THE CRITICAL POLYNOMIAL OF A UNIFORM-GRID VARIANCE-DEFICIT OBJECTIVE: ONE SIGN CHANGE FOR $3 \le T \le 300$, AND A CLOSED FORM AT $x = 1$

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ABSTRACT. For an integer step count $T \ge 1$ put $b_0(x) = T$, $b_j(x) = jx + (T - j)$ for $1 \le j \le T$, and

$$H_T(x) \;=\; T^2 x \sum_{i=1}^{T} \frac{1}{i\, b_i(x)\, b_{i-1}(x)^2}, \qquad x > 0.$$

This is the finite-step variance deficit of a Schrödinger-bridge sampler on the uniform grid, in the form derived by Fallah and Yang, who prove that H_T has a unique positive critical point for $2 \le T \le 120$ by exhibiting an integer polynomial with exactly one sign change in its nonzero coefficient sequence, report a numerical scan finding one local minimum for $T \le 200$ and for $T \in \{500, 1000, 5000\}$, and leave the general case open, reducing it to the conjecture that the coefficient sequence has one sign change for every T. Their polynomial is not printed, and the sign-variation count is a property of the polynomial chosen and not of the critical set, so we construct one: an explicit $R_T \in \mathbb{Z}[x]$, of degree $3T - 3$ for $T \ge 2$, given by a quadratic-time integer recurrence, for which we prove — for every $T \ge 1$ and every $x > 0$, with no restriction on T — that $H_T'(x) = 0$ if and only if $R_T(x) = 0$. We then prove that the nonzero coefficient sequence of R_T has exactly one sign change for every $3 \le T \le 300$; that R_T consequently has exactly one positive real root there; and that H_T has exactly one positive critical point for $3 \le T \le 300$, which is the source's uniqueness statement on 2.5 times its proved range and 1.5 times its numerical scan. Two further results are uniform in T and involve no computation: $H_T'(1) = \big((T+2)h_T - 3T\big)/T^2$ for every $T \ge 1$, where h_T is the T-th harmonic number, and $H_T'(1) > 0$ for every $T \ge 3$, so that $T = 2$ — where $R_2 = 16X^2(X - 1)$ and the critical point is exactly $x = 1$ — is the only step count $T \ge 2$ at which $x = 1$ is critical. The band results rest on a single exhaustive computation in exact integer arithmetic over the 298 step counts $T = 3, \dots, 300$, whose largest instance has degree 897 and coefficients of up to 9436 bits. Every theorem, proposition and lemma below is machine-checked in Lean 4; Section 7 records exactly what the kernel checks, and which of the remaining statements are computations outside the formal development.

1. INTRODUCTION

1.1. **The objective.** Fallah and Yang [1] design the reference process of a Schrödinger-bridge model by minimising a finite-step variance deficit. On a grid $0 = \rho_0 < \rho_1 < \cdots < \rho_T = 1$ with T solver steps, the deficit is a function of a single dimensionless colour parameter $x = v/P > 0$; restricted to the *uniform* grid $\rho_i = i/T$, it reduces to the one-variable rational function that their appendix writes as

$$H(x) = \frac{x\,\mathcal{S}(x)}{T}, \qquad \mathcal{S}(x) = \frac{1}{T}\sum_{i=1}^{T} \frac{1}{\rho_i\, \psi_i\, \psi_{i-1}^2}, \qquad \psi_i = \psi(i/T) = 1 - b\tfrac{i}{T}, \quad b = 1 - x. \tag{1}$$

Writing $\psi_j = b_j/T$ with

$$b_0(x) = T, \qquad b_j(x) = jx + (T - j) \quad (1 \le j \le T), \tag{2}$$

one has $\rho_i \psi_i \psi_{i-1}^2 = i\, b_i b_{i-1}^2/T^4$, and (1) becomes the form we use throughout:

$$H_T(x) \;=\; T^2 x \sum_{i=1}^{T} \frac{1}{i\, b_i(x)\, b_{i-1}(x)^2}, \qquad x > 0. \tag{3}$$

The quantity actually minimised in [1] is a Kullback–Leibler functional $\Phi = \frac{1}{2}(u - 1 - \ln u)$ with $u = 1 - H_T(x) \in (0, 1)$, which is strictly decreasing in u; minimising Φ is therefore the same as minimising H_T, and this reduction is the source's own. The objective (3) tends to 1 at both ends of

$(0,\infty)$: as $x \to 0^+$ the term $i = T$ contributes $T^2x/(T \cdot Tx \cdot b_{T-1}^2) \to 1$ and every other term is $O(x)$, while as $x \to \infty$ the term $i = 1$ contributes $T^2x/(1 \cdot b_1 \cdot T^2) \to 1$ and every other term is $O(x^{-2})$. Since $\Phi \to \infty$ as $H_T \to 1$, the objective has an interior minimum, and the question is whether it has only one. (These two limits are elementary and are stated here only for orientation; nothing below uses them.)

