CORNER-VECTOR DESIGNS ON $\mathbb{S}^3$: THE MAXIMUM DEGREE IS EXACTLY 13

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Abstract. For $a > 0$ and $s \in \{0, \dots, n-1\}$ let the *generalized corner vector* $v_{a,s} \in \mathbb{S}^{n-1}$ be the unit vector along $(a, 1, \dots, 1, 0, \dots, 0)$ with s ones. Its hyperoctahedral orbit $v_{a,s}^{B_n}$ is called *proper* when $a \neq 1$; a weighted spherical t-design which is a union of such orbits, with one positive weight per orbit, is a *design of type* (1). Tanino, Tamaru, Hirao and Sawa proved that $t \leq 15$ for every $n \geq 4$, exhibited 11- and 13-designs on $\mathbb{S}^3$ numerically, and asked in their Problem 6.1 whether a 15-design on $\mathbb{S}^3$ of this type with more than five proper orbits exists. We answer this negatively and in a stronger form: on $\mathbb{S}^3$ there is no 15-design of type (1) at all, with any number of orbits, proper or not. Such designs are centrally symmetric, so a 14-design is a 15-design; the uniform bound therefore sharpens at $n = 4$ from $t \leq 15$ to $t \leq 13$. The value 13 is attained: we give a 13-design with nine proper orbits and 656 points, all of whose parameters and weights are rational and exact, where the examples in the literature have four proper orbits and six printed digits. Hence the maximum degree of a design of type (1) on $\mathbb{S}^3$ is exactly 13. The proof is linear-programming duality rather than elimination: in the squared coordinates a design is a point of the convex hull of four explicit moment curves, and the obstruction is a single integer functional on the eleven moment conditions which is strictly positive at every point of every curve. Its positivity amounts to three degree-7 polynomial inequalities, which we prove from exact factorizations; no statement here rests on an exhaustive search. Every theorem, proposition and lemma below other than the reduction to the moment system and two one-line corollaries has been formally verified in Lean 4 against *Mathlib*; the remarks, which report how the certificate was found, what was tried and abandoned, and what the published configurations do when recomputed, are labelled as lying outside that development.

1. Introduction

Let B_n be the hyperoctahedral group of signed permutations of $\mathbb{R}^n$ and write x^{B_n} for the B_n-orbit of $x \in \mathbb{S}^{n-1}$. Following Tanino, Tamaru, Hirao and Sawa [9, Definition 2.19], for $a > 0$ and $s \in \{0, 1, \dots, n-1\}$ the *generalized corner vector* is

$$v_{a,s} = \frac{1}{\sqrt{a^2 + s}} \left(a\, e_1 + e_2 + \cdots + e_{s+1} \right) \in \mathbb{S}^{n-1},$$

and $v_{a,s}^{B_n}$ is called a *proper orbit* when $a \neq 1$; at $a = 1$ the vector $v_{1,s}$ is the classical corner vector v_{s+1}, the barycentre of an $(s+1)$-dimensional face of the cross-polytope. A *spherical cubature of degree t*, in the sense of [9, Definition 2.2], is a finite $X \subset \mathbb{S}^{n-1}$ together with a weight function $w : X \to \mathbb{R}_{>0}$ — the positivity is part of the definition — for which $|\mathbb{S}^{n-1}|^{-1} \int_{\mathbb{S}^{n-1}} f \, d\rho = \sum_{x \in X} w(x) f(x)$ for every polynomial f of degree at most t; the pair (X, w) is a spherical t-design, called *weighted* there when w is not the constant $1/|X|$. The object of this paper is the class of designs that [9] calls designs of type (1) and which we shall write as

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(y)\, d\rho = \sum_{i=1}^{k} W_i \sum_{x \in v_{a_i, s_i}^{B_n}} f(x) \qquad \text{for every polynomial } f \text{ with } \deg f \leq t, \tag{1}$$

with $a_i > 0$, $s_i \in \{0, \dots, n-1\}$ and $W_i > 0$: a weighted t-design whose support is a union of B_n-orbits of generalized corner vectors and whose weight is constant on each orbit. We call the largest t for which such a configuration exists the *maximum degree* of a design of type (1) on $\mathbb{S}^{n-1}$.

The positivity of the W_i is not a convenience: [9, Section 6] points out, citing Keast [5], that 15-designs with negative weights do exist, so (1) and the same identity with signs allowed are different questions. Proposition 5.2 below makes that distinction exact for the case at hand.

What is known. The corner-vector method proper — $a_i = 1$ for all i — cannot reach beyond degree 7, by Bajnok [1, Proposition 15], quoted as [9, Theorem 2.18]. Allowing one proper orbit reaches degree 11 and no further: this is Sawa and Xu [8], as reported in [9, Section 6], and the bound is attained on $\mathbb{S}^3$ by a formula of Schur ([9, Example 2.5], cf. [3, p. 721]). In dimension three the question is settled at 17: [9, Remark 3.2] reports, on the strength of [4, Theorem 2.1], that a design of type (1) on $\mathbb{S}^2$ has degree at most 17, and [9, Section 6] exhibits a 17-design on $\mathbb{S}^2$, taken from [4]. (Theorem 2.1 of [4] is itself a necessary condition on the *structure* of a fully symmetric cubature formula, in the classification of [4]; the bound 17 on designs of type (1) is the form in which [9, Remark 3.2] reports it, and it is that report we quote. We reproduce the 17-design of [9, Section 6] in Remark 6.2(b).) For $n \geq 4$ the state of the art is [9, Theorem 3.1]: if (1) holds then $t \leq 15$. Two further partial results concern two orbits: [9, Theorem 5.7] rules out three of the five cases of its own characterisation [9, Proposition 5.5] of two-orbit 9-designs, and the companion manuscript [2] rules out two-orbit 9-designs on $\mathbb{S}^3$ outright, working in the harmonic basis $f_4, f_6, f_{8,1}, f_{8,2}$.

