

RESTRICTED SCHUR MULTIPLICITY: EXACT MINIMA AND THREE-BLOCK MINIMISERS

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Fix integers $k \geq 2$ and $2 \leq \ell \leq k$, and call a solution of $x_1 + \dots + x_k = x_{k+1}$ in positive integers a solution of level ℓ when its $k+1$ entries take exactly $\ell+1$ distinct values. Let $m(k, \ell; n)$ be the least number of monochromatic level- ℓ solutions inside $[n] = \{1, \dots, n\}$, minimised over all 2-colourings of $[n]$, solutions being counted as multisets. The level ℓ is Gaiser’s, who used it to index a *threshold* — the restricted generalized Schur number $S_2(k; \ell)$, the least n forcing one such solution — and who never counts; counting is the Schur-triple multiplicity problem of Graham–Rödl–Ruciński and Robertson–Zeilberger, which does not leave $k = 2$ and does not restrict the number of distinct values. We compute $m(k, \ell; n)$ exactly at 74 triples spread over seven pairs (k, ℓ) with $k \leq 5$ and $n \leq 55$; ten of them, the row $k = \ell = 2$, reproduce Robertson–Zeilberger’s own objective function and are carried as an external control. At every one of the 74 triples the minimum is attained by a colouring with at most three blocks, $[1, a] \cup (b, n]$ against $(a, b]$. We also prove that $m(k, \ell; \cdot)$ is non-decreasing; that it is positive exactly from $n = S_2(k; \ell)$ on, so that the multiplicity function recovers the threshold as its own zero set and re-proves $S_2(3; 2) = 15$, $S_2(3; 3) = 24$ and $S_2(4; 3) = 35$ by a route that runs through no threshold search at all; and that a two-block colouring gives $m(k, \ell; n) = O((n/k)^\ell)$ for every n . Every statement is machine-checked in Lean 4, except at the few places where we say otherwise; the lower bounds are replays of LRAT refutations of a cardinality-constrained encoding.

1. INTRODUCTION

Two questions about one equation. Let

$$(1) \quad x_1 + x_2 + \dots + x_k = x_{k+1}$$

and let $[n] = \{1, 2, \dots, n\}$. Schur’s theorem is the case $k = 2$ of the statement that for every r there is an n such that every r -colouring of $[n]$ admits a monochromatic solution of (1); the least such n is a generalized Schur number. Two refinements of the question have been pursued independently, and the present paper is about the point where they meet.

The first refinement counts distinct values. For a solution $S = (x_1, \dots, x_{k+1})$ write $c(S)$ for the number of distinct integers among its entries. Gaiser [6] defines, for $r, k, \ell \in \mathbb{N}$ with $\ell \leq k$ and $k \geq 2$, the *restricted generalized Schur number* $S_r(k; \ell)$ to be the least n , if it exists, such that every r -colouring of $[n]$ has a monochromatic solution S of (1) with $c(S) = \ell + 1$, and notes that $S_r(k; k) = S_r^*(k)$, the number that requires the x_i to be distinct — the weak Schur number. The level ℓ interpolates between the fully degenerate solutions and the all-distinct ones. Gaiser’s paper is a threshold paper throughout: it asks when a monochromatic solution of a given level must exist, and never how many there are.

The second refinement counts solutions. Graham, Rödl and Ruciński [7] asked for the asymptotically least number of monochromatic Schur triples in a 2-colouring of $[n]$ — Robertson and Zeilberger [13] report it as the \$100 problem Ron Graham proposed at SOCA 96 — and answered it in [13]: the minimum is $n^2/22 + O(n)$, attained by a colouring with three blocks. Datskovsky [4] obtained the same constant under the ordered convention, where it reads $n^2/11$; Schoen [16] obtained an exact value along a progression, in our own convention (see below); and Koutschan and Wong [9] determined the exact minimum $\lfloor (n^2 - 4n + 6)/11 \rfloor$ under the ordered, degeneracy-allowed convention. This literature is about $k = 2$. For $k \geq 3$ very little is known over $[n]$: Butler, Costello and Graham [3, p. 408] write of $x + y = z$ that “this is currently the only known situation in which the minimal

number of monochromatic solutions in a coloring is known, namely there are at least $(1/22)n^2 + O(n)$ monochromatic solutions”, and Dong, Mani, Pham and Tidor [5], who prove a lower bound of order δn^{k-1} for $a_1x_1 + \dots + a_kx_k = 0$ with $k \geq 3$ and coefficients of both signs, still describe the $[n]$ setting as “far less understood” than its finite-field analogue. The one direction in which the $k = 2$ result has been pushed is a different one: Thanatipanonda and Wong [17] determine the minimum for $x + ay = z$ with $a \geq 2$ to be $n^2/(2a(a^2 + 2a + 3)) + O(n)$, but that is again a three-variable equation, not (1) with $k \geq 3$. In the finite-field setting the corresponding question has a name, the Rado multiplicity problem, and a recent treatment by Rué and Spiegel [15]; that treatment normalises to a density and restricts to all-distinct entries, never to an intermediate level. The transfer of multiplicity questions to further hosts is active: Katona and Mao [8] pose it for partially ordered sets.

The quantity studied here. This paper counts monochromatic solutions of (1) *at a prescribed level* ℓ . For a 2-colouring c of $[n]$ let $M_n(k, \ell; c)$ be the number of monochromatic solutions of (1) inside $[n]$ with exactly $\ell + 1$ distinct entries, counted as multisets (Section 2 fixes the convention), and set

$$m(k, \ell; n) = \min_{c: [n] \rightarrow \{0,1\}} M_n(k, \ell; c).$$

The two parents are the two extreme readings of this quantity: ℓ is Gaiser’s, the minimisation is Graham–Rödl–Ruciński’s, and the pair $(k, \ell) = (2, 2)$ is exactly Robertson–Zeilberger’s problem. That last coincidence is not an analogy. The objective function of [13] is, verbatim,

$$(2) \quad F(x_1, \dots, x_n) = \sum_{\substack{1 \leq i < j \leq n \\ i+j \leq n}} [x_i x_j x_{i+j} + (1 - x_i)(1 - x_j)(1 - x_{i+j})],$$

and the inequality $i < j$ is strict, so every triple F counts has three distinct entries. Hence $M_n(2, 2; c) = F(c)$ identically, and the $k = 2$ row of our tables is a control against published mathematics rather than a contribution.