1.2. **The source's proposition, and what it leaves open.** Proposition `prop:uniqueness` of [1] reads, verbatim — the one elision being an internal cross-reference to the source's own appendix:

> "At $T = 2$, Φ is strictly unimodal. On uniform grids, uniqueness is proved for $2 \le T \le 120$ and numerically certified for $T \le 200$ and $T \in \{500, 1000, 5000\}$ […]; extending the proof to general T remains an open problem."

The proof for $2 \le T \le 120$ is a Descartes argument. In the appendix's own words:

> "For each integer $2 \le T \le 120$ we derives an explicit polynomial $P_T \in \mathbb{Z}[x]$ whose positive roots are exactly the critical points of H. The sequence of nonzero coefficients of P_T has exactly one sign change, so by Descartes' rule (S5) P_T has at most one positive root [2, Sec. 2.2.1]."

and then:

> "Beyond that certified range, a tolerance-robust numerical scan finds exactly one local minimum for every $T = 2, \dots, 200$ and for $T \in \{500, 1000, 5000\}$. A general analytic proof remains open. It reduces it either to the conjecture that the relevant coefficient sequence has one sign change for every T, or to verifying $\mathrm{Var}_p(z) < 1/12$ at each critical point, where $1/12$ is the limiting value in the continuum."

Two features of that passage set the problem addressed here. First, the open question is stated as a conjecture about *the coefficient sequence of a polynomial*, and the polynomial is never printed, nor is its construction described. Second — and this is why the omission matters — the number of sign variations is a property of the particular polynomial chosen, not of the critical set: two polynomials with the same positive roots can have different sign-variation counts, and multiplying a certificate by a fixed positive polynomial can change the count in either direction. A sign-change statement is therefore meaningless until a polynomial is fixed. So the first task is to build one, and to prove that its positive roots really are the positive critical points of H_T — for every T, not on a band.

1.3. **Results.** Section 2 constructs an explicit $R_T \in \mathbb{Z}[X]$ of degree $3T - 3$ by a quadratic-time integer recurrence (Definition 2.1), together with a companion $G_T \in \mathbb{Z}[X]$ that is strictly positive on $x > 0$. The first result is the bridge between R_T and the source's question, and it is uniform in T:

$$H_T'(x) = \frac{T^2\, R_T(x)}{G_T(x)\, b_T(x)^2} \quad (x > 0), \qquad \text{so} \qquad H_T'(x) = 0 \iff R_T(x) = 0 \quad (x > 0) \tag{4}$$

for every $T \ge 1$ (Lemma 2.2 and Proposition 2.3). This is ordinary calculus, but it is what makes a sign-change statement about R_T a statement about [1]'s objective.

The main results are then a certified band and two identities uniform in T. Write $\mathrm{v}(P)$ for the number of sign changes in the sequence of nonzero coefficients of a nonzero real polynomial P (Section 2.3).

- $\mathrm{v}(R_T) = 1$ for every $3 \le T \le 300$ (Theorem 3.1). This is the source's conjecture, for the polynomial constructed here, on 2.5 times the range it proves and 1.5 times the range it scans numerically.
- R_T has exactly one positive real root for every $3 \le T \le 300$ (Theorem 3.2), and that root lies in $(0, 1)$ (Corollary 3.3).
- H_T has exactly one positive critical point for every $3 \le T \le 300$ (Theorem 3.4) — the source's uniqueness proposition, on the extended range.
- $H_T'(1) = \big((T+2)h_T - 3T\big)/T^2$ for every $T \ge 1$, with $h_T = \sum_{k=1}^{T} 1/k$ (Theorem 5.1); and $H_T'(1) > 0$ for every $T \ge 3$ (Corollary 5.2). Neither is in [1], which gives for the minimiser only the asymptotic law $x^*(T) = (2 \ln T)^{-1/2}(1 + o(1))$ and a numerical table; both are exact and hold for all T at once.

- $T = 2$ is degenerate: $R_2 = 16X^2(X-1)$, and $x = 1$ is the unique positive critical point of H_2 (Proposition 4.1). Together with Corollary 5.2 this says that $T = 2$ is the *only* step count $T \geq 2$ at which the critical point sits at $x = 1$; [1] handles $T = 2$ by an elementary convexity argument, and we record it because it is exactly the case our finite certificate excludes.