Against the bound $t \leq 15$, the examples in [9, Section 6] reach only degree 13: an 11-design on $\mathbb{S}^3$ with three proper orbits, and two 13-designs with five orbits of which four are proper, all printed to six digits and with no exact data. The paper records that the extremal case resisted its numerical search, and asks:

> **Problem 6.1** [9]. *Does there exist a weighted* 15*-design on* $\mathbb{S}^3$ *of type* (1) *with more than* 5 *proper orbits?* [Their type (1) is the display (1) above.]

What is settled here. The problem, in a stronger form than it is posed, together with the exact value of the maximum degree in dimension four.

> *On* $\mathbb{S}^3$ *there is no weighted* 15*-design of type* (1)*: not with more than five proper orbits, and not with any number of orbits, proper or not. There is a* 13*-design of type* (1) *on* $\mathbb{S}^3$ *with nine proper orbits,* 656 *points, and rational* a_i^2 *and* W_i*. Hence the maximum degree of a design of type* (1) *on* $\mathbb{S}^3$ *is exactly* 13*.*

These are Theorems 3.5, 4.1 and Corollary 4.3. Since $-I \in B_4$, every configuration of type (1) is centrally symmetric and the odd degrees are automatic (as [9, Section 3] also notes), so a 14-design of this type is literally the same thing as a 15-design; the nonexistence statement therefore sharpens the bound of [9, Theorem 3.1] at $n = 4$ from $t \leq 15$ to $t \leq 13$, and Theorem 4.1 shows that this is sharp. Problem 6.1 asks for more than five proper orbits at degree 15; Theorem 4.1 provides nine of them at degree 13.

The method. The reason a linear-programming argument settles what a numerical search could not is that the problem is convex, once it is written in the right coordinates. Every orbit is invariant under sign changes, so a design condition only ever sees the *squared* coordinates of a point, and these lie on the simplex $p_1 = 1$; on that simplex the algebra of test functions is generated by the power sums p_2, p_3, p_4. In these coordinates the whole degree-15 condition on $\mathbb{S}^3$ is eleven polynomial equations (5), one orbit contributes one point of one of four explicit curves — one curve for each $s \in \{0, 1, 2, 3\}$, parametrized by $A = a^2 > 0$ — and a design exists if and only if a fixed rational vector of sphere moments lies in the convex hull of those four curves. That is a linear programme. Its dual is a separating functional, and a separating functional is a certificate: an integer vector $c \in \mathbb{Z}^{11}$ with $\sum_\beta c_\beta L_\beta = 0$ on the right-hand sides and $G(s, A) = \sum_\beta c_\beta M_\beta(s, A) > 0$ at every orbit. Applying c to a putative design gives $\sum_i w_i\, G(s_i, A_i) = 0$ with all $w_i > 0$ and all $G(s_i, A_i) > 0$, which is absurd.

Section 3 exhibits such a c. Finding it needed a search — an exact-rational cutting-plane loop over a finite sample of the four curves — but the search supports nothing: once written down, c is checked by four polynomial identities and three inequalities, and those are what the proof uses. The three inequalities are the delicate part: the certificate polynomials come within a few parts in 10^4 of their coefficient scale of vanishing on $A > 0$, so no crude positivity test sees them (Remark 3.8 lists what was measured and abandoned), and each is proved instead from an exact factorization that isolates its near-double root as a square.

Provenance. The source [9] is quoted from the published article, read on 3 September 2026; the version-2 e-print of arXiv:2501.11437, read on 2 September 2026 when the arXiv version list showed no later version, is where the recomputations of Section 6 were first made. Every numbered item cited here — Definitions 2.2 and 2.19, Example 2.5, Theorems 2.13, 2.18, 3.1 and 5.7, Propositions 2.14 and 5.5, Remark 3.2 and Problem 6.1 — carries the same number in both, and the source's display labelled `eq:invariant0`, which is our (1), is numbered (1) in both. Two of the numerical examples of its Section 6 differ between the two versions, and Remark 6.2 says how. A full-text sweep of a local arXiv corpus of 369 shards, incremental through the announcement window of August 2026, finds the string `2501.11437` in exactly one shard and only inside the source's own record, so no citing paper is present there; *generalized corner vector* and *corner-vector method* occur only inside the source. Searches of the arXiv metadata return only the source for *corner-vector method*, and nothing at all for 15-*design* together with *spherical design*. OpenAlex reports zero citing works for the e-print's DOI and zero for the journal DOI, re-checked on 3 September 2026; Semantic Scholar returns exactly two citing papers, one on the Prouhet–Tarry–Escott problem and one on tight t-fusion frames, neither of which contains a nonexistence statement for corner-vector designs. The published bounds listed above all cover strictly less than Theorem 3.5: Bajnok's is for $a_i = 1$, Sawa and Xu's for one proper orbit, Heo and Xu's for $n = 3$, and [9, Theorems 3.1 and 5.7] for $t \le 15$ and for two orbits. We did not consult MathSciNet, zbMATH or Google Scholar, nor the full texts of Sawa–Xu [8] or Keast [5], nor non-arXiv venues, nor anything posted after 3 September 2026. Heo–Xu [4] was read directly, for Remark 6.2(b) only.