Results. Table 2 lists 74 exact values of $m(k, \ell; n)$, over seven pairs (k, ℓ) and $9 \leq n \leq 55$. Each value is two-sided: an explicit minimising colouring and a certified exhaustive lower bound. The $k = 2$ row is the control just described (Proposition 5.1); the remaining six rows, 64 values, are Theorems 5.2–5.7. Beyond the table:

- *Three-block minimisers* (Theorem 6.1). At every one of the 74 triples the minimum is attained by a colouring of the shape $[1, a] \cup (b, n]$ against $(a, b]$. For $k = 2$ this is Robertson–Zeilberger’s optimal colouring $Z_0 = 0^{4n/11} 1^{6n/11} 0^{n/11}$; for $k \geq 3$ we have not found the statement in the literature, and it is not a fact one may assume, since for the neighbouring multiplicity problem over $[n]$ — monochromatic 3-term arithmetic progressions — not even the leading constant is known [12].
- *The threshold is the zero set of the count* (Theorem 3.5). One has $m(k, \ell; n) \geq 1$ if and only if $n \geq S_2(k; \ell)$. This ties the whole table to Gaiser’s numbers, and yields second, independent proofs of $S_2(3; 2) = 15$, $S_2(3; 3) = 24$ and $S_2(4; 3) = 35$ (Corollary 3.7).
- *Monotonicity and a general upper bound* (Theorems 3.1 and 3.2). The function $m(k, \ell; \cdot)$ is non-decreasing, and the two-block colouring of $[n]$ that isolates $[1, t]$ with $t \leq n < k(t + 1)$ leaves *exactly* the solutions inside $[1, t]$ monochromatic; taking the least admissible t gives $m(k, \ell; n) = O((n/k)^\ell)$ for every n . This is the only asymptotic statement we prove, and Remark 3.4 shows it is far from tight.

The lower half of each value is the assertion that *no* 2-colouring of $[n]$ does better, which is a statement about 2^n colourings with a cardinality constraint on top. We obtain it by refuting a CNF encoding with a SAT solver and replaying the resulting LRAT certificate inside a proof assistant; Section 4 describes the encoding and Section 9 the verification. What is new mathematically is the values and the shape of the minimisers, not the method.

source	ordered?	$x = y$?	entries distinct?	constant
Robertson–Zeilberger [13] ($i < j$)	no	no	yes, forced	$n^2/22$
Schoen [16], as reported in [14]	no	no	yes	$\frac{1}{22}n^2 - \frac{7}{22}n$, exact
Datskovsky [4]	yes	yes	no	$n^2/11$
Koutschan–Wong [9] (exact)	yes	yes	no	$\lfloor (n^2 - 4n + 6)/11 \rfloor$
here, $k = \ell = 2$	no	no	yes	$n^2/22$

TABLE 1. Counting conventions for monochromatic Schur triples. The bottom row is the top row: $M_n(2, 2; \cdot)$ is the function F of (2).

Conventions matter. A monochromatic Schur triple can be counted in at least three ways — as an ordered triple $(x, y, x + y)$, as an unordered one with $x \leq y$, or as an unordered one with $x < y$ — and the constants differ by factors of two accordingly. Table 1 records which source uses which. Ours is Robertson–Zeilberger’s, the last of the three.

The one source we were unable to obtain is Schoen [16], and secondary accounts of his convention disagree. Parczyk and Spiegel [11] write that in [16] the two summands are “additionally” required to be distinct; Koutschan and Wong [9] write that [16] counts “Schur triples $(x, y, x + y)$ with $x \leq y$ ”. The disagreement can be settled without the primary text, and it is worth settling, because under the first reading Schoen’s theorem is an exact statement about the very function $m(2, 2; \cdot)$ that the $k = 2$ row of Table 2 tabulates.

Rosta’s survey [14] defines a Schur triple as a triple x, y, z with “ $x \neq y$ and $x + y = z$ ” and reports Schoen’s exact value as $S(N) = \frac{1}{22}N^2 - \frac{7}{22}N$. That expression is an integer exactly when $N \equiv 0$ or $7 \pmod{11}$ — which is why [9] correctly describe Schoen’s exact result as holding only along a progression — and it is a sharp fingerprint of the convention: at $N = 11, 18$ and 22 it returns 2, 9 and 15, whereas the minima under the $x \leq y$ and the ordered conventions are 5, 14, 21 and 7, 23, 36 at those N . (Those six numbers are exhaustive searches over all 2^{N-1} colourings, run outside the formal development; the ordered three are $\lfloor (N^2 - 4N + 6)/11 \rfloor$, so [9] checks them.) The first of the three sequences is ours: $m(2, 2; 11) = 2$ and $m(2, 2; 18) = 9$ are entries of Table 2, machine-checked, and the value 15 at $N = 22$ is one more exhaustive search. On this evidence Schoen’s convention is the strict one and [11] is right, though we have not seen his text. We note in passing that the sentence in [9] which ascribes $x \leq y$ to [16] ascribes it to Robertson–Zeilberger as well, and there the primary text is available and is explicit: the sum in (2) runs over $i < j$ strictly. We still rely on [16] nowhere.

2. SETUP

Throughout, $k \geq 2$ and $2 \leq \ell \leq k$ are integers and $n \geq 1$. It is convenient to let a colouring be a map c from the positive integers to $\{0, 1\}$; every count below inspects only the values of c on $[n]$, so “a 2-colouring of $[n]$ ” and “a colouring, read on $[n]$ ” are interchangeable, and we may compare the counts of one colouring at two different n .

Solutions in normal form. Gaiser’s Lemma 2.1 [6] puts a solution of (1) of level ℓ into normal form; read as a statement about individual solutions rather than about a set containing one, it says that a solution S of (1) in positive integers with $c(S) = \ell + 1$, taken as a multiset, is the same datum as a pair (u, a) where $u = (u_1 < u_2 < \dots < u_\ell)$ are the distinct values occurring on the left-hand side and $a = (a_1, \dots, a_\ell)$ are their multiplicities, so that $a_i \geq 1$, $\sum_i a_i = k$, and the right-hand side is $\langle a, u \rangle = \sum_i a_i u_i$. (That $\langle a, u \rangle$ is distinct from every u_i , so that the level really is $\ell + 1$ and not ℓ , is automatic: $\langle a, u \rangle \geq \sum_i u_i > u_\ell$ when $\ell \geq 2$.)

