

**RANK-METRIC INTERSECTING CODES AT THE OPEN CELL $[8, 3]_{q^6/q}$:
CERTIFIED WITNESSES, EVERY PUNCTURING
OF THE EXTREMAL EXAMPLE, AND THE EXCLUSION
OF $\mathbb{F}_4$ -LINEAR SYSTEMS AT $q = 2$**

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ABSTRACT. A rank-metric code $C \subseteq \mathbb{F}_{q^m}^n$ is *rank-metric intersecting* when the rank supports of any two nonzero codewords meet nontrivially (Bartoli, Borello, Marino and Scotti, arXiv:2507.00569); a nondegenerate $[n, k, d]_{q^m/q}$ code has this property if and only if its q -system $U \subseteq \mathbb{F}_{q^m}^k$ is not 2-spannable, i.e. $U \neq (H_1 \cap U) + (H_2 \cap U)$ for every pair of $\mathbb{F}_{q^m}$ -hyperplanes. Borello, Polverino and Zullo (arXiv:2604.02004) settle the admissible lengths for $k = 3$, $m = 6$ except one: such codes exist for $5 \leq n \leq 7$ and $n = 9$, cannot exist for $n < 5$ or $n > 9$, and the existence of an $[8, 3]_{q^6/q}$ rank-metric intersecting code is left open; they report that an extensive search for $[8, 3]_{64/2}$ codes obtained by puncturing their extremal $[9, 3, 5]_{64/2}$ example fails. We study the cell at $q = 2$ through a finite model of $\text{PG}(2, 64)$ and a criterion that decides 2-spannability from the largest hyperplane sections alone. We certify the known part of the row (the $[9, 3, 5]_{64/2}$ example, whose q -system is maximum scattered, and explicit $[5, 3, 3]$, $[6, 3, 3]$, $[7, 3, 4]$ codes over $\mathbb{F}_{64}/\mathbb{F}_2$), turn the reported puncturing search into a theorem — all 511 rank-metric puncturings of the example are 2-spannable, each with an explicit spanning pair of hyperplanes — and prove that no $[8, 3]_{64/2}$ rank-metric intersecting code has an $\mathbb{F}_4$ -linear q -system, exhaustively over 1365 systems that cover every equivalence class of such systems. The exclusion itself is routine, as we point out: an $\mathbb{F}_4$ -linear system of $\mathbb{F}_2$ -dimension 8 in $\mathbb{F}_{64}^3$ meets some hyperplane in dimension 6, so its code has $d \leq 2 < k$; what the exhaustive theorem adds is the certificates and the completeness of the list. A double count yields an identity for the hyperplane weights of every 8-dimensional $\mathbb{F}_2$ -subspace of $\mathbb{F}_{64}^3$, which forces an intersecting $[8, 3]_{64/2}$ system to have exactly $31 + 4n_2 + 28n_3$ hyperplanes of section size 16, pairwise meeting nontrivially; the best candidate produced by a search over $5.5 \cdot 10^7$ subspaces is scattered, has exactly 31 such hyperplanes and still fails on 108 of their 465 pairs, and it is certified as such. The cell itself remains open; non-existence is not claimed. Every finite statement is verified in Lean 4, without Mathlib.

1. INTRODUCTION

Rank-metric codes were introduced by Delsarte [4] as sets of matrices over a finite field with the rank distance, and independently by Gabidulin [5] and Roth [6] in the vector form used here: an $\mathbb{F}_{q^m}$ -subspace $C \subseteq \mathbb{F}_{q^m}^n$ with $d(x, y)$ the rank over $\mathbb{F}_q$ of the matrix expansion of $x - y$. Codes meeting the Singleton-like bound (MRD codes) and their constructions — Gabidulin's, and the twisted Gabidulin codes of Sheekey [7] — are the best studied part of the theory; the geometric viewpoint that we use, in which a nondegenerate code is replaced by an $\mathbb{F}_q$ -subspace of $\mathbb{F}_{q^m}^k$ called its q -system, goes back to Randrianarisoa [13] and Alfaro, Borello, Neri and Ravagnani [14], and is the vector-space form of the theory of linear sets (Polverino [16]; the duality between rank-metric codes and linear sets is in Sheekey and Van de Voorde [15]). Scattered subspaces, introduced by Blokhuis and Lavrauw [17], and their relation to MRD codes [18] play a central role below.