Organisation. Section 2 reduces (1) at $n = 4$, $t = 15$ to eleven polynomial equations. Section 3 gives the separating invariant, its positivity, and the nonexistence theorem. Section 4 gives the 13-design and the exact value of the maximum degree. Section 5 records the controls that keep the moment system honest, including the exact sense in which positivity of the weights is what fails. Section 6 reports, outside the formal development, what happened when the published configurations of [9, Section 6] were recomputed, and what the same pipeline returns on $\mathbb{S}^2$. Section 7 describes the formal verification.

2. The moment system

Throughout this section $n = 4$. For a unit vector $v \in \mathbb{S}^3$ write $y(v) = (v_1^2, v_2^2, v_3^2, v_4^2)$, a point of the simplex $\{y \ge 0 : \ p_1(y) = 1\}$, where $p_j(y) = \sum_\ell y_\ell^j$ is the j-th power sum. Let E denote the expectation of a function of $y(v)$ for v uniform on $\mathbb{S}^3$. For an orbit $v_{a,s}^{B_4}$ put

$$A = a^2 > 0, \qquad T = A + s, \qquad P_j = A^j + s \quad (j = 2, 3, 4), \tag{2}$$

so that $y(v_{a,s}) = (A, 1, \dots, 1, 0, \dots, 0)/T$ with s ones, and hence

$$p_j\big(y(v_{a,s})\big) = \frac{A^j + s}{(A + s)^j} = \frac{P_j}{T^j} \qquad (j = 2, 3, 4), \tag{3}$$

with no case distinction on s. Finally, for $\alpha, \beta, \gamma \ge 0$ set $L_{\alpha\beta\gamma} = E\big[p_2^\alpha p_3^\beta p_4^\gamma\big]$. The eleven values needed below are

$$\begin{aligned} &L_{000} = 1, \quad L_{100} = \frac{1}{2}, \quad L_{200} = \frac{11}{40}, \quad L_{300} = \frac{93}{560}, \quad L_{010} = \frac{5}{16}, \\ &L_{110} = \frac{3}{16}, \quad L_{210} = \frac{219}{1792}, \quad L_{020} = \frac{123}{896}, \quad L_{001} = \frac{7}{32}, \quad L_{101} = \frac{9}{64}, \end{aligned} \qquad L_{011} = \frac{111}{1024}. \tag{4}$$

Proposition 2.1. *Let $k \ge 0$, let $a_i > 0$ and $s_i \in \{0, 1, 2, 3\}$ for $1 \le i \le k$, and let $W_i > 0$. Put $A_i = a_i^2$, $T_i = A_i + s_i$, and let*

$$w_i = \frac{W_i \,\big|v_{a_i,s_i}^{B_4}\big|}{T_i^7} > 0.$$

Then the configuration $\big(\bigcup_i v_{a_i,s_i}^{B_4},\, W\big)$ is a weighted 15-design on $\mathbb{S}^3$ if and only if

$$\sum_{i=1}^{k} w_i\,(A_i^2 + s_i)^\alpha (A_i^3 + s_i)^\beta (A_i^4 + s_i)^\gamma (A_i + s_i)^{7 - 2\alpha - 3\beta - 4\gamma} = L_{\alpha\beta\gamma} \tag{5}$$

for the eleven triples (α,β,γ) with $2\alpha+3\beta+4\gamma\le 7$. It is a weighted 13-design if and only if the nine of those equations with $2\alpha+3\beta+4\gamma\le 6$ hold.

Proof. By Sobolev's theorem [9, Theorem 2.13] — the weights being constant on each orbit and positive — the design identity (1) for all f of degree at most t is equivalent to the same identity for all B_4-invariant f of degree at most t. The algebra of B_4-invariant polynomials is generated by the elementary symmetric functions of $x_1^2,\dots,x_4^2$, so a B_4-invariant polynomial is a symmetric polynomial in the squared coordinates; restricted to the sphere the relation $p_1=1$ holds, and the residual algebra is generated by p_2,p_3,p_4. A B_4-invariant f of degree at most t therefore lies in the span of the monomials $p_2^\alpha p_3^\beta p_4^\gamma$ with $2\alpha+3\beta+4\gamma\le\lfloor t/2\rfloor$. For $t=15$ there are exactly eleven of them and for $t=13$ exactly nine, which matches the dimension count from the harmonic Molien series $\big((1-\lambda^4)(1-\lambda^6)(1-\lambda^8)\big)^{-1}$ of [9, Proposition 2.14], which gives $\dim\mathcal{P}_{15}(\mathbb{S}^3)^{B_4}=1+1+1+2+1+3+2=11$ and $\dim\mathcal{P}_{13}(\mathbb{S}^3)^{B_4}=9$; so the monomials in question are moreover linearly independent on the simplex, and testing on them is testing on a basis. Testing (1) on $f=p_2^\alpha p_3^\beta p_4^\gamma$ and using (3) gives

$$\sum_i W_i\big|v^{B_4}_{a_i,s_i}\big|\,\frac{(A_i^2+s_i)^\alpha(A_i^3+s_i)^\beta(A_i^4+s_i)^\gamma}{T_i^{2\alpha+3\beta+4\gamma}}=L_{\alpha\beta\gamma},$$

and multiplying the i-th term above and below by T_i^7 replaces the factor $W_i\big|v^{B_4}_{a_i,s_i}\big|\,T_i^{-2\alpha-3\beta-4\gamma}$ by $w_i\,T_i^{7-2\alpha-3\beta-4\gamma}$, which is (5). ☐

Three remarks on (5), all used below. First, the equation indexed by $(0,0,0)$ reads $\sum_i w_iT_i^7=1$: the total mass. Second, the system does not know how many distinct orbits there are, nor whether they are proper: k is arbitrary and repetitions among the pairs (s_i,A_i) are allowed, so a nonexistence statement about (5) is a nonexistence statement about all configurations at once. Third, every monomial of degree 15 has an odd exponent and hence integrates to 0 against a centrally symmetric configuration, so on $\mathbb{S}^3$ a design of type (1) of degree 14 is exactly one of degree 15.