Definition 2.1. $\text{Sol}(k, \ell; n)$ is the set of pairs (u, a) as above with $u_1 \geq 1$ and $\langle a, u \rangle \leq n$; we write $N(k, \ell; n) = |\text{Sol}(k, \ell; n)|$. A pair $(u, a) \in \text{Sol}(k, \ell; n)$ is *monochromatic* under a 2-colouring c of $[n]$

when $c(u_1) = \dots = c(u_\ell) = c(\langle a, u \rangle)$, and

$$M_n(k, \ell; c) = \#\{ (u, a) \in \text{Sol}(k, \ell; n) : (u, a) \text{ is monochromatic under } c \},$$

$$m(k, \ell; n) = \min_c M_n(k, \ell; c).$$

Spelling the normal form out makes the level restriction concrete. The multiplicity vectors available at a given (k, ℓ) are the compositions of k into ℓ positive parts, of which there are $\binom{k-1}{\ell-1}$, so a level- ℓ solution inside $[n]$ is a choice of one of those together with an ℓ -subset $u_1 < \dots < u_\ell$ of $[n]$ whose weighted sum is again at most n . For $(k, \ell) = (2, 2)$ there is one composition and the solutions are $u_1 + u_2 = u_3$ with $u_1 < u_2$, which is precisely the range of the sum in (2). For $(k, \ell) = (3, 2)$ there are two, and the solutions are

$$2u_1 + u_2 = u_3 \quad \text{and} \quad u_1 + 2u_2 = u_3, \quad u_1 < u_2;$$

for $(k, \ell) = (3, 3)$ there is again one, $u_1 + u_2 + u_3 = u_4$ with $u_1 < u_2 < u_3$, the weak Schur case; and for $(k, \ell) = (4, 2)$ there are three, $3u_1 + u_2 = u_3$, $2u_1 + 2u_2 = u_3$ and $u_1 + 3u_2 = u_3$.

Two different multiplicity vectors over the same value tuple are two different solutions, even when they share a right-hand side: their underlying multisets differ. The smallest instance is $k = 6$, $\ell = 3$, $u = (1, 2, 3)$ with $a = (2, 3, 1)$ and $a = (3, 1, 2)$, both giving right-hand side 11 and the distinct multisets $\{1, 1, 2, 2, 2, 3\}$ and $\{1, 1, 1, 2, 3, 3\}$. Any enumeration that records only $(u, \langle a, u \rangle)$ therefore repeats entries, which is harmless for an existence question and fatal for a counting one. Proposition 2.4 records the consequence.

Proposition 2.2 (Statement fidelity). *There is an explicit finite list $L(k, \ell, n)$, obtained by pairing each strictly increasing ℓ -tuple of positive integers with sum at most n against each composition of k into ℓ positive parts and discarding the pairs with $\langle a, u \rangle > n$, such that $(u, a) \in L(k, \ell, n)$ if and only if $(u, a) \in \text{Sol}(k, \ell; n)$, and such that $L(k, \ell, n)$ has no repeated entry. Consequently $M_n(k, \ell; c)$ is the number of monochromatic elements of $\text{Sol}(k, \ell; n)$, not merely the number of monochromatic entries of a list.*

Proof sketch. Soundness and completeness are a single equivalence between membership in the list and the defining conjunction of Definition 2.1; both directions are routine, the only step worth naming being that $\sum_i u_i \leq \langle a, u \rangle \leq n$, which is what makes the sum budget in the tuple enumerator exact. Absence of repetitions follows because both factors are repetition-free — each is generated as a disjoint union over its first entry — and a product of repetition-free lists, filtered, is again repetition-free. No computation is involved. $\square$

Proposition 2.3 (Colour swap). *For every colouring c and every n , $M_n(k, \ell; 1 - c) = M_n(k, \ell; c)$.*

Proof. Being monochromatic does not refer to which colour, so the two counts are counts of the same subset of $\text{Sol}(k, \ell; n)$. $\square$

Proposition 2.3 is not decoration: it licenses fixing the colour of the integer 1, which halves the search space and, in the encoding of Section 4, was measured to roughly halve the size of the certificates.

Proposition 2.4. *At $(k, \ell, n) = (6, 3, 11)$, the enumeration that records a solution as $(u, \langle a, u \rangle)$, forgetting a , has exactly as many entries as $L(6, 3, 11)$, but it is not repetition-free: the entry $((1, 2, 3), 11)$ occurs twice. Meanwhile $L(6, 3, 11)$ is repetition-free.*

Proof. Two finite evaluations at $(k, \ell, n) = (6, 3, 11)$, plus Proposition 2.2 instantiated there. $\square$

The two lengths agreeing is the point: a defect of this kind is invisible to any count taken over the shorter enumeration, and shows only when repetitions are looked for. A multiplicity question therefore cannot reuse an enumeration written for an existence question, which is why Proposition 2.2 carries the repetition-freeness clause.

Colourings by blocks. For $0 \leq a \leq b \leq n$ let $\chi_{a,b}$ be the 2-colouring of $[n]$ given by

$$\chi_{a,b}(x) = 1 \iff a < x \leq b,$$

that is, the middle block $(a, b]$ against $[1, a] \cup (b, n]$. We call these the *three-block* colourings, with the convention that a block may be empty — $a = 0$ gives a two-block colouring, and so does $b = n$. Robertson–Zeilberger’s optimal colouring $Z_0 = 0^{4n/11}1^{6n/11}0^{n/11}$ is of this shape with $(a, b) \approx (4n/11, 10n/11)$. The special case $b = n$, which we write $\lambda_t = \chi_{t,n}$ — the block $[1, t]$ against $(t, n]$ — carries Theorem 3.2.

3. STRUCTURAL FACTS

Theorem 3.1 (Monotonicity). *If $n \leq n'$ then $M_n(k, \ell; c) \leq M_{n'}(k, \ell; c)$ for every colouring c , and consequently $m(k, \ell; n) \leq m(k, \ell; n')$. In particular, a lower bound $m(k, \ell; n) \geq V$ implies $m(k, \ell; n') \geq V$ for every $n' \geq n$.*

Proof. $\text{Sol}(k, \ell; n) \subseteq \text{Sol}(k, \ell; n')$, monochromaticity of a pair does not depend on n , and by Proposition 2.2 both counts are counts of sets, so the inclusion gives the inequality colouring by colouring. Passing to the minimum preserves it. $\square$

The operational content of Theorem 3.1 is that only the *smallest* n at which each value occurs needs an exhaustive lower-bound proof; the 74 values of Table 2 rest on 41 such proofs, and each of them is the cheapest one for its cardinality bound rather than the most expensive.