Theorem 3.1 and its two consequences rest on a single exhaustive computation: the coefficient list of R_T is built in exact integer arithmetic for each of the 298 values $T = 3, 4, \ldots, 300$, and a six-part Boolean test is evaluated on it (Section 3). Nothing else in the paper is computational: the bridge (4), the harmonic identity and its corollary are proved for all T by hand-checkable arguments, and the degenerate case $T = 2$ needs one evaluation.

1.4. **What is not claimed.** We do not prove the conjecture of [1]. Theorem 3.1 is a band, not a theorem for all T; Section 6 records what the coefficient sequence looks like beyond it and which uniform statements are within reach, but none of that is proved here.

We do not identify our R_T with the source's P_T, which is not printed there. Our statements are about the explicit polynomial of Definition 2.1; they bear on [1] through Proposition 2.3, which ties that polynomial to the source's objective, and not through any claim about the source's own certificate.

We do not reprove the source's asymptotic law for the minimiser, its non-uniqueness results on general grids, or anything else in [1]; and although Corollary 3.3 places the critical point in $(0, 1)$ for $3 \leq T \leq 300$, we prove no bound on the minimiser valid for all T.

Finally, we have no proof that the objects here are new in the sense of appearing nowhere else. We searched for the objective, for the polynomial and for its coefficient sequences, and found only [1]; but a negative search is a negative search.

2. Preliminaries

2.1. **The derivative of H_T.** Fix $T \geq 1$ and let b_j be as in (2); note $b_j(x) > 0$ for every $0 \leq j \leq T$ and every $x > 0$, and $b_T(x) = Tx$. Differentiating a single term of (3) and clearing one factor of b_{i-1},

$$\frac{d}{dx}\left[\frac{x}{i\, b_i\, b_{i-1}^2}\right] = \frac{A_i(x)}{i\, b_i^2\, b_{i-1}^3},$$

where

$$A_i(x) \;=\; b_i b_{i-1} - x\big(i\, b_{i-1} + 2(i-1) b_i\big) \;=\; (T-i)(T-i+1) - (i-1)(T-i)\, x - 2i(i-1)\, x^2. \tag{5}$$

In particular $A_1 = T(T-1)$ is constant and $A_T(x) = -2T(T-1)x^2$. Summing,

$$H_T'(x) \;=\; T^2 \sum_{i=1}^{T} \frac{A_i(x)}{i\, b_i(x)^2\, b_{i-1}(x)^3}, \qquad x > 0. \tag{6}$$

2.2. **The polynomial.** The denominators in (6) are cleared by a single accumulation, which is what makes the construction quadratic in T rather than exponential.

Definition 2.1. For $T \geq 1$ define pairs $(N_k, G_k) \in \mathbb{Z}[X]^2$, $1 \leq k \leq T$, by

$$N_1 = A_1, \qquad G_1 = T^3,$$
$$N_{k+1} = (k+1)\, N_k\, b_k\, b_{k+1}^2 + A_{k+1}\, G_k, \qquad G_{k+1} = (k+1)\, G_k\, b_k^3,$$

and set $R_T := N_T$ and G_T as produced by the recurrence, so that

$$G_T = T!\, T^3 \prod_{j=1}^{T-1} b_j^3.$$

The recurrence is designed around the invariant

$$N_k = \Big(G_k\, b_k^2\Big) \sum_{i=1}^{k} \frac{A_i}{i\, b_i^2\, b_{i-1}^3} \qquad (1 \leq k \leq T,\ x > 0), \tag{7}$$

which is immediate at $k = 1$ and propagates by one line of algebra. Everything in Definition 2.1 is an integer polynomial, so $R_T \in \mathbb{Z}[X]$. For $T \geq 2$ its degree is $3T - 3$, since $\deg N_{k+1} = \deg N_k + 3$ while $\deg(A_{k+1}G_k) = 3k - 1 < 3k$, and its leading coefficient is positive, since that same comparison gives $\mathrm{lc}(N_{k+1}) = k(k+1)^3 \,\mathrm{lc}(N_k)$ with $\mathrm{lc}(N_1) = A_1 = T(T-1) > 0$. (At $T = 1$ the objective (3) is the constant 1 and $R_1 = 0$, so (4) holds with both sides identically 0; the case plays no further role.)

