are realized by disjoint real pairs; for the linking number itself, Remark 24 gives the stronger statement by a different route.

9. Verification

Everything in Sections 4–6 except the four pieces of linear algebra named in Section 1.6 is formalised in Lean 4 [13] against Mathlib [14] and checked by the Lean kernel: the common-secant system in the form (2), the three types as the sign conditions of Proposition 3, Lemmas 4 and 5, the chord-plane minors of Section 5 as they are computed from the matrix, Lemma 8, Proposition 9, and, for each of the nine configurations, its ten applications of Lemma 5 and the resulting statement that there are exactly ten common secants with the listed types. That is the whole content of Theorems 12–17, 18, 19 and 22. What is not formalised is listed in Section 1.6, together with Remark 2, the six controls of Section 7 and the whole of Section 8, which report computations made in PARI/GP [15].

The toolchain is `leanprover/lean4:v4.32.0` and Mathlib is taken at revision `81a5d257c8e410db227a6665ed08f64fea08e997`. The finite data — for each configuration its sixteen matrix entries, the nine integer coefficient lists E, D, C_e, A_1, A_2, $H_1, \ldots, H_4$, and the cofactors of the identities (C1)–(C4) — and the 549 finite checks over them (61 per configuration: four identities, twenty integer evaluations, twenty Pos certificates, and seventeen identities and non-vanishing facts for the count) live in modules that import nothing beyond the list arithmetic, so every product, Möbius transform and sign sweep is decided by the kernel on core integer and list operations; the real content of Sections 2–5 is supplied separately over Mathlib. In Lean's terms every decision procedure used is `decide +kernel`, and *no use of* `native_decide` *occurs anywhere*, so neither a compiler nor a runtime is part of the trusted base; there is no `sorry`, and every formalised statement depends only on `propext`, `Classical.choice` and `Quot.sound`, the three axioms of Lean's standard foundation. The development is about 5 260 lines and checks in well under one processor-hour in all; the peak resident set of the import-free certificate modules ranges from about 1.0 to 5.9 GB, and the Mathlib-facing modules sit at Mathlib's own baseline of roughly 6.6 GB.

The development also carries fourteen negative controls, formalised alongside the theorems to guard against statements that are true for the wrong reason. Four of them are carried over from the earlier version of this paper: that the common-secant system is not satisfied by an arbitrary quadruple; that the three types are pairwise exclusive; that configuration B does not realize $(10, 0, 0)$, one of its secants being partially real; and that it does not realize $(0, 10, 0)$, another of its secants being totally real.

Four more guard the count on configuration A: that nine is not an upper bound for the number of common secants; that eleven is not the number; that the pair does have a common secant, so the count is not a statement about the empty set; and that the tangent line to C_0 at $t = 0$, which is the chord $(1 : 0 : 0)$, is not a common secant, so the rank condition of Proposition 7 is not vacuous.

Six more negative controls guard the two new configurations: the same too-small and too-large refutations for configuration H; the non-vacuity of its system and of its rank condition; the fact that H does not realize $(10, 0, 0)$, one of its secants being partially real and hence not totally real; and the fact that configuration J has a totally real common secant and therefore does not realize $(0, 10, 0)$, so that the two new cells are genuinely different.

Two independent checks stand outside the formalisation. All 308 integer certificates of the first seven configurations and the 3-tuples they produce were recomputed in a second, unrelated implementation, which agreed in every case. And the chord-model data of Section 5 is not entered by hand at all: the minors are recomputed inside the formal development from the sixteen matrix entries by the route $M \mapsto \operatorname{adj} M \mapsto \Lambda^2(\operatorname{adj} M) \mapsto N \mapsto m_k$, so a disagreement between the generator and the formal development can only make a check fail, never make one succeed wrongly. The repository is private at the time of writing; repository reference to be added at submission.

Acknowledgement of provenance

The problem, the classification of real common secants into totally, partially and minimally real, the count of 161 admissible 3-tuples, the realization of 128 of them and the list of the remaining 33 are due to Aslam, Faust, Hauenstein, Lopez Garcia, Kagy, Regan, Wampler and Zhang [1], as is the numerical work behind them; the count of ten common secants for a general pair is classical [3], as is the chord-plane model of Section 5 and the description of the common secants as a rank locus with answer ten (Remark 11), and the count of ten at an explicit special pair was first carried out — by a different method, and for a different matrix — in the proof of Proposition 2.2 of [1]. What is offered here is the realization of eight of the 33 cells,

two of them in the open problem of Remark 1.8 of [1]; the exact certificate that replaces the numerical one, including the exact count for all nine explicit pairs; the erratum of Remark 2; the report of Section 8 and the two negative remarks that close the topological route; and the two corrections of Section 1.5 to the earlier version of this paper. We have not touched the monodromy theorem of [1], which is a certified path-tracking computation and not a finite check of this kind.