Theorem 3.2 (The two-block construction is exact). *Let $t \leq n < k(t+1)$. Then*

$$M_n(k, \ell; \lambda_t) = N(k, \ell; t), \quad \text{hence} \quad m(k, \ell; n) \leq N(k, \ell; t).$$

Proof. A solution monochromatic under λ_t has either all its $\ell+1$ integers in $[1, t]$ or all of them in $(t, n]$. In the first case its right-hand side is at most t , so it lies in $\text{Sol}(k, \ell; t)$. The second case is impossible: if every $u_i \geq t+1$ then $\langle a, u \rangle \geq (\sum_i a_i)(t+1) = k(t+1) > n$. Conversely every element of $\text{Sol}(k, \ell; t)$ has all its integers in $[1, t]$ and so is monochromatic under λ_t . The two sets therefore coincide. $\square$

Corollary 3.3. *Put $t = \lceil (n+1)/k \rceil - 1$. For every $n \geq 1$,*

$$m(k, \ell; n) \leq N(k, \ell; t) \leq \binom{t}{\ell} \binom{k-1}{\ell-1};$$

in particular, for fixed k and ℓ , $m(k, \ell; n) = O(n^\ell/k^\ell)$.

Proof. One has $t \leq n$ and $k(t+1) \geq n+1 > n$, so Theorem 3.2 applies and gives the first inequality. For the second, a member of $\text{Sol}(k, \ell; t)$ is determined by an ℓ -subset of $[1, t]$ together with a composition of k into ℓ positive parts. Finally $t \leq (n+1)/k$. The counting step here is elementary and is not part of the formal development; Theorem 3.2 is. $\square$

Remark 3.4. The bound of Theorem 3.2 is far from tight. At $(k, \ell, n) = (3, 2, 20)$ the value $t = 7$ is admissible and $N(3, 2; 7) = 7$, whereas $m(3, 2; 20) = 3$. In particular the hypothesis $t \leq n < k(t+1)$ is satisfiable, so the statement is not vacuous, and the two-block family does not contain a minimiser in general — which is what makes Theorem 6.1 a statement about *three* blocks.

The threshold as the zero set of the count. Gaiser’s number $S_2(k; \ell)$ is the least n such that every 2-colouring of $[n]$ has a monochromatic solution of level ℓ . The next theorem says that the multiplicity function knows it.

Theorem 3.5. *For all k, ℓ, n with $\ell \geq 1$,*

$$m(k, \ell; n) \geq 1 \iff \text{every 2-colouring of } [n] \text{ has a monochromatic level-}\ell \text{ solution,}$$

and when $S_2(k; \ell)$ exists — for $r = 2$ colours that is exactly when $\ell \geq 2$, by Gaiser’s Theorem 2.4 — both are equivalent to $n \geq S_2(k; \ell)$. Consequently, if c is a colouring with $M_{N-1}(k, \ell; c) = 0$ and $m(k, \ell; N) \geq 1$, then $S_2(k; \ell) = N$.

Proof sketch. The first equivalence is the assertion that $M_n(k, \ell; c) > 0$ if and only if some colour class of c contains a solution of level ℓ ; both directions go through the normal form of Section 2, and we prove them against Gaiser’s own predicate (“some colour class contains $x_1, \dots, x_k$ positive with $x_1 + \dots + x_k$ of the same colour and exactly ℓ distinct values among the x_i ”) rather than against our enumeration, so that a mismatch between the two would surface here. The second equivalence is the definition of $S_2(k; \ell)$ once colourings by $\{0, 1\}$ and colourings by a 2-element index set are identified; that identification is bookkeeping. No computation is involved. $\square$

Proposition 3.6. $M_{14}(3, 2; \chi_{3,12}) = 0$, $M_{23}(3, 3; \chi_{5,20}) = 0$ and $M_{34}(4, 3; \chi_{6,30}) = 0$. In particular $m(3, 3; 23) = 0$, so the lower bound $m(3, 3; 23) \geq 1$ is false.

Proof. Three finite evaluations of M_n over the enumeration of Proposition 2.2. $\square$

Proposition 3.6 is a control rather than a result, and a necessary one: without it the whole of Table 2 would be consistent with a count that never vanishes, in which case Theorem 3.1 would force positivity everywhere and every value in the table would be an artefact. It is also worth noting that all three avoiding colourings are themselves three-block colourings — the degenerate, level-0 end of the pattern of Theorem 6.1.

Corollary 3.7. $S_2(3; 2) = 15$, $S_2(3; 3) = 24$ and $S_2(4; 3) = 35$.

Proof. Combine Theorem 3.5 with Proposition 3.6 and the lower bounds $m(3, 2; 15) \geq 1$, $m(3, 3; 24) \geq 1$ and $m(4, 3; 35) \geq 1$, which are the first entries of Theorems 5.2, 5.3 and 5.5. $\square$

None of these three values is new to us. The first is Gaiser’s Theorem 1.1, which gives $S_2(k; 2) = k^2 + 2k$ for $k \geq 3$; the second is the weak Schur number $S_2^*(3) = 24$ tabulated by Ahmed, Boza, Revuelta and Sanz [1] and recorded as OEIS A365190 [10]; the third we obtained previously by an exhaustive threshold search, in work not yet published, and we have not found it in the literature. What Corollary 3.7 adds is a second, independent derivation of each: the positive halves are refutations of a *cardinality* formula at bound 0, a different formula from the one a pure threshold search uses — different variables, different clause order, a counting tree on top — so the agreements are confirmations rather than restatements. Four further thresholds fall out of the same computation outside the formal development and also agree with the tabulated values: $S_2(2; 2) = 9$, $S_2(4; 2) = 24$, $S_2(4; 4) = 52$ and $S_2(5; 2) = 35$.

4. THE ENCODING

The upper half of each value in Table 2 is a single colouring, and evaluating $M_n(k, \ell; \chi_{a,b})$ against the enumeration of Proposition 2.2 settles it. The lower half — *no* 2-colouring of $[n]$ has fewer than V monochromatic solutions — is the hard side. We encode its negation as a propositional formula and refute the formula.

Fix k, ℓ, n , a bound B (so that the target is $m(k, \ell; n) \geq B + 1$), and a depth d with $N(k, \ell; n) \leq 2^d$; put $\text{cap} = B + 1$. The variables are

- x_i for $i \in [n]$: “ i has colour 1”;
- $o_{e,i,j}$: “at least $j + 1$ of the solutions whose index lies in $[i \cdot 2^e, (i + 1) \cdot 2^e)$ are monochromatic”, for $0 \leq e \leq d$, $0 \leq i < 2^{d-e}$ and $0 \leq j < \text{cap}$.