Lemma 2.2 (the derivative in closed form)**.** *For every $T \geq 1$ and every $x > 0$,*

$$G_T(x) > 0 \qquad \text{and} \qquad H_T'(x) = \frac{T^2\, R_T(x)}{G_T(x)\, b_T(x)^2}.$$

Proof. Positivity of G_T on $x > 0$ follows from $G_1 = T^3 > 0$ and $b_k(x) > 0$ by induction on the recurrence. The displayed identity is (6) combined with the invariant (7) at $k = T$, after dividing by $G_T(x)b_T(x)^2 \neq 0$. The invariant itself is proved by induction on k: at $k = 1$ both sides equal A_1, because $G_1 b_1^2$ cancels the denominator $1 \cdot b_1^2 b_0^3 = b_1^2 T^3$; and the inductive step is the identity

$$\Big(S_k + \frac{A_{k+1}}{(k+1)b_{k+1}^2 b_k^3}\Big) G_{k+1} b_{k+1}^2 = (k+1)\big(S_k G_k b_k^2\big) b_k b_{k+1}^2 + A_{k+1} G_k, \qquad S_k = \sum_{i \leq k} \frac{A_i}{i\, b_i^2 b_{i-1}^3},$$

which is $G_{k+1} = (k+1)G_k b_k^3$ substituted and the fractions cleared. □

Proposition 2.3 (the bridge)**.** *For every $T \geq 1$ and every $x > 0$: x is a critical point of H_T if and only if $R_T(x) = 0$.*

Proof. By Lemma 2.2, $H_T'(x)$ is $T^2 R_T(x)$ divided by the quantity $G_T(x)b_T(x)^2$, which is strictly positive for $x > 0$; and $T^2 \neq 0$. □

Proposition 2.3 is elementary, and we state it as a result only because [1] asserts the existence of such a polynomial without printing it or saying how it is built: it is the load-bearing link between any coefficient-sign statement and the source's actual question.

2.3. **Sign variations and Descartes' rule.** For a nonzero $P \in \mathbb{R}[X]$ let $\mathrm{v}(P)$ denote the number of sign changes in the sequence of its nonzero coefficients, read in any fixed order of the exponents (we read from the leading coefficient down; the count is the same either way). Descartes' rule of signs, in the form we use, is:

$$\#\{\text{positive real roots of } P, \text{ with multiplicity}\} \;\leq\; \mathrm{v}(P), \tag{8}$$

for every nonzero $P \in \mathbb{R}[X]$. This is Theorem 2.33 of [2, Sec. 2.2.1] — the reference, at the section number, that [1] itself gives for this step — where it appears in the sharper form that $\mathrm{v}(P) - \mathrm{pos}(P)$ is non-negative and even, $\mathrm{pos}(P)$ counting positive roots with multiplicity. Only the inequality is used below, and we use it in the machine-checked form of [3].

3. The certified band

3.1. **The finite test.** Write $R_T = \sum_{k \geq 0} r_k X^k$. For a given T the test evaluated is the conjunction of six conditions on the integer coefficient list of R_T:

$$\text{(i) } r_{3T-3} \neq 0; \quad \text{(ii) } \mathrm{v}(R_T) = 1; \quad \text{(iii) } r_0 = 0; \quad \text{(iv) } r_1 = 0; \quad \text{(v) } r_2 < 0; \quad \text{(vi) } R_T(1) > 0. \tag{9}$$

Conditions (i) and (ii) drive the Descartes half of the argument; (iii)–(vi) drive the existence half. Condition (vi) is exactly what fails at $T = 2$, where $R_2(1) = 0$: the test is not vacuous, and its failure at $T = 2$ is the degenerate geometry of Proposition 4.1 and not a defect of the certificate.

Theorem 3.1 (one sign change)**.** *For every integer T with $3 \leq T \leq 300$, the nonzero coefficient sequence of R_T has exactly one sign change:* $\mathrm{v}(R_T) = 1$.

Proof sketch. For each of the 298 integers $T = 3, 4, \ldots, 300$ the coefficient list of R_T is built from Definition 2.1 in exact integer arithmetic — $T - 1$ steps of the recurrence, each a convolution of integer lists — and the test (9) is evaluated on it. All 298 instances pass; in particular (ii) holds throughout. This is the whole of the exhaustive computation behind this section: no sampling, no floating point, no

search over anything but T. Its largest instance is $T = 300$, where R_T has degree 897 and its largest coefficient has 9436 bits. The computation is organised in three blocks, $T = 3, \dots, 120$ (the range [1] proves), $T = 121, \dots, 200$ (the range it scans numerically) and $T = 201, \dots, 300$ (beyond both), for the reason given in Section 7: each block is one invocation of compiled evaluation, and keeping them separate keeps the three ranges independently auditable. □

Theorem 3.2 (exactly one positive root)**.** *For every integer T with $3 \le T \le 300$, the polynomial R_T has exactly one positive real root.*

Proof sketch. At most one. By (i) of (9), $R_T \neq 0$; by Theorem 3.1, $\mathrm{v}(R_T) = 1$; so (8) bounds the number of positive roots, counted with multiplicity, by 1. Two distinct positive roots would give a set of two elements inside the multiset of positive roots, contradicting that bound.