The eleven values (4) were computed from the closed form $E\big[\prod_\ell x_\ell^{2\alpha_\ell}\big]=\prod_\ell(2\alpha_\ell-1)!!/\prod_{j=0}^{m-1}(4+2j)$, $m=\sum_\ell\alpha_\ell$, and checked by the identity $\sum_{|\alpha|=m}\binom{m}{\alpha}E\big[x^{2\alpha}\big]=1$, the expansion of $(\sum_\ell x_\ell^2)^m=1$, for every $m\le 7$.

3. The separating invariant, and nonexistence at degree 15

Definition 3.1. For $s,A\in\mathbb{R}$ put $T=A+s$ and $P_j=A^j+s$ as in (2), and let

$$\begin{aligned}G(s,A)={}&-12444714240\,P_2T^5+37932384000\,P_2^2T^3-211196160\,P_2^3T\\&+12551974400\,P_3T^4-96068582400\,P_2P_3T^2+4690291200\,P_2^2P_3\\&+53365329920\,P_3^2T+1937671680\,P_4T^3+23550589440\,P_2P_4T\\&-26452392960\,P_3P_4+1149102503\,T^7.\end{aligned}\tag{6}$$

The eleven coefficients of (6) are the entries of an integer vector c indexed by the eleven monomials of (5), and they were chosen so that

$$\sum_{\alpha\beta\gamma}c_{\alpha\beta\gamma}L_{\alpha\beta\gamma}=0\tag{7}$$

for the values (4); that single rational identity is what makes G vanish on designs.

Lemma 3.2. *Let $g_1,g_2,g_3\in\mathbb{Z}[A]$ be*

$$\begin{aligned}g_1(A)={}&152461+68589787A-604501359A^2+1150639175A^3\\&+1150639175A^4-604501359A^5+68589787A^6+152461A^7,\end{aligned}$$

$$\begin{aligned}g_2(A)={}&1229760128-12298217792A+35535024672A^2-23128619920A^3\\&+11433787960A^4-2184041916A^5+137179574A^6+152461A^7,\end{aligned}$$

$$g_3(A) = 10725025167 - 105419925597A + 294565656483A^2 - 172116729225A^3 + 42548243325A^4 - 4738621671A^5 + 205769361A^6 + 152461A^7.$$

Then, identically in A,

$$G(0, A) = 457383\,A^7, \qquad G(1, A) = 3g_1(A), \qquad G(2, A) = 3g_2(A), \qquad G(3, A) = 3g_3(A).$$

Proof. Four polynomial identities in one variable, each verified by expanding (6) at the integer value of s. ☐

Note that g_1 is palindromic, as it must be: the orbit $v_{a,1}^{B_4}$ is unchanged when a is replaced by $1/a$, that is, when $A \mapsto 1/A$, so the $s = 1$ curve is invariant under that involution and $A^7 g_1(1/A) = g_1(A)$.

Proposition 3.3. *$g_1(A) > 0$, $g_2(A) > 0$ and $g_3(A) > 0$ for every real $A > 0$.*

Proof. Each of the three is an exact polynomial identity followed by an inspection of signs. For g_1 we use the palindromic structure: $g_1(A)/A^3$ is a cubic in $u = A + A^{-1}$, and completing the square in that cubic at $u = 2$ gives, identically in A,

$$277408368\, g_1(A) = (A+1)\Big(42293957193648\,(A-1)^6 + A\big(138704184(A-1)^2 - 397817232A\big)^2 + 12314324993833728\, A^3\Big). \tag{8}$$

For $A > 0$ the factor $A+1$ is positive, the first summand is nonnegative, the second is a positive multiple of a square, and the third is strictly positive; hence $g_1(A) > 0$.

For g_2 and g_3 the near-double root is captured by rational quadratic factors of the form $(cA-m)^2+d$ with $d > 0$, leaving a linear cofactor and a remainder with only nonnegative coefficients:

$$\begin{aligned} 6103515625000000000000\, g_2(A) = {} & \big((5000A - 1054)^2 + 15000\big)\big((2500A - 17205)^2 + 13635000\big) \\ & \times \big((2500A - 2002)^2 + 20587500\big)\big(87241566437 + 95288125A\big) \\ & + 2499057059688782007804679524800 \\ & + 8278938951035086960187837200000\, A \\ & + 9635276441674633523452000000000\, A^2 \\ & + 6369551438140723126328125000000\, A^3 \\ & + 3987542086241925781250000000000\, A^4 \\ & + 4351379369357421875000000000000\, A^5, \end{aligned} \tag{9}$$

$$\begin{aligned} 25000000000000000000000000000\, g_3(A) = {} & \big((10000A - 2175)^2 + 20000\big) \\ & \times \big((10000A - 37204)^2 + 229950000\big) \\ & \times \big((10000A - 74735)^2 + 1091490000\big) \\ & \times \big(523122369777 + 381152500A\big) \\ & + 301205216683758270262167984330000000 \\ & + 112941589840746983577092759500000000\, A \\ & + 621814520407279503044539425000000000\, A^2 \\ & + 171999532403802909490875000000000000\, A^3 \\ & + 131606109341245695300000000000000000\, A^4 \\ & + 148411641281284500000000000000000000\, A^5. \end{aligned} \tag{10}$$