References

[1] S. Aslam, M. Faust, J. D. Hauenstein, J. Lopez Garcia, B. Kagy, M. H. Regan, C. W. Wampler and A. Zhang, *Common Real Secants to Pairs of Real Twisted Cubic Curves*, arXiv:2603.25003v1 (26 March 2026), 20 pages. As of 3 September 2026 there is no later version, no journal reference and no DOI other than the arXiv one, so all theorem, proposition, remark, problem, figure, equation and table references above are to `arXiv:2603.25003v1`.

[2] *Secants of two cubics mrc2025*, 2025–2026. The software and data repository listed as reference [1] of `arXiv:2603.25003v1` — printed there without an author line and in exactly this form — at `https://github.com/mattfaust/SecantsofTwoCubicsMRC2025`. It is where that paper says its code, sample parameter points and certified solution data live. Consulted at its commit `82e60c8` of 27 April 2026, which was still its last commit on 3 September 2026; the files used above are its running tally of 3-tuples produced by the group's sampling, the accompanying list of 3-tuples ever produced, and the `Bertini` input file printed in its README.

[3] D. Eisenbud and J. Harris, *3264 and All That: A Second Course in Algebraic Geometry*, Cambridge University Press, Cambridge, 2016. (Chapter 3, Keynote Question (c) and Proposition 3.14.)

[4] D. J. Bates, J. D. Hauenstein, A. J. Sommese and C. W. Wampler, *Bertini: Software for Numerical Algebraic Geometry*, `https://bertini.nd.edu`.

[5] J. D. Hauenstein and F. Sottile, *Algorithm 921: alphaCertified: certifying solutions to polynomial systems*, ACM Trans. Math. Software **38** (2012), no. 4, Art. 28, 20 pp. This article, and not only the software, is what [1] cites as its reference [11].

[6] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www2.macaulay2.com`. The system used for the saturation computation in the proof of Proposition 2.2 of [1].

[7] T. Duff, N. Hein and F. Sottile, *Certification for polynomial systems via square subsystems*, J. Symbolic Comput. **109** (2022), 367–385; arXiv:1812.02851.

[8] J. D. Hauenstein, N. Hein and F. Sottile, *A primal-dual formulation for certifiable computations in Schubert calculus*, Found. Comput. Math. **16** (2016), no. 4, 941–963.

[9] A. Cossidente, J. W. P. Hirschfeld and L. Storme, *Applications of line geometry, III: The quadric Veronesean and the chords of a twisted cubic*, Australas. J. Combin. **16** (1997), 99–111. The correspondence is set up there over a finite field, for a twisted cubic in $\mathrm{PG}(3, q)$; Proposition 6 above is the same map over $\mathbb{C}$.

[10] K. Hulek and K. Ranestad, *Abelian surfaces with two plane cubic curve fibrations and Calabi–Yau threefolds*, arXiv:math/9810120; published in: Complex Analysis and Algebraic Geometry (T. Peternell and F.-O. Schreyer, eds.), de Gruyter, Berlin, 2000, 275–316. Lemma 4.3 there, attributed to Reye, is the correspondence between the common secant lines of two twisted cubics and the rank-two quadrics of the associated web; that a general web of rank-four quadrics carries ten rank-two quadrics, by Porteous' formula, is stated in the paragraph immediately preceding it, and the proof of the lemma recovers the same ten from the bidegree $(1, 3)$ of the congruence of secant lines of a twisted cubic.

[11] A. B. Coble, *Algebraic Geometry and Theta Functions*, American Mathematical Society Colloquium Publications **10**, American Mathematical Society, Providence, 1929. The same count of ten appears here. The attribution, and the page, are those of I. Dolgachev, *Quartic surfaces with icosahedral symmetry*, Adv. Geom. **18** (2018), no. 1, 119–132 (arXiv:1604.02567v3), whose remark on the nodes of the symmetric discriminant quartic reads "According to A. Coble [...], p. 250, they are the ten common secants of the rational normal curves C_1 and C_2". We have not checked the page against a copy of Coble's volume.

[12] J. Chen, A. Edelman and J. Urschel, *The largest 5th pivot may be the root of a 61st degree polynomial*, arXiv:2602.20390v2 (15 March 2026). The subject of the companion development

from which the certificate layer of Section 4 is taken unchanged; that layer is not a contribution of the present paper.

[13] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE-28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635.

[14] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381.

[15] The PARI Group, *PARI/GP*, version 2.13.3, Univ. Bordeaux, 2021. Used for the elimination of Section 3, for the construction of the cofactors of Section 5, for the search of Section 8 and for the independent checks reported in Section 7. Available at `https://pari.math.u-bordeaux.fr/`.