At least one. By (iii) and (iv), $R_T = X^2 Q$ for a polynomial Q with $Q(0) = r_2 < 0$ by (v). Since Q is continuous, $Q < 0$ on some interval $(0, \delta)$, hence $R_T(x) = x^2 Q(x) < 0$ for $0 < x < \delta$; and $R_T(1) > 0$ by (vi). The intermediate value theorem on $[x_0, 1]$, for any $x_0 \in (0, \min(\delta, 1))$, produces a root in $(x_0, 1)$. □

Corollary 3.3. *For $3 \le T \le 300$ the unique positive root of R_T lies in $(0, 1)$.*

Proof. The existence half of the previous proof produces a root in $(0, 1)$, and the uniqueness half says there is no other. □

Theorem 3.4 (uniqueness of the critical point)**.** *For every integer T with $3 \le T \le 300$, the objective H_T has exactly one positive critical point.*

Proof. Immediate from Theorem 3.2 and Proposition 2.3: for $x > 0$ the two conditions $H_T'(x) = 0$ and $R_T(x) = 0$ are equivalent, so the two solution sets coincide, and one of them is a singleton. □

Theorem 3.4 is the uniqueness assertion of [1] for uniform grids, proved on $3 \le T \le 300$: 2.5 times the range proved there and 1.5 times the range of its numerical scan. The step counts $T \le 120$ reproduce the source's own certified range from an independently constructed polynomial.

4. The degenerate step count $T = 2$

Proposition 4.1 ($T = 2$)**.** *$R_2 = 16X^2(X - 1)$, and for $x > 0$ one has $H_2'(x) = 0$ if and only if $x = 1$. In particular H_2 has exactly one positive critical point, and it is $x = 1$.*

Proof. Running Definition 2.1 at $T = 2$ gives the coefficient list $(0, 0, -16, 16)$, i.e. $R_2 = 16X^3 - 16X^2 = 16X^2(X - 1)$; this is one evaluation of the recurrence. For $x > 0$, Proposition 2.3 turns $H_2'(x) = 0$ into $16x^2(x - 1) = 0$, and $x^2 \neq 0$ leaves $x = 1$. □

So $R_2(1) = 0$, which is why condition (vi) of (9) fails at $T = 2$ and why the band of Section 3 starts at $T = 3$. The failure is the geometry, not the method: at $T = 2$ the critical point sits exactly at the point $x = 1$ where the existence half of the certificate tests the sign. [1] proves the case $T = 2$ separately as well, by an elementary convexity argument: there $H(x) = z_1 + (1 - z_1)^2$ with $z_1 = x/(x + 1)$, and $z \mapsto z + (1 - z)^2$ is strictly convex, giving the minimiser $z_1 = \frac{1}{2}$, that is, $x = 1$.

5. Two identities uniform in T

The results of this section hold for every T, involve no computation, and are not in [1], which describes the minimiser only through the asymptotic law $x^*(T) = (2 \ln T)^{-1/2}(1 + o(1))$ and a numerical table of values. Let

$$h_T = \sum_{k=1}^{T} \frac{1}{k}$$

be the T-th harmonic number.

Theorem 5.1 (a closed form at $x = 1$)**.** *For every integer $T \ge 1$,*

$$H_T'(1) \;=\; \frac{(T + 2)\, h_T - 3T}{T^2}.$$

Proof. At $x = 1$ every grid factor collapses: $b_j(1) = j + (T - j) = T$ for all $0 \le j \le T$. The numerators collapse too, by a cancellation visible only after expanding (5):

$$A_i(1) = (T - i)(T - i + 1) - (i - 1)(T - i) - 2i(i - 1) = T\big(T + 2 - 3i\big).$$

Substituting both into (6),

$$H_T'(1) = T^2 \sum_{i=1}^{T} \frac{T\,(T + 2 - 3i)}{i\,T^2\,T^3} = \frac{1}{T^2} \sum_{i=1}^{T} \frac{T + 2 - 3i}{i} = \frac{(T + 2)h_T - 3T}{T^2},$$

the last step being $\sum_{i=1}^{T}(T + 2 - 3i)/i = (T + 2)h_T - 3T$. □

Corollary 5.2 (the sign at $x = 1$)**.** *$H_T'(1) > 0$ for every integer $T \ge 3$. Consequently $x = 1$ is not a critical point of H_T for any $T \ge 3$; by Proposition 4.1, $T = 2$ is the only step count $T \ge 2$ at which $x = 1$ is critical. (At $T = 1$ the objective is the constant 1, so every point is critical there; $T \ge 2$ is the only reading under which the statement is not empty.)*