In the Hamming metric a linear code is *intersecting* when the supports of any two nonzero codewords share a coordinate; the notion is due to Katona and Srivastava [8] and Miklós [9], was developed by Cohen and Lempel [10] and Cohen and Zémor [11], and has a projective-geometric description [12]. Bartoli, Borello, Marino and Scotti [2] introduced the rank-metric analogue: C is *rank-metric intersecting* when $\text{supp}(c) \cap \text{supp}(c') \neq \{0\}$ for all nonzero $c, c' \in C$, the support being the rank support (the row space of the matrix expansion). They proved that a nondegenerate $[n, k, d]_{q^m/q}$ code is rank-metric intersecting if and only if its q -system is not 2-spannable, i.e. is not the sum of its intersections with two $\mathbb{F}_{q^m}$ -hyperplanes (their Theorem 3.4), that such a code satisfies $2k - 1 \leq n \leq 2m - 3$ (their Corollary 2.4

and Theorem 4.7) and $k \leq d \leq m$ (their Theorem 4.2), and that codes exist for $m \leq n \leq 2m - 2k + 1$ whenever $2 \leq k \leq \lfloor (m+1)/2 \rfloor$ (their Theorem 4.4, restated as $2k - 1 \leq n \leq 2m - 2k + 1$ in [1, Theorem 2.5]); their Question 4.11 asks about the lengths $2m - 2k + 2 \leq n \leq 2m - 3$. Conca, Jany and Ravagnani [3] relate the property to the vertical connectivity of the induced q -matroid.

Borello, Polverino and Zullo [1] then showed that the dual q -system of a rank-metric intersecting code with $k \geq 3$ is $(1, 2m - n - 2)$ -evasive (their Theorem 3.2), whence $\max\{k, n - m + 2\} \leq d \leq m$ (Corollary 3.3) and $n \leq 2m - \lfloor (k+4)/2 \rfloor$ (Theorem 3.4); that length $2m - 3$ occurs exactly when $k = 3$, $m \geq 6$ and the dual system is scattered, in which case $d = m - 1$ (Corollary 3.5); that such codes exist for every even $m \geq 6$ (Corollary 4.2), with an explicit $[9, 3, 5]_{64/2}$ code (Example 4.3); and that $[6, 3, 3]_{q^5/q}$ rank-metric intersecting codes do not exist for any prime power q (Theorem 5.4). For $k = 3$ and $m = 6$ this leaves one length undecided. Their Remark 4.5 reads, verbatim:

“Let us remark that, for $m = 6$, we know the existence of $[n, 3]_{q^6/q}$ rank-metric intersecting codes for $5 \leq n \leq 7$ and for $n = 9$. We also know that for $n < 5$ and $n > 9$, an $[n, 3]_{q^6/q}$ rank-metric intersecting code cannot exist. However, the existence of an $[8, 3]_{q^6/q}$ rank-metric intersecting code remains open. A natural question is whether such a code can be obtained by extending or puncturing (in the rank-metric sense) known constructions. However, this does not seem to be an easy task: an extensive search for $[8, 3]_{64/2}$ codes obtained by puncturing the code in Example 4.3 fails to produce a rank-metric intersecting one.”

and their Conclusions repeat that “the existence of $[2m - 4, 3]_{q^m/q}$ rank-metric intersecting codes is open also in the case where m is even.” (Statement numbers of [1] and [2] throughout are those of the arXiv PDFs; for [1] they were also read off a compilation of the submitted e-print, and both quotations are verbatim from it.)