The clauses are, in this order: two per solution, forcing its leaf output when all of its $\ell + 1$ integers agree in colour; the unit clause $\neg x_1$, the colour-swap symmetry break licensed by Proposition 2.3; the merge clauses of a totalizer [2] on the leaf outputs, in the form $o_{e-1,2i,\alpha-1} \wedge o_{e-1,2i+1,\beta-1} \rightarrow o_{e,i,\alpha+\beta-1}$ with $1 \leq \alpha + \beta \leq \text{cap}$; and the unit clause $\neg o_{d,0,\text{cap}-1}$, saying that the root does not see $B + 1$ monochromatic solutions. The leaf array is padded from $N(k, \ell; n)$ up to 2^d with dummy solutions that are never monochromatic.

Theorem 4.1 (Encoding fidelity). *Let $\ell \geq 1$, $n \geq 1$ and $N(k, \ell; n) \leq 2^d$. If c is a 2-colouring of $[n]$ with $c(1) = 0$ and $M_n(k, \ell; c) \leq B$, then the assignment reading $x_i = c(i)$ and $o_{e,i,j} = 1$ exactly when*

k	ℓ	n	$m(k, \ell; n)$
2	2	9, ..., 18	1, 2, 2, 3, 4, 5, 6, 7, 8, 9
3	2	15, ..., 28	1, 1, 1, 2, 3, 3, 4, 4, 4, 5, 6, 7, 8, 9
3	3	24, ..., 34	1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 6
4	2	24, ..., 37	1, 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7
4	3	35, ..., 43	1, 1, 1, 1, 2, 3, 3, 3, 4
4	4	52, ..., 55	1, 1, 1, 1
5	2	35, ..., 46	1, 1, 1, 1, 1, 2, 3, 3, 3, 3, 4, 5

TABLE 2. The 74 exact values. Each row begins at $n = S_2(k; \ell)$, the first n at which the count is positive. The $k = 2$ row is Robertson–Zeilberger’s quantity and is carried as a control.

at least $j + 1$ of the solutions in the window $[i \cdot 2^e, (i + 1) \cdot 2^e)$ are monochromatic under c satisfies the formula above. Consequently, if the formula is unsatisfiable then $m(k, \ell; n) \geq B + 1$.

Proof sketch. The leaf clauses hold because a monochromatic solution forces its leaf variable, and the merge clauses hold because the window counts add: the count over a window is the sum of the counts over its two halves. The root unit holds because the total count is at most B . The hypothesis $c(1) = 0$ is removed in the second sentence by Proposition 2.3: if $c(1) = 1$, apply the first sentence to $1 - c$, which has the same count.

Only the implication “colouring $\Rightarrow$ satisfying assignment” is proved, because that is the only direction a refutation needs; the totalizer accordingly carries only its upward implications, and an omitted clause can only make the solver’s task harder. The proof needs no fresh-variable bookkeeping: the padding to 2^d leaves makes the index map $(e, i, j) \mapsto n + \text{cap} \cdot (2^d e + i) + j$ rectangular, so it is inverted by two divisions. $\square$

Remark 4.2. The choice of a totalizer over a sequential counter is the design decision of the encoding, and it was made by measurement rather than by taste. On the single cell $k = 3, \ell = 3, n = 33, V = 5$, with the same solver on the same machine, a sequential counter produced 28.8 MB of certificate, a sequential counter with the symmetry unit 16.3 MB, an unpadded totalizer with the unit 1.29 MB, and the padded totalizer used here 2.14 MB — a factor of 13.5 against the plain counter, which is what brought the $\ell = 3$ rows inside our budget. Padding costs a factor of 1.7 in certificate size against the unpadded tree and buys the rectangular layout above. These are measurements of an implementation, not theorems.

5. THE TABLES

Each of the following theorems asserts the corresponding row of Table 2. We state them separately because their standing differs: the first is a control against published mathematics, for the remaining six we have not found the values in the literature.

Proposition 5.1 (Control row). *For $n = 9, 10, \dots, 18$,*

$$m(2, 2; n) = 1, 2, 2, 3, 4, 5, 6, 7, 8, 9.$$

By (2) these are values of Robertson–Zeilberger’s minimum, and it is their theorem $\min F = n^2/22 + O(n)$ that they must be consistent with. They are. Outside the formal development we ran an independent minimisation over three-block colourings at larger n ; it returns 145, 423, 847, 1416 at $n = 60, 100, 140, 180$, which are upper bounds for $m(2, 2; n)$ and, by [13], asymptotically correct ones. The corresponding ratios to n^2 are 0.0403, 0.0423, 0.0432, 0.0437, against $1/22 = 0.04545$; the optimal break points are $a/n = 0.350, 0.360, 0.357, 0.361$ and $b/n = 0.900, 0.900, 0.900, 0.906$, against Robertson–Zeilberger’s $4/11 = 0.3636$ and $10/11 = 0.9091$. Datskovsky’s constant $n^2/11$ [4] is the same number under the ordered convention. We stress that the exact small values above are offered as a control and never as a result; we could not find them in a published table, since Koutschan and

Wong’s $\lfloor (n^2 - 4n + 6)/11 \rfloor$ (OEIS A321195, [9, 10]) counts the ordered, degeneracy-allowed solutions — their own $n = 6$ example lists $(2, 2, 4)$ and $(3, 3, 6)$, which the level restriction discards — and Schoen’s exact formula, in the form [14] reports it, is defined only where $\frac{1}{22}n^2 - \frac{7}{22}n$ is an integer. At the two such n in the range above, $n = 11$ and $n = 18$, it returns 2 and 9, which are our values. Whether Schoen’s theorem in fact covers those two n we cannot say — [9] report its range as $n = 22k$ — but the agreement is a check either way (Section 1).

Theorem 5.2. For $n = 15, 16, \dots, 28$,

$$m(3, 2; n) = 1, 1, 1, 2, 3, 3, 4, 4, 4, 5, 6, 7, 8, 9.$$

Theorem 5.3. For $n = 24, 25, \dots, 34$,

$$m(3, 3; n) = 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 6.$$

Theorem 5.4. For $n = 24, 25, \dots, 37$,

$$m(4, 2; n) = 1, 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7.$$

Theorem 5.5. For $n = 35, 36, \dots, 43$,

$$m(4, 3; n) = 1, 1, 1, 1, 2, 3, 3, 3, 4.$$

Theorem 5.6. $m(4, 4; n) = 1$ for $n = 52, 53, 54, 55$.