In each case every quadratic factor is positive for all real A, the linear cofactor is positive for $A > 0$, and every coefficient of the remainder is positive; so the right-hand side, and with it the left, is strictly positive for $A > 0$. ☐

Theorem 3.4. *$G(s,A) > 0$ for every $s \in \{0,1,2,3\}$ and every real $A > 0$.*

Proof. Immediate from Lemma 3.2 and Proposition 3.3, the case $s = 0$ being $457383A^7 > 0$. ☐

Theorem 3.5. *There is no weighted 15-design on $\mathbb{S}^3$ of type (1): for no $k \geq 0$ do there exist $s_i \in \{0,1,2,3\}$, $a_i > 0$ and $W_i > 0$ satisfying (1) with $n = 4$, $t = 15$. Equivalently, by Proposition 2.1, the system (5) has no solution with $s_i \in \{0,1,2,3\}$, $A_i > 0$ and $w_i > 0$. The orbits are not assumed distinct, and no orbit is assumed proper.*

Proof. Suppose $(s_i, A_i, w_i)_{i \leq k}$ solves (5) with all $A_i > 0$ and all $w_i > 0$. If $k = 0$ the equation indexed by $(0,0,0)$ reads $0 = 1$. So let $k \geq 1$. Multiplying the eleven equations of (5) by the eleven coefficients of (6) and adding them gives, by (7),

$$\sum_{i=1}^{k} w_i\, G(s_i, A_i) = \sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma} L_{\alpha\beta\gamma} = 0.$$

But each term of the left-hand sum is a product of $w_i > 0$ with $G(s_i, A_i) > 0$ (Theorem 3.4), and $k \geq 1$, so the sum is strictly positive. This contradiction proves the theorem. ☐

Corollary 3.6. *There is no weighted 14-design on $\mathbb{S}^3$ of type (1) either, and every design of type (1) on $\mathbb{S}^3$ has degree at most 13. In particular the bound $t \leq 15$ of [9, Theorem 3.1] can be sharpened at $n = 4$ to $t \leq 13$.*

Proof. Every monomial of degree 15 has an odd exponent and both sides of (1) vanish on it, since $-I \in B_4$; so a 14-design of this type is a 15-design, and Theorem 3.5 applies. ☐

Remark 3.7 (How the certificate was found, and what the search does and does not support). The vector c was produced by a cutting-plane loop in exact rational arithmetic: sample the four curves $\{(s, A) : A > 0\}$ finitely, run a phase-1 simplex on the resulting finite linear programme, take the Farkas dual of the infeasible primal, test the four resulting univariate polynomials for positivity on $A > 0$ by Sturm sequences over $\mathbb{Q}$, add the violating sample points to the sample, and repeat. Starting from 60 points on each of the three nontrivial curves — a sample of 181 points in all — the loop converged after four rounds with a sample of 186 points; rounding the floating-point dual to the common denominator 10^5 gave an integer vector which passes the exact test, and that vector is the c of (6). Its positivity was then verified twice and independently before anything else was done with it: by Sturm sequences over $\mathbb{Q}$, and by counting the positive real roots of g_1, g_2, g_3 numerically at high precision (zero in each case). The whole search was then re-run from a different starting sample (37 points per curve instead of 60, and hence a different sequence of cuts); it converged in four rounds to a *different* rational functional, whose three polynomials also pass both exact tests. So the conclusion does not depend on one particular vertex of the dual cone. We stress that none of this search enters the proof. A grid computation can never establish infeasibility — Remark 6.3 below gives a concrete instance in which the true witness has irrational parameters and is invisible to any rational grid — and the only things Theorem 3.5 uses are the identity (7), the four identities of Lemma 3.2, and the three identities (8), (9), (10).

Remark 3.8 (Why the positivity proofs look the way they do). Each g_s has a local minimum near $A \approx 0.21$ whose value is between $1.8 \cdot 10^{-4}$ and $6.1 \cdot 10^{-4}$ times the largest absolute coefficient of g_s — numerically, the three polynomials have complex root pairs whose imaginary parts are between $1.6 \cdot 10^{-2}$ and $3.9 \cdot 10^{-2}$ — and this shallowness is what forces the exact factorizations (8)–(10). The following cheaper routes were measured and are all unusable, and are recorded here so that they are not retried. None of the three g_s has nonnegative coefficients. For Pólya's method the least N making $(1+A)^N g_1(A)$ coefficientwise nonnegative is $N = 203$, giving a polynomial of degree 210 with 69-digit coefficients; g_2 and g_3 are no better. Bernstein expansion and interval subdivision fail for the same reason: resolving a minimum that small against the second derivative needs a Bernstein degree of order 10^6. Finally, writing g_s in the cone spanned by $A^i(A - r)^{2j}$ for a single rational r is infeasible for all three, by an exact linear programme over a 1000-point grid of values of r; the near-double root needs a genuine quadratic factor, not a monomial-weighted square. A better-centred functional — a maximum-margin certificate rather than the Farkas vertex — would shrink the constants above by orders of magnitude, but the margin linear programme is degenerate and we did not obtain one.

4. A 13-design with nine proper orbits

Theorem 3.5 is only half of an exact answer; the other half is that degree 13 is reached. That 13-designs of type (1) exist on $\mathbb{S}^3$ is due to [9, Section 6], which exhibits two, each with five orbits of which four are proper, to six printed digits and with no exact data. The following is exact, and has nine proper orbits.