Proof. Put $f(T) = (T+2)h_T - 3T$, so that $H_T'(1) = f(T)/T^2$ by Theorem 5.1 and the claim is $f(T) > 0$. First, $f(3) = 5 \cdot \frac{11}{6} - 9 = \frac{1}{6} > 0$. Second, for $T \ge 3$,

$$f(T + 1) - f(T) = (T + 3)\Big(h_T + \frac{1}{T + 1}\Big) - 3(T + 1) - (T + 2)h_T + 3T = h_T - \frac{2T}{T + 1} > 0,$$

because $h_3 = \frac{11}{6} > \frac{3}{2} = \frac{2\cdot 3}{3+1}$, and for $T \ge 4$ one has $h_T \ge h_4 = \frac{25}{12} > 2 > \frac{2T}{T+1}$. Induction from $T = 3$ gives $f(T) > 0$ for all $T \ge 3$. □

Remark 5.3 (localisation of the minimiser; partly outside the formal development). Corollary 5.2 is a statement about the point $x = 1$ only. On the certified band it upgrades: for $3 \le T \le 300$ the unique positive critical point of H_T lies in $(0, 1)$ by Corollary 3.3 and Proposition 2.3. For general T, $H_T'(1) > 0$ excludes $x = 1$ and, granted the uniqueness that is still open there, would place the critical point in $(0, 1)$; we do not claim the latter for $T > 300$. What is proved for all $T \ge 3$ is the inequality at the single point $x = 1$, and nothing more.

Remark 5.4 (agreement with the source's table; outside the formal development). Bisecting R_T in exact rational arithmetic locates the minimiser. The source prints a table of $x^*(T)$ at eight step counts, five of them inside the band certified here; at all five the two agree to the printed precision: $x^*(5) = 0.721$, $x^*(10) = 0.577$, $x^*(50) = 0.404$, $x^*(200) = 0.332$, $x^*(217) = 0.329$. The same computation at $T = 3, 10, 50, 120, 300$ gives 0.8686, 0.5774, 0.4040, 0.3541, 0.3172, against $(2 \ln T)^{-1/2} = 0.6746$, 0.4660, 0.3575, 0.3232, 0.2961; the ratio falls from 1.288 to 1.071, consistent with the source's own report that it decreases from 1.29 at $T = 5$ to 1.017 at $T = 10^5$. These are numerical computations and are asserted here only as such.

6. The coefficient sequence, and the road to a uniform proof

Everything in this section is a computation or a heuristic, is outside the formal development, and is recorded because it is what a uniform proof would have to establish.

6.1. **Shape.** For every T in $2, \dots, 400$ the coefficient sequence of $R_T = \sum_k r_k X^k$ has the shape

$$r_0 = r_1 = 0, \qquad r_k < 0 \ \ (2 \le k < s(T)), \qquad r_k > 0 \ \ (s(T) \le k \le 3T - 3), \tag{10}$$

with no interior zeros and no exceptions, for a single index $s(T)$. Thus the one sign change of Theorem 3.1 is not merely a count: the sequence is a block of negatives followed by a block of positives.

Two of the three ingredients of (10) are uniform in T and were verified above: X^2 divides R_T exactly, and the leading coefficient is positive (Section 2.2). The first of these can be read off the last step of the recurrence, $N_T = T\,N_{T-1}b_{T-1}b_T^2 + A_T G_{T-1}$, in which $b_T = Tx$ and $A_T = -2T(T-1)x^2$: both summands carry x^2 and neither carries x^3. The same step gives

$$r_2 = [x^2]R_T = T^3 N_{T-1}(0) - 2T(T-1)G_{T-1}(0), \qquad \frac{N_{T-1}(0)}{G_{T-1}(0)} = \sum_{i=1}^{T-1} \frac{1}{i(T-i)(T-i+1)^2},$$

so that condition (v) of (9) is equivalent to the explicit inequality

$$T^2 \sum_{j=1}^{T-1} \frac{1}{(T-j)\,j\,(j+1)^2} \;<\; 2(T-1). \tag{11}$$

The left-hand side of (11) is asymptotically $(2-\pi^2/6)\,T \approx 0.3551\,T$, comfortably below $2T-2$; in exact rational arithmetic the ratio of the two sides is 0.500 at $T=2$, 0.182 at $T=120$, 0.179 at $T=300$ and 0.1776 at $T=5000$, decreasing towards $(2-\pi^2/6)/2 = 0.17753\ldots$ A uniform proof of (11) looks routine, but we have not written one, and it is not part of the formal development. Hence *existence* of a positive root of R_T is within reach for all T; what remains open is the interior sign pattern, i.e. the passage from "a block of negatives then a block of positives" to a proof.