Theorem 5.7. For $n = 35, 36, \dots, 46$,

$$m(5, 2; n) = 1, 1, 1, 1, 1, 2, 3, 3, 3, 3, 4, 5.$$

Two of these rows deserve a word. The row $\ell = k = 4$ of Theorem 5.6 is the *diagonal*, all-distinct level: it begins at the weak Schur number $S_2^*(4) = 52$ [1, 10], so its starting point is a published number, and what is not published is that the multiplicity then stays at 1 for four consecutive n . The all-distinct restriction is exactly the one Rué and Spiegel [15] impose, but they work in $\mathbb{F}_q^n$ and normalise to a density. The row $\ell = 3$ of Theorem 5.3 sits at a level the classical literature has no analogue of at all, since at $k = 2$ the only non-degenerate level is $\ell = 2$. Neither row is monotone in any evident way, and we know no closed form for any of the six.

Proof of Proposition 5.1 and Theorems 5.2–5.7. Every value rests on exhaustive computation, in both halves, and we say precisely which.

Upper bounds. For each of the 74 triples (k, ℓ, n) we exhibit a three-block colouring $\chi_{a,b}$ and evaluate $M_n(k, \ell; \chi_{a,b})$ by walking the list $L(k, \ell, n)$ of Proposition 2.2; the value of that evaluation is the tabulated number, so $m(k, \ell; n)$ is at most it. The break points (a, b) are listed in Table 4. They were found by an exhaustive search over all $O(n^2)$ pairs $0 \leq a \leq b \leq n$ outside the formal development; that search is not trusted, only the resulting witness is checked.

Lower bounds. For each of the 41 pairs (n, V) listed in Table 3 — the smallest n at which the value V occurs, for each value V occurring in the row — the formula of Section 4 at $B = V - 1$ was refuted by a SAT solver and the refutation replayed against Theorem 4.1, giving $m(k, \ell; n) \geq V$. The refutation is an exhaustive statement about all 2^{n-1} colourings with $c(1) = 0$ (the halving is Proposition 2.3), together with the cardinality constraint; there is no case-by-case enumeration of colourings anywhere in the argument, but the certificate is a complete resolution-style proof of the exhaustion. Theorem 3.1 then carries each lower bound up to the next break point, which is why 41 refutations settle 74 values.

The two halves meet. At each triple the upper bound and the lower bound coincide, and the common value is the entry of Table 2. $\square$

Remark 5.8. The wall this method meets is the cardinality bound B , not n . On the row $k = 3, \ell = 3$ the certificate grows from 2.14 MB at $B = 4$ ($n = 33$) to 3.31 MB at $B = 5$ ($n = 34$), 5.12 MB at $B = 6$ and 83.6 MB at $B = 12$ ($n = 40$); on $k = \ell = 2$ from 0.96 MB at $B = 8$ ($n = 18$) to 52.4 MB at $B = 18$ ($n = 24$) and 416.8 MB at $B = 26$ ($n = 28$). Each row can therefore be extended until

k	ℓ	break points $n(V)$, and $[N(k, \ell; n); \text{clauses}; \text{LRAT}]$
2	2	9(1), 10(2), 12(3), 13(4), 14(5), 15(6), 16(7), 17(8), 18(9) [72; 7 004; 0.96 MB]
3	2	15(1), 18(2), 19(3), 21(4), 24(5), 25(6), 26(7), 27(8), 28(9) [173; 14 118; 1.26 MB]
3	3	24(1), 27(2), 30(3), 31(4), 33(5), 34(6) [865; 29 353; 3.31 MB]
4	2	24(1), 28(2), 29(3), 32(4), 33(5), 36(6), 37(7) [273; 18 433; 0.74 MB]
4	3	35(1), 39(2), 40(3), 43(4) [2 593; 62 518; 1.83 MB]
4	4	52(1) [8 778; 50 324; 1.99 MB]
5	2	35(1), 40(2), 41(3), 45(4), 46(5) [385; 10 992; 0.35 MB]

TABLE 3. The 41 refutations. Entry $n(V)$ means: the formula at (k, ℓ, n) with bound $B = V - 1$ is unsatisfiable, hence $m(k, \ell; n) \geq V$. The bracket gives, for the last break point of each row, the number of solutions, the number of clauses and the certificate size. Total certificate size over all 41: 19.3 MB; total solver time under four seconds.

its value reaches roughly 7, and not much further without a better encoding. New pairs (k, ℓ) , whose rows start at $B = 0$, are the cheap direction.

Remark 5.9. For orientation only, and with the caveat that the four values are taken at four different n : at the top of each $\ell = 2$ row the ratio $m(k, 2; n)/n^2$ is 0.0278, 0.0115, 0.0051, 0.0024 for $k = 2, 3, 4, 5$ at $n = 18, 28, 37, 46$. This is an indication of the k -dependence that an asymptotic treatment will have to explain, not a limit.

6. THREE-BLOCK MINIMISERS

Theorem 6.1. *At every one of the 74 triples (k, ℓ, n) of Table 2 there are $0 \leq a \leq b \leq n$ with*

$$M_n(k, \ell; \chi_{a,b}) = m(k, \ell; n),$$

namely the pairs listed in Table 4. That is, the three-block colourings contain a minimiser at every computed cell, across seven pairs (k, ℓ) and $9 \leq n \leq 55$.

Proof. For each triple, the upper-bound half of Proposition 5.1 and Theorems 5.2–5.7 is precisely the evaluation $M_n(k, \ell; \chi_{a,b}) = V$ for the listed pair (a, b) , and the lower-bound half is $m(k, \ell; n) \geq V$; together they give the displayed equality. So the theorem is exactly as strong as the table it accompanies, and rests on the same exhaustive computations. $\square$

Remark 6.2 (A second, independent check). The proof above obtains three-block optimality from the certified minimum. We also checked it the other way round, outside the formal development and by two independent exhaustive searches: a branch-and-bound minimisation of $M_n(k, \ell; \cdot)$ over all 2^{n-1} colourings of $[n]$ (colour of 1 fixed by Proposition 2.3), and an exhaustive minimisation over the $O(n^2)$ three-block colourings only. The two agree at all 74 triples, with no mismatch. The first of these searches is what produced the values before any of them was proved.

For $k = 2$ the shape in Theorem 6.1 is Robertson–Zeilberger’s own optimal colouring $Z_0 = 0^{4n/11}1^{6n/11}0^{n/11}$ [13], so the content of the theorem is its $k \geq 3$ half. We have not found a rule-by-rule or shape statement for $k \geq 3$ in the literature; that negative should be read as “not found”, not as “new”. It is worth saying why it is not a statement one may assume. For the nearest neighbour of this problem — the minimum number of monochromatic 3-term arithmetic progressions in a 2-colouring of $[n]$, a multiplicity question over $[n]$ whose configurations are automatically distinct — not even the leading constant is known: Parrilo, Robertson and Saracino [12] bracket it between $(1675/32768)n^2(1 + o(1))$ and $(117/2192)n^2(1 + o(1))$. “Some simple shape is optimal” is therefore not a principle one may invoke in this area; at the 74 triples above it is a checked fact.