Theorem 4.1. *Let the nine pairs (s_i, A_i) and the nine weights $\mathrm{Wt}_i = W_i\left|v^{B_4}_{a_i,s_i}\right|$ be as in the table*

s_i	$A_i = a_i^2$	Wt_i	$\left\lvert v^{B_4}_{a_i,s_i}\right\rvert$
1	2/3	1954804877/36225116928	48
1	8	16063378712721/131580280832000	48
1	9	602638961/17985897216	48
2	1/4	1952993965141161/6422006743040000	96
2	2/3	159060731/6803094375	96
2	6	59206541/316103375	96
2	8	236516625/19196215808	96
3	3	81/640	64
3	4	16807/122880	64

Then $\left(\bigcup_{i=1}^9 v^{B_4}_{a_i,s_i},\, W\right)$ is a weighted 13-design on $\mathbb{S}^3$ of type (1). All nine orbits are proper and pairwise disjoint, so the design has $3\cdot 48 + 4\cdot 96 + 2\cdot 64 = 656$ points, and every a_i^2 and every W_i is rational.

Proof. By Proposition 2.1 it suffices to check the nine equations of (5) with $2\alpha + 3\beta + 4\gamma \le 6$ for $w_i = \mathrm{Wt}_i/(A_i+s_i)^7$; all nine hold with defect exactly 0 in rational arithmetic. Every A_i differs from 1, so every orbit is proper. For distinctness: orbits with different s have different numbers of zero coordinates, and within a fixed s the multiset of absolute values of the coordinates of $v_{a,s}$ determines $\{A\}$ when $s \in \{2,3\}$ and determines $\{A, 1/A\}$ when $s = 1$; the three values $2/3, 8, 9$ occurring at $s = 1$ contain no reciprocal pair, so the nine orbits are pairwise distinct and hence disjoint, and the sizes are 48, 96 and 64 for $s = 1, 2, 3$ respectively. □

Proposition 4.2. *The configuration of Theorem 4.1 is not a 15-design: it fails the equation of (5) indexed by $(\alpha,\beta,\gamma) = (2,1,0)$.*

Proof. The left-hand side of that equation differs from $L_{210} = 219/1792$ by

$$-\frac{46557443971}{189147463680000} \neq 0.$$

□

Both equations of weight 7 in fact fail, the one indexed by $(0,1,1)$ by $-7476802853/54042132480000$; as they must, by Theorem 3.5. That second computation is exact rational arithmetic outside the formal development.

Corollary 4.3. *The maximum degree of a weighted spherical design on $\mathbb{S}^3$ of type (1) is exactly 13. Moreover a design of type (1) on $\mathbb{S}^3$ with more than five proper orbits exists at degree 13, with nine.*

Proof. Theorem 4.1 gives a 13-design with nine proper orbits, and Corollary 3.6 forbids degree 14 and above. □

5. Controls

A nonexistence theorem about a linear system is only as good as the system, and an argument that turns on positivity is only as good as the evidence that positivity is what fails. This section records the three checks that settle both points. The first is a compute-first control: it was run before any of the above was claimed.

Proposition 5.1 (Schur's 11-design)**.** *Let* $(s_i, A_i)_{i\le 4} = \big((2,4),(0,1),(1,1),(3,1)\big)$ *with weights* $w = \big(\frac{1}{414720}, \frac{1}{15}, \frac{1}{1024}, \frac{1}{122880}\big)$*. Then all six equations of* (5) *with* $2\alpha + 3\beta + 4\gamma \le 5$ *hold.*

This is the classical formula of Schur printed as [9, Example 2.5], namely $v_{2,2}^{B_4} \cup v_1^{B_4} \cup v_2^{B_4} \cup v_4^{B_4}$ with weights $\frac{9}{640}, \frac{1}{60}, \frac{1}{96}, \frac{1}{60}$; those printed weights have total mass 2 against the orbit sizes $96, 8, 24, 16$, so we have halved them, which is the same normalisation correction recorded independently in [2]. By Proposition 2.1 the six equations say exactly that this configuration is an 11-design, so the moment system (5) is the right one and is not vacuous at degree 11. It is an 11-design and no more: the three equations of weight 6 fail, by 3/1120, 1/1344 and 1/640 for $(\alpha,\beta,\gamma) = (3,0,0)$, $(0,2,0)$ and $(1,0,1)$ respectively — a computation outside the formal development.

Proposition 5.2 (positivity is what fails)**.** *All eleven equations of* (5) *are satisfied by*

s_i	1	0	1	2	2	2	3	3	3	3	1
A_i	2	1	1	1	2	3	1	2	3	4	3
w_i	$\frac{-5}{14336}$	$\frac{351}{10240}$	$\frac{1193}{430080}$	$\frac{-1849}{2580480}$	$\frac{1}{5120}$	$\frac{-1}{53760}$	$\frac{-167}{860160}$	$\frac{47}{430080}$	$\frac{-89}{2580480}$	$\frac{1}{172032}$	$\frac{37}{860160}$

with all $s_i \in \{0,1,2,3\}$ *and all* $A_i > 0$*; five of the eleven weights are negative.*

So Theorem 3.5 is not an emptiness statement about the linear system: over $\mathbb{Q}$ the eleven equations are solvable, and what fails is exactly the positivity of the weights. This is the formal counterpart of the remark in [9, Section 6] that 15-designs with negative weights exist, for which that paper cites Keast [5].

Proposition 5.3 (the hypothesis $A > 0$ is not vacuous)**.** $g_1(-1000) < 0$*, and* $g_1(-1) = 0$.

The first statement refutes the reading of Theorem 3.4 in which the restriction to $A > 0$ could be dropped; the second is forced, g_1 being palindromic of odd degree and hence divisible by $A+1$, and it shows in addition that the ray in Theorem 3.4 cannot even be widened to $A \ge -1$. For the record, $g_1(-1000) = -832655621530687449486634539$.