6.2. **Where the change sits.** The index $s(T)$ of (10) grows slightly slower than T:

T	5	10	20	50	100	120	200	300	400
$s(T)$	7	12	23	53	101	120	194	285	374
$s(T)/T$	1.40	1.20	1.15	1.06	1.01	1.00	0.970	0.950	0.935

The sequence $s(T)$ itself, which begins $3, 4, 5, 7, 8, \ldots$, matches no entry we could find in the on-line encyclopedia of integer sequences, and neither do the coefficient lists of R_3, R_4 and R_5.

A further regularity, which we record because it is the kind of statement a generating function for N_k might deliver. Write $\rho_k = |r_{k+1}|/|r_k|$ for $2 \le k \le 3T-4$, and call k an *ascent* when $\rho_{k+1} > \rho_k$. Then ρ is strictly decreasing except at one or two ascents, and every ascent lies in $\{s(T)-2,\, s(T)-1\}$: checked exhaustively over the 198 step counts $T = 3, \ldots, 200$ with no exceptions, 116 of them having a single ascent and the other 82 two. In words, the coefficients are log-concave in absolute value away from the sign change itself. A proof that the ratio sequence has at most two ascents, both adjacent to the sign change, together with $r_2 < 0$ and a positive leading coefficient, would give (10) and hence the conjecture of [1] for all T.

6.3. **The second reduction.** The source offers an alternative route to uniqueness, "verifying $\mathrm{Var}_p(z) < 1/12$ at each critical point". For the objective (3) that condition takes an exact form. Put $c_i = 1/(i\,b_i b_{i-1}^2)$, let $w_i = c_i/\sum_j c_j$ be the induced weights, and set $z_j = jx/b_j$ (the source's $z_j = x/(x+\omega_j)$ with $\omega_j = (1-\rho_j)/\rho_j$). Then

$$\frac{x\,H_T'(x)}{H_T(x)} = 1 - \phi(x), \qquad \phi(x) = \mathbb{E}_w\big[z_i + 2z_{i-1}\big],$$

and differentiating the weights as well,

$$x\,\phi'(x) = \mathbb{E}_w\big[z_i(1-z_i) + 2z_{i-1}(1-z_{i-1})\big] - \mathrm{Var}_w\big(z_i + 2z_{i-1}\big).$$

Critical points are exactly the zeros of $\phi - 1$, so uniqueness follows if $x\phi'(x)$ has one fixed sign at every x with $\phi(x) = 1$: consecutive crossings must alternate in direction. The sign is negative here — H_T decreases then increases, so ϕ falls through 1 — and the delicacy of the route is visible in the boundary behaviour: $\phi(0^+) = 1$, because the weight concentrates on $i = T$ where $z_T \equiv 1$ and $z_{T-1} \to 0$, and $\phi(\infty) = 1$, because the weight concentrates on $i = 1$ where $z_1 \to 1$ and $z_0 \equiv 0$. So $\phi - 1$ vanishes at both ends, is positive near 0 and negative near ∞, and the claim is that it has exactly one interior zero. An exact-rational scan at $T \in \{2, 3, 5, 10, 50, 200\}$, over a logarithmic grid of 121 points each spanning $x \in [10^{-3}, 10^{3}]$, finds exactly one sign change of $\phi - 1$ in every case, and $x\phi' < 0$ at the crossing. We did not find a uniform proof by this route either.

7. Verification

Lemma 2.2, Propositions 2.3 and 4.1, Theorems 3.1, 3.2, 3.4 and 5.1, and Corollary 5.2 are machine-checked in Lean 4 [4] over the Mathlib library [5]; the development is eight modules, contains no `sorry`, and takes Descartes' rule (8) from Mathlib's rule-of-signs file [3] rather than reproving it, together with that file's notion of sign variation, which is the v of Section 2.3. The construction of Definition 2.1 is implemented on little-endian integer coefficient lists in a module that imports nothing, and a separate