Question 6.3. Is it true for all $k \geq 3$, $2 \leq \ell \leq k$ and all sufficiently large n that $m(k, \ell; n)$ is attained by a three-block colouring, and if so, what is $\lim_{n \rightarrow \infty} m(k, \ell; n)/n^\ell$?

k	ℓ	(a, b) for n increasing through the row of Table 2
2	2	(3, 8), (0, 4), (4, 10), (4, 10), (4, 11), (4, 12), (5, 13), (5, 14), (6, 15), (6, 16)
3	2	(0, 4), (4, 14), (4, 15), (4, 15), (4, 15), (5, 18), (0, 6), (6, 20), (6, 21), (6, 21), (6, 21), (6, 22), (7, 24), (7, 24)
3	3	(5, 20), (6, 22), (6, 23), (6, 23), (7, 25), (7, 26), (7, 26), (7, 27), (8, 29), (8, 29), (8, 30)
4	2	(0, 5), (5, 22), (5, 23), (5, 24), (5, 24), (5, 24), (6, 27), (6, 28), (6, 28), (6, 28), (7, 31), (7, 32), (7, 32), (7, 32)
4	3	(6, 30), (7, 32), (7, 33), (7, 34), (7, 34), (7, 35), (8, 37), (8, 38), (8, 38)
4	4	(9, 45), (10, 47), (10, 48), (10, 49)
5	2	(0, 6), (6, 32), (6, 33), (6, 34), (6, 35), (6, 35), (6, 35), (7, 38), (7, 39), (7, 40), (7, 40), (7, 40)

TABLE 4. A minimising three-block colouring $\chi_{a,b}$ at each of the 74 triples: colour 1 on $(a, b]$, colour 0 on $[1, a] \cup (b, n]$. Entries with $a = 0$ are two-block colourings. These are the pairs our search returned; we do not claim they are the only minimising pairs.

If the answer to the first half is yes, the second half becomes a two-variable minimisation of a piecewise polynomial in the break-point fractions $\alpha = a/n$ and $\beta = b/n$, and the missing ingredient is a closed form for $M_n(k, \ell; \chi_{a,b})$ as a function of n — a lattice-point count in a polytope, one per composition of k into ℓ parts. At $(k, \ell) = (2, 2)$ the count is three such regions and can be done by hand.

Proposition 6.4. *Let $0 \leq a \leq b \leq n$ satisfy $2a \leq b$, $2b \geq n$ and $n - b - 1 \leq a$. Then*

$$M_n(2, 2; \chi_{a,b}) = \left\lfloor \frac{(a-1)^2}{4} \right\rfloor + \left\lfloor \frac{(b-2a-1)^2}{4} \right\rfloor + \frac{(n-b)(n-b-1)}{2},$$

the three summands counting, respectively, the monochromatic solutions with all three entries in $[1, a]$, those with all three entries in $(a, b]$, and those with $x \in [1, a]$ and $y, x + y \in (b, n]$.

Proof. Write $A = [1, a]$, $B = (a, b]$ and $C = (b, n]$, so that $\chi_{a,b}$ gives B one colour and $A \cup C$ the other. By the discussion after Definition 2.1 a level-2 solution inside $[n]$ is a pair of positive integers $x < y$ with $x + y \leq n$, and it is monochromatic when x, y and $x + y$ share a colour. We use throughout the elementary count

$$(3) \quad \#\{(x, y) : 1 \leq x < y, x + y \leq M\} = \left\lfloor \frac{(M-1)^2}{4} \right\rfloor \quad (M \geq 0),$$

which holds because the pairs with $x + y = s$ number $\lfloor (s-1)/2 \rfloor$, so that the left side of (3) equals $\sum_{t \leq M-1} \lfloor t/2 \rfloor = \lfloor (M-1)^2/4 \rfloor$.

A solution monochromatic in B is a pair with $x, y, x + y \in (a, b]$. There $x + y > 2a \geq a$ holds automatically, and $x + y \leq b$ with $x \geq 1$ forces $y < b$; so these solutions are exactly the pairs $a < x < y$ with $x + y \leq b$. Writing $x = a + i$ and $y = a + j$ turns that into (3) with $M = b - 2a \geq 0$, which is the second summand.

For the colour $A \cup C$, note that every element of A is smaller than every element of C , so $x < y$ leaves three positions for the pair: both in A , one in each, or both in C . If $x, y \in A$ then $x + y \leq 2a - 1 < b$, so $x + y$ cannot lie in C and the solution is monochromatic exactly when $x + y \leq a$, which is (3) with $M = a$: the first summand. (This is where $2a \leq b$ is used.) If $x, y \in C$ then $x + y \geq 2b + 3 > n$ because $2b \geq n$, so there is no such solution. Finally, if $x \in A$ and $y \in C$ then $x + y > y > b$, so $x + y$ lies in C as soon as it is at most n , and the only constraint is $x + y \leq n$. Summing the number $\max(0, n - b - x)$ of admissible y over $x = 1, \dots, a$, and using $n - b - 1 \leq a$ to see that no admissible x is omitted, gives $\sum_{m=1}^{n-b-1} m = (n-b)(n-b-1)/2$: the third summand. $\square$

In terms of $\alpha = a/n$ and $\beta = b/n$ the three summands are $(\frac{1}{4}\alpha^2 + \frac{1}{4}(\beta - 2\alpha)^2 + \frac{1}{2}(1 - \beta)^2)n^2 + O(n)$, a positive definite quadratic whose unique critical point is $\alpha = 4/11$, $\beta = 10/11$ — in the interior of the region where Proposition 6.4 applies — and whose value there is $1/22$. So on that part of the two-parameter family the minimum is $n^2/22 + O(n)$, attained at Robertson–Zeilberger’s break points: their value at their colouring, as it must be. The numerics agree: at $(n, a, b) = (60, 21, 54)$, $(100, 36, 90)$, $(140, 50, 126)$ and $(180, 65, 163)$ — the optimal three-block colourings found outside the formal development — Proposition 6.4 returns 145, 423, 847 and 1416, the values quoted after Proposition 5.1. Proposition 6.4 is elementary and, like the rest of this discussion, is not part of the formal development.

Note also that Corollary 3.3 bounds $m(k, \ell; n)$ above by a constant times n^ℓ — the same order as the total number of level- ℓ solutions inside $[n]$, so the two-block construction saves only a constant factor — while no lower bound of order n^ℓ is available to us. The δn^{k-1} bound of Dong, Mani, Pham and Tidor [5] does not supply one: it is proved for $a_1x_1 + \dots + a_kx_k = 0$ with all variables free, whereas the level- ℓ solutions counted here number only $\Theta(n^\ell)$ in total.