6. Recomputing the published configurations

The computations reported in this section were carried out in exact rational arithmetic on the printed data, but they are numerical statements about six-digit numbers, they are not part of the formal development, and nothing above depends on them. We report them because the first group is the compute-first gate that preceded every claim in this paper, and because the second group concerns three small points a reader reproducing [9, Sections 5.2 and 6] will meet, two of which separate the version-2 e-print of arXiv:2501.11437 from the published article.

Remark 6.1 (The gate)*.* Before any certificate was searched for, the configurations printed in [9] were recomputed through the machinery of Section 2 and through a second, independent implementation using raw monomial moments and explicit orbit averages. Schur's 11-design reproduces exactly (Proposition 5.1); the 11-design of [9, Section 6] with three proper orbits reproduces with design defect $9.9\cdot 10^{-9}$; its two 13-designs reproduce with defects $4.7\cdot 10^{-8}$ and $3.0\cdot 10^{-8}$; its $\mathbb{S}^2$ 11-design with three proper orbits reproduces with defect $2.6\cdot 10^{-7}$; and the first case of its Table 3 ($n = 3$) reproduces with defect $2.7\cdot 10^{-7}$. All of these are at the limit set by six printed digits. In the coordinates of Section 2 the first 13-design of [9, Section 6] satisfies the nine equations of weight at most 6 to $1.9\cdot 10^{-7}$ and misses both equations of weight 7 by $7.5\cdot 10^{-5}$ — the shape Theorem 3.5 predicts — and the separating invariant evaluated on it is $+1.6\cdot 10^{6}$, strictly positive as it must be on any positively weighted configuration.

Remark 6.2 (Three readings, and what the published article already fixes)*.* Three items of [9, Sections 5.2 and 6] did not reproduce when the recomputation was first run against the version-2 e-print. Two of the three are already corrected in the published article. None of them affects any theorem of [9], and none is part of the formal development here. The defects below are the largest residuals of the conditions of Section 2, in the form appropriate to the sphere in question.

(a) *Corrected in print.* The first of the two 13-designs is displayed in the e-print with first orbit $v_{1,4}^{B_4}$, which is outside the range of the notation $v_{a,s}$ at $n = 4$, where $s \leq 3$. Reading it as the corner vector $v_4 = v_{1,3}$ makes the configuration reproduce with defect $4.7 \cdot 10^{-8}$, whereas the three in-range readings $v_{1,0}, v_{1,1}, v_{1,2}$ give $9.5 \cdot 10^{-3}$, $9.5 \cdot 10^{-3}$ and $1.9 \cdot 10^{-2}$. The published article prints exactly $v_{1,3}^{B_4}$ there.

(b) *Corrected in print, and traced.* The two versions quote *different* 17-designs on $\mathbb{S}^2$ from [4]. Table 2.1 of [4] lists two fully symmetric formulae of degree 17, both with 110 nodes: one of structure $[1; 1, 0, 3; 1, 0]$ with all weights positive, and one of structure $[1; 1, 0, 2; 2, 0]$ marked as having a negative weight. The published article quotes the first, and it reproduces: total mass 1.0000000 and defect $2.9 \cdot 10^{-7}$, at the limit of six printed digits. The e-print quotes the second, whose numerical values are the ones tabulated in the appendix of [4, p. 278], where the weight of the six-point orbit $v_{1,0}^{B_3}$ is $\mu_1 = -0.002664002664$; the e-print prints it as $+0.00266400$, and the total mass is then $1 + 12 \cdot 0.002664002664 = 1.031968$, which is the 1.0320 the recomputation returned, with defect $3.2 \cdot 10^{-2}$. The e-print's four proper parameters there, $2.41421 = 1 + \sqrt{2}$, $0.414214 = \sqrt{2} - 1$, 0.645826 and 0.228616, are the ratios of the node coordinates printed there, to all six digits the e-print gives. Two smaller points survive into the published article: the example is called one with six proper orbits, although two of its six orbits have $a = 1$; and its numerical values are not in Table 2.1 of [4], which lists structures and node counts, but in the appendix of [4] and in the technical report cited there.

(c) *Not a misprint.* The second case of [9, Table 3] ($n = 3$, $s_1 = s_2 = 2$, $a_1 = 0.389041$, $a_2 = 3.89103$, $W_1 = 0.0193973$, $W_2 = 0.0222694$) is printed identically in both versions, and it *is* a 9-design — but only when the two weights are read against the other parameter. Pairing W_1 with a_1 gives defect $3.4 \cdot 10^{-2}$; pairing W_1 with a_2 and W_2 with a_1 gives $8 \cdot 10^{-7}$, the same size as the first case of the same table, which reproduces as printed ($8 \cdot 10^{-7}$, and $2.6 \cdot 10^{-2}$ under the transposition). So the two weights of the second row are transposed, and the paper's own three displayed equations for that case confirm it: both $a_1^2 = 0.151353$ and $a_2^2 = 15.14015$ are roots of the printed quartic $A^4 - 20A^3 + 66A^2 + 116A - 19$; the printed relation $-175 + 12A_1 - 47A_2 + 19A_2^2 - A_2^3 = 0$ carries $A_2 = 15.14015$ to $A_1 = 0.151353$; and the printed relation $-3956 + 2291A_2 - 349A_2^2 + 13A_2^3 + 7770W_1 = 0$ then gives $W_1 = 0.534466$, which is $24 \cdot 0.0222694$ — the table's W_2. This is a three-dimensional case and is outside the scope of the present paper.