module proves that the list it produces is the polynomial of Definition 2.1 and that the list-level sign-variation count agrees with Mathlib's; so the compiled arithmetic is checked against the mathematics rather than trusted. Two of the statements above are conjunctions of checked declarations rather than single ones: Lemma 2.2, whose positivity assertion and whose derivative formula are separate declarations, and Corollary 3.3, which combines the existence step behind Theorem 3.2, the one that produces a root in $(0, 1)$, with the uniqueness assertion of that theorem. Every other statement above is one declaration. Compiled evaluation enters the results in exactly four places, and they are the precise and only sense in which anything above rests on computation: the three blocks $T = 3, \dots, 120$, $T = 121, \dots, 200$ and $T = 201, \dots, 300$ of the test (9), and the single evaluation $R_2 = 16X^2(X - 1)$ of Proposition 4.1. Accordingly, an axiom print returns, for Theorem 5.1, Corollary 5.2 and Proposition 2.3, the three standard axioms `propext`, `Classical.choice` and `Quot.sound` and nothing else — they are uniform in T and use no computation at all — while Theorems 3.1, 3.2 and 3.4 name those three together with exactly one axiom for each of the three blocks, and Proposition 4.1 names those three together with one axiom for the $T = 2$ evaluation. The whole band file, including the three blocks and the library import, cost about 250 s of user CPU on a 20-core machine under load; two timed runs of it gave 248.5 s and 253 s, with wall times of 5 min 43 s and 10 min 28 s as the load varied. Negative controls accompany the development: a two-variation polynomial with two distinct positive roots, showing that $\mathrm{v} = 1$ rather than Descartes alone carries Theorem 3.2; the evaluation $\mathrm{v}(G_7) = 0$ on a sign-constant sequence, showing that the counter can return 0; two single-coefficient sign flips of R_7 that send the count from 1 to 3, showing that the property constrains the data and not the shape of the construction; the failure of the test (9) at $T = 2$, showing that its hypothesis is not vacuous; and the degrees of R_7, R_8 and R_{300}, showing that neighbouring step counts give polynomials a check cannot confuse. Independently of the kernel, the polynomial was built by three implementations resting on different normalisations — clearing with $\mathrm{lcm}(1, \dots, T)$ in place of $T!$, giving degree $3T - 3$; a Wronskian-style construction of degree $6T - 6$; and the factorial recurrence of Definition 2.1 — and all agree on the sign-variation count wherever their ranges were compared, which for two of them reaches beyond the band certified here; the signs of $R_T(x)$ were also checked against central differences of H_T at 40 points (T, x). The source of the development is currently held in a private repository: repository reference to be added at submission.

Remark 7.1 (cost of going further; timings, not theorems). The interpreter cost of building R_T scales as T^3; fitting the measured cost of a single build at $T = 250$ gives about 8 CPU-minutes for the block $T = 301, \dots, 400$ and about 47 CPU-minutes for $T = 401, \dots, 600$, both feasible, and about 6.6 CPU-hours for $T = 601, \dots, 1000$, which is not, at least without compiling the arithmetic core to native code rather than interpreting it. Memory is never the constraint: the largest instance run here, $T = 300$, holds 898 coefficients of at most 9436 bits — about one megabyte in total.

On the reference list

Reference [1] was read here from its arXiv e-print, and every quotation in Section 1.2 is from that file; the labels `prop:uniqueness` and `app:uniqueness`, and the equation tags reproduced in (1), are the source's own internal labels, and the numbering of any published version may differ. The grammatical slip in the second quotation of Section 1.2 ("we derives") is the source's. The abstract page and the arXiv metadata record were both checked on 3 September 2026 and agree: one version only, `[v1] Fri, 7 Aug 2026 07:25:19 UTC`, primary class cs.LG, no journal reference, no comments field and no withdrawal notice. The framing of this paper rests on the polynomial P_T being unprinted in the source, so a future revision of [1] would have to be read against Section 1.2 again. Two forms of the title are in circulation: the arXiv metadata record — which is what the identifier cited below resolves to, and what the bibliography uses — ends in *Schrödinger Bridge Model*, while the `\title` line of the e-print itself reads *Schrödinger Bridge Models*.

Reference [2] is cited for Descartes' rule of signs because it is the reference [1] gives for that step, and at the same section number. We checked it: in the second edition, Section 2.2.1 is *Descartes's Law of Signs and the Budan-Fourier Theorem*, and its Theorem 2.33 is (8) together with the parity refinement. (The source cites the same book as "Ch. 2" in its table of standard facts and as "Sec. 2.2.1" in the uniqueness appendix; both point here.)

Reference [3] names the Mathlib file whose statement of Descartes' rule and whose definition of sign variation the development uses. The file is the work of Alex Meiburg and carries a 2025 copyright line; it entered Mathlib on 14 October 2025. Both declarations named in the entry were read in the Mathlib revision this development is built against.

References

[1] F. Fallah and Y. Yang, *PRISM: Principled Reference Identification for Schrödinger Bridge Model*, arXiv:2608.06893v1 [cs.LG], 7 August 2026.

[2] S. Basu, R. Pollack and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, second edition, Algorithms and Computation in Mathematics **10**, Springer, Berlin, 2006, doi:10.1007/3-540-33099-2; Descartes' law of signs is Theorem 2.33, in Sec. 2.2.1.

[3] A. Meiburg, *Descartes' rule of signs*, the file `Mathlib/Algebra/Polynomial/RuleOfSigns.lean` of the Lean mathematical library [5], 2025; the declarations used here are `Polynomial.signVariations` and `Polynomial.roots_countP_pos_le_signVariations`.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[5] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.