7. NEGATIVE CONTROLS

A development in which every theorem confirms can be satisfied by a misplaced quantifier. We therefore record refutations at both ends of the table.

Proposition 7.1. *At $(k, \ell, n) = (3, 3, 24)$: the claims $m = 2$ and $m = 0$ are both false; no 2-colouring of [24] avoids all level-3 solutions, so the upper bound 0 is false; and some 2-colouring achieves 1, so the lower bound 2 is false. At the top of the $k = 4$, $\ell = 3$ row, $(4, 3, 43)$: the claims $m = 3$ and $m = 5$ are both false. Finally the objective is nowhere near constant: at $(3, 3, 24)$ the all-one colouring makes all $N(3, 3; 24) = 274$ solutions monochromatic, against the minimum 1.*

Proof. Each refutation is one of the two halves of a value in Table 2 combined with a strict inequality; the last statement is a finite evaluation of M_{24} over $L(3, 3, 24)$. $\square$

The last sentence of Proposition 7.1 is the scale control: it shows that the minimisation is not vacuous, the objective ranging from 1 to 274 at that single cell.

8. FURTHER DIRECTIONS

Besides Question 6.3, two directions seem to us both natural and within reach.

A lower bound of the right order. The level- ℓ solutions inside $[n]$ number $\Theta(n^\ell)$ in total, and Corollary 3.3 gives a matching upper bound for the minimum; we have no lower bound of that order. As noted above, the δn^{k-1} bound of [5] does not apply, its hypotheses concerning the equation with all variables free. The route that suggests itself is to split the count according to the greatest common divisor of a solution. Call $(u, a) \in \text{Sol}(k, \ell; n)$ *primitive* when $\gcd(u_1, \dots, u_\ell) = 1$, write $P_n(k, \ell; \gamma)$ for the number of primitive solutions inside $[n]$ that are monochromatic under γ , and put $c_d(x) = c(dx)$. Factoring $u = gu'$ with u' primitive is a bijection of $\text{Sol}(k, \ell; N)$ onto the disjoint union over $d \geq 1$ of the primitive solutions inside $[\lfloor N/d \rfloor]$, and (du', a) is monochromatic under c exactly when (u', a) is monochromatic under c_d ; hence

$$M_N(k, \ell; c) = \sum_{d \geq 1} P_{\lfloor N/d \rfloor}(k, \ell; c_d).$$

If there were a *primitive threshold* $T = T(k, \ell)$ — a least n such that every 2-colouring of $[n]$ has a monochromatic primitive level- ℓ solution — then each of the $\lfloor N/T \rfloor$ values of d with $\lfloor N/d \rfloor \geq T$ would contribute at least one to the sum, giving $m(k, \ell; N) \geq \lfloor N/T \rfloor$: linear in N against a truth of order N^ℓ . It is worth recording that even this is not available, because *no primitive threshold exists when k is even*. Colour $[n]$ by parity. A primitive solution cannot have all of $u_1, \dots, u_\ell$ even; and if all of them are odd then $\langle a, u \rangle \equiv \sum_i a_i = k \equiv 0 \pmod{2}$, so the right-hand side takes the other colour. Either way the solution is not monochromatic, so for even k the parity colouring of $[n]$ has, for *every* n , no monochromatic primitive level- ℓ solution at all, and the route above says nothing on four of the

seven rows of Table 2. For odd k the parity obstruction disappears, but we do not know that $T(k, \ell)$ exists. This paragraph is elementary and is not part of the formal development.

More colours. Nothing in Definition 2.1 needs two colours, and $m_r(k, \ell; n)$, the minimum over r -colourings, is the obvious next object; the threshold control of Theorem 3.5 becomes a control against $S_r(k; \ell)$. At $r = 3$, $k = \ell = 2$ there is again a published statement to check against: the final section of [13] gives the r -colour upper bound $n^2/(2^{2r-3} \cdot 11) + O(n)$. On the computational side the colour-swap symmetry break of Proposition 2.3 becomes a break by the full symmetric group on r letters, and the level $\ell = 1$ existence question changes shape.

9. VERIFICATION

Every result of this paper is formally verified in Lean 4 against *Mathlib*, with the four exceptions listed at the end of this section; the repository reference is to be added at submission. The development is free of `sorry`. The structural results — Propositions 2.2 and 2.3, Theorems 3.1, 3.2, 3.5 and 4.1 — are ordinary proofs depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`; in particular the encoding fidelity of Theorem 4.1 contains no appeal to compiled evaluation at all, so a bug in the encoding could not manufacture a lower bound, only make a formula harder or easier to refute. What is delegated to compiled evaluation, through Lean’s `native_decide`, is exactly the finite searches: one evaluation of $M_n(k, \ell; \chi_{a,b})$ over the enumeration of Proposition 2.2 for each of the 74 triples; at each of the 41 break points of Table 3, one replay of an LRAT certificate through Lean’s verified checker together with one evaluation of the depth bound $N(k, \ell; n) \leq 2^d$; and eight further evaluations, for Propositions 2.4, 3.6 and 7.1 and Remark 3.4. The refutations were produced by the solver `cadical` and are checked, not trusted: the certificate is replayed against the formula that Lean itself builds. Because an LRAT certificate names input clauses by position, the clause order of the external generator that produced the files must agree with the in-Lean formula byte for byte; that agreement was checked by diffing a dump of the in-Lean formula against the generator at $(k, \ell, n, B) = (3, 2, 15, 0)$, $(3, 3, 30, 2)$, $(4, 2, 28, 1)$ and $(2, 2, 13, 3)$, covering the levels $\ell = 2$ and 3 and the tree depths 6, 8 and 10. Every value of Table 2 was first produced by two independent implementations in C — one enumerating solutions as non-decreasing k -tuples, one as normal forms — which agree with each other and with the formal statements at all 74 triples.

Four things in this paper are deliberately *not* machine-checked, and are marked as such where they occur: the elementary counting step of Corollary 3.3, Proposition 6.4, and the dilation identity and parity obstruction of Section 8, all of which are ordinary hand proofs; the large- n numerics after Proposition 5.1 and after Proposition 6.4, together with the convention-discriminating searches of Section 1, all of which are computations outside the formal development; the certificate-size measurements, which are properties of an implementation rather than statements about m ; and the exhaustive searches reported in the remark after Theorem 6.1, which are checks on results the formal development proves anyway.

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