At $q = 2$ the open cell is a finite question: *is there an $\mathbb{F}_2$ -subspace $U \subseteq \mathbb{F}_{64}^3$ with $\dim_{\mathbb{F}_2} U = 8$, spanning $\mathbb{F}_{64}^3$ over $\mathbb{F}_{64}$, which is not 2-spannable?* This paper does not answer it. It records what can be settled about it by exact computation, in a form that a machine has checked, and it measures the obstruction to the exhaustive direction. Concretely:

- (a) A finite model of $\text{PG}(2, 64)$ over the field $\mathbb{F}_{64} = \mathbb{F}_2[\alpha]/(\alpha^6 + \alpha^4 + \alpha^3 + \alpha + 1)$ of [1, Example 4.3], with its 4161 hyperplanes, is set up and its faithfulness is certified (Proposition 3.4); and a criterion decides 2-spannability from the largest hyperplane sections alone (Lemma 3.2), which is what makes each witness a few seconds of computation rather than a search over 4161^2 pairs.
- (b) The known part of the row $k = 3$, $m = 6$ is certified at $q = 2$: the $[9, 3, 5]_{64/2}$ code of [1, Example 4.3] has a maximum scattered q -system and is rank-metric intersecting (Theorem 4.1), and explicit $[5, 3, 3]_{64/2}$, $[6, 3, 3]_{64/2}$ and $[7, 3, 4]_{64/2}$ rank-metric intersecting codes are exhibited (Theorem 4.2).
- (c) The puncturing search reported in [1, Remark 4.5] becomes a theorem: all 511 rank-metric puncturings of Example 4.3 are 2-spannable, each with an explicit spanning pair of hyperplanes (Theorem 5.1).
- (d) No $[8, 3]_{64/2}$ rank-metric intersecting code has an $\mathbb{F}_4$ -linear q -system (Theorem 7.2); $\mathbb{F}_4$ is the only proper subfield of $\mathbb{F}_{64}$ over which an 8-dimensional $\mathbb{F}_2$ -subspace can be linear. The proof is exhaustive over 1365 systems that cover every equivalence class, with an explicit spanning pair for each. The statement is in fact routine (Remark 7.3): such a system meets some hyperplane in $\mathbb{F}_2$ -dimension 6, so its code has $d \leq 2 < k$, which [1, Theorem 2.6] forbids. What the exhaustive theorem adds is the certificates and the completeness of the list.
- (e) A double count gives an identity for the hyperplane weights of every 8-dimensional $\mathbb{F}_2$ -subspace of $\mathbb{F}_{64}^3$ (Proposition 6.1, proved by hand and not formalised), which forces an intersecting $[8, 3]_{64/2}$ system to have exactly $31 + 4n_2 + 28n_3$ hyperplanes of section size 16, all of whose pairs meet nontrivially and which are never concurrent.
- (f) The best candidate found by a search over $5.5 \cdot 10^7$ subspaces is certified: it is scattered, has exactly the 31 maximal hyperplanes the identity predicts, and 108 of their 465 pairs still span it (Proposition 8.1). The search protocol, its outcome and the price of an exhaustive search are

recorded (Remark 8.2); the exhaustive direction is out of reach not for want of processor time to check orbit representatives but for want of a canonical form to enumerate them.

(g) Negative controls and non-vacuity checks (Section 9).