Remark 6.3 (The method control: dimension three). The sharpest available external check on the pipeline of Section 3 is to run it in the dimension where the answer is published. On $\mathbb{S}^2$ the orbit types are $s \in \{0, 1, 2\}$, the test algebra is generated by p_2, p_3, the test space for degree t is spanned by $\{p_2^\alpha p_3^\beta : 2\alpha + 3\beta \leq \lfloor t/2 \rfloor\}$, and the published maximum degree of a design of type (1) is 17: the bound is the one [9, Remark 3.2] reports from [4, Theorem 2.1], and a 17-design is quoted in [9, Section 6] from [4] — the one recomputed in Remark 6.2(b). The same code returns exactly that: the primal is feasible at $t = 9, 11, 13, 15$, producing exact rational designs on a rational grid of values of A, and at $t = 17$ after one cutting-plane point off that grid, while at $t = 19$ and $t = 21$ the cutting-plane loop returns separating functionals. The row $t = 17$ is the important one, because there a design is known to exist and a method that could manufacture a spurious certificate would do so there; instead every candidate functional fails the exact positivity test until the primal turns feasible. The same row is why a grid can never prove infeasibility: the primal at $t = 17$ is infeasible on the grid $A \in \{j/4 : 1 \leq j \leq 60\}$ and turns feasible only after a cutting-plane point off it, and the four proper parameters of the 17-design quoted in [9, Section 6] are $A = a^2 = 0.296743$, 27.1819, 4.38693 and 0.0978383 to six digits, none of them on that grid. This is why Theorem 3.5 is stated only on the strength of the exactly verified separating functional, and never on the strength of a grid search.

At $t = 19$ the functional the loop returns passes the exact test: its constant term is positive and its two degree-9 polynomials are Sturm-positive on $A > 0$ over $\mathbb{Q}$, so the separation on $\mathbb{S}^2$ at degree 19 is exact, as it is on $\mathbb{S}^3$ at degree 15. Since $-I \in B_3$, the argument of Theorem 3.5 would then give the bound 17 on $\mathbb{S}^2$ from scratch. We do not state that as a theorem: the bound is not new, it is [4, Theorem 2.1] as reported in [9, Remark 3.2], and the $\mathbb{S}^2$ computation is outside the formal development of Section 7, which is about $\mathbb{S}^3$ throughout. At $t = 21$ the returned functional was checked in floating point only.

7. Verification

Nine statements of Sections 3–5 — Lemma 3.2, Propositions 3.3, 4.2, 5.1, 5.2, 5.3 and Theorems 3.4, 3.5, 4.1 — have been formally verified in Lean 4 [6] against *Mathlib* [7]. The development contains no `sorry`; every one of its declarations depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, so nothing is delegated to compiled evaluation and the tactics `native_decide` and `decide` do not occur in it. There is nothing to delegate: the polynomial identities of Lemma 3.2 and of (8)–(10) are certified by ring normalisation, the rational arithmetic of Theorem 4.1 and Propositions 5.1 and 5.2 by normalisation of numerals over $\mathbb{Q}$, and the sign reasoning by linear arithmetic and positivity; no exhaustive search occurs anywhere in the development. What is done by hand is flagged here as such. The reduction of Proposition 2.1 is classical — it is Sobolev's theorem [9, Theorem 2.13], the invariant theory of B_4 and the dimension count of [9, Proposition 2.14] — and it enters the formal development as a *definition*: what is machine-checked is the statement about the system (5), in the form given as the second sentence of Theorem 3.5 and as the hypothesis lists of Theorem 4.1 and Proposition 5.1. The values (4) of the sphere moments are likewise taken as data, computed and cross-checked as described at the end of Section 2, not derived inside Lean. Finally the orbit sizes, the disjointness of the nine orbits and the point count 656 in Theorem 4.1, the halving of the printed weights before Proposition 5.1, the two one-line deductions Corollaries 3.6 and 4.3, the failure of the three weight-6 equations after Proposition 5.1 and of the second weight-7 equation after Proposition 4.2, everything in Remarks 3.7 and 3.8, and everything in Section 6, were done outside the formal development. Repository reference to be added at submission.

References

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[4] S. Heo and Y. Xu, *Constructing fully symmetric cubature formulae for the sphere*, Math. Comp. **70** (2001), no. 233, 269–279; DOI `10.1090/S0025-5718-00-01198-4`. Consulted directly for Remark 6.2(b) only: its Table 2.1 (p. 275) lists the structures and node counts of its formulae, and its appendix (p. 278) tabulates the nodes and weights of three of them, including the degree-17 formula of structure $[1; 1, 0, 2; 2, 0]$. The bound 17 in dimension three is quoted here as [9, Remark 3.2] reports it from Theorem 2.1 of this paper; that theorem is a necessary condition on the structure of a fully symmetric formula, and we have not reworked the deduction of the bound from it.

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[9] K. Tanino, T. Tamaru, M. Hirao and M. Sawa, *More on the corner-vector construction for spherical designs*, Algebr. Comb. **8** (2025), no. 5, 1387–1414; DOI `10.5802/alco.441`; arXiv:2501.11437. Quotations here are from the published article, read on 2026-09-03; the e-print source of version 2 (2025-05-29, still the latest version on 2026-09-03) was read on 2026-09-02 and is where the recomputations of Section 6 were first made. The display labelled `eq:invariant0` in the e-print source, quoted here as (1), is numbered (1) in both versions; every theorem-like item cited above carries the same number in both. Its Section 6 differs between the two versions in the two places recorded in Remark 6.2(a),(b).