What is new and what is not. The mathematics of (b) is that of [1] and [2]: the example is theirs, and existence for $5 \leq n \leq 7$ is [1, Theorem 2.5], asserted there without a printed $q = 2$ witness; the objects and their certification are ours. Statement (c) is the theorem form of a search that [1] reports having run; the 511 certificates are ours. Statement (d) was not found in the literature (the search is described in Section 10), but it is *routine*: as Remark 7.3 shows, a five-line dimension count over $\mathbb{F}_4$ together with the bound $k \leq d$ of [2, Theorem 4.2] excludes every $\mathbb{F}_4$ -linear system, and we claim no novelty for the exclusion itself. What the exhaustive theorem adds is a certified spanning pair for each of the 1365 systems and a kernel-checked proof that those 1365 exhaust the classes. The identity (e) is elementary double counting and is stated here as ordinary mathematics, outside the formal development. Item (f) is a computation about one explicit subspace, and the search figures are measurements: they are evidence about the search, never about the cell. The question of [1, Remark 4.5] remains exactly as open as it was left there.

2. PRELIMINARIES

2.1. Rank-metric codes and q -systems. Let q be a prime power and $\mathbb{F}_{q^m} \supseteq \mathbb{F}_q$. For $x = (x_1, \dots, x_n) \in \mathbb{F}_{q^m}^n$ and an ordered $\mathbb{F}_q$ -basis Γ of $\mathbb{F}_{q^m}$ let $\Gamma(x) \in \mathbb{F}_q^{m \times n}$ be the matrix whose j -th column holds the coordinates of x_j ; the *rank support* of x is $\text{supp}(x) = \text{rowsp} \Gamma(x) \subseteq \mathbb{F}_q^n$, which does not depend on Γ , and the rank weight is $\text{wt}(x) = \dim \text{supp}(x)$. A (linear, vector) rank-metric code is an $\mathbb{F}_{q^m}$ -subspace $C \subseteq \mathbb{F}_{q^m}^n$ with $d(x, y) = \text{wt}(x - y)$; it is an $[n, k, d]_{q^m/q}$ code when $k = \dim_{\mathbb{F}_{q^m}} C$ and d is its minimum distance, and it is *nondegenerate* when the columns of a (any) generator matrix are $\mathbb{F}_q$ -linearly independent. Two codes are equivalent when $C' = CA$ for some $A \in \text{GL}(n, q)$.

An $[n, k, d]_{q^m/q}$ *system* is an $\mathbb{F}_q$ -subspace $U \subseteq \mathbb{F}_{q^m}^k$ with $\dim_{\mathbb{F}_q} U = n$, $\langle U \rangle_{\mathbb{F}_{q^m}} = \mathbb{F}_{q^m}^k$ and

$$d = n - \max\{\dim_{\mathbb{F}_q}(U \cap H) : H \text{ an } \mathbb{F}_{q^m}\text{-hyperplane of } \mathbb{F}_{q^m}^k\};$$

two systems are equivalent when one is the image of the other under some $\varphi \in \text{GL}(k, q^m)$. If G is a generator matrix of a nondegenerate code C and U is the $\mathbb{F}_q$ -span of its columns, then $\text{wt}(xG) = n - \dim_{\mathbb{F}_q}(U \cap x^\perp)$ for $x \in \mathbb{F}_{q^m}^k$ ([1, Theorem 1.1], after [13, 14]), and this gives a bijection between equivalence classes of nondegenerate $[n, k, d]_{q^m/q}$ codes and of $[n, k, d]_{q^m/q}$ systems; U is a q -system associated with C .

For an $\mathbb{F}_q$ -subspace U of $\mathbb{F}_{q^m}^k$ and an $\mathbb{F}_{q^m}$ -subspace W the *weight* of W is $\text{wt}_U(W) = \dim_{\mathbb{F}_q}(U \cap W)$. U is *scattered* when every one-dimensional $\mathbb{F}_{q^m}$ -subspace has weight at most 1; a scattered $\mathbb{F}_q$ -subspace of $\mathbb{F}_{q^m}^k$ has $\mathbb{F}_q$ -dimension at most $mk/2$ ([1, Theorem 1.2], after [17]), and one of dimension $mk/2$ is *maximum scattered*. The $\mathbb{F}_q$ -linear set of U is $L_U = \{\langle u \rangle_{\mathbb{F}_{q^m}} : u \in U \setminus \{0\}\} \subseteq \text{PG}(k-1, q^m)$. Finally, for a nondegenerate reflexive bilinear form σ on $\mathbb{F}_{q^m}^k$, the form $\sigma' = \text{Tr}_{q^m/q} \circ \sigma$ is a nondegenerate $\mathbb{F}_q$ -bilinear form, and $U^{\perp'}$ denotes the orthogonal complement of U with respect to σ' ; it has $\mathbb{F}_q$ -dimension $mk - \dim_{\mathbb{F}_q} U$.

2.2. Rank-metric intersecting codes.

Definition 2.1 ([1, Definitions 2.1 and 2.2]). A code C is *rank-metric intersecting* if $\text{supp}(c) \cap \text{supp}(c') \neq \{0\}$ for all $c, c' \in C \setminus \{0\}$. An $[n, k]_{q^m/q}$ system U is *2-spannable* if there are two $\mathbb{F}_{q^m}$ -hyperplanes H_1, H_2 of $\mathbb{F}_{q^m}^k$ with $U = (H_1 \cap U) + (H_2 \cap U)$.

Theorem 2.2 ([2, Theorem 3.4]; [1, Theorem 2.3]). A nondegenerate $[n, k, d]_{q^m/q}$ code is rank-metric intersecting if and only if an associated q -system is not 2-spannable.

The bounds used below are: $2k - 1 \leq n \leq 2m - 3$ [1, Theorem 2.4]; existence for $2k - 1 \leq n \leq 2m - 2k + 1$ [1, Theorem 2.5] (which restates [2, Theorem 4.4], itself stated for $m \leq n \leq 2m - 2k + 1$; the remaining lengths $2k - 1 \leq n < m$ carry MRD codes, for which $2d > n$, so [2, Theorem 2.5] applies); $k \leq d \leq m$ [1, Theorem 2.6], i.e. [2, Theorem 4.2]; and, for $k \geq 3$, $\max\{k, n - m + 2\} \leq d \leq m$ [1,

Corollary 3.3]. For $k = 3$ and $m = 6$ the admissible lengths are therefore $5 \leq n \leq 9$, the lengths 5, 6, 7 are covered by existence, $9 = 2m - 3$ by [1, Corollary 4.2 and Example 4.3], and an $[8, 3]_{q^6/q}$ rank-metric intersecting code, if it exists, has $d \geq n - m + 2 = 4$, hence $d \in \{4, 5, 6\}$.

2.3. The model at $q = 2$, $m = 6$, $k = 3$. Throughout, $\mathbb{F}_{64} = \mathbb{F}_2[\alpha]/(\alpha^6 + \alpha^4 + \alpha^3 + \alpha + 1)$, the field of [1, Example 4.3], in which α is primitive. Vectors of $\mathbb{F}_{64}^3$ are written $v = (v_1, v_2, v_3)$, the bilinear form is the standard one, $\langle a, v \rangle = a_1 v_1 + a_2 v_2 + a_3 v_3$, and for $a \neq 0$

$$H_a = \{v \in \mathbb{F}_{64}^3 : \langle a, v \rangle = 0\}$$

is the $\mathbb{F}_{64}$ -hyperplane orthogonal to a ; $H_a = H_{\lambda a}$ for $\lambda \in \mathbb{F}_{64}^\times$, so the hyperplanes are indexed by the $(64^3 - 1)/63 = 4161$ points of $\text{PG}(2, 64)$, and we take as representatives the *normalised* vectors, those whose first nonzero coordinate is 1. A point $P = \langle v \rangle_{\mathbb{F}_{64}}$ and a hyperplane H of $\mathbb{F}_{64}^3$ have, for an $\mathbb{F}_2$ -subspace $U \subseteq \mathbb{F}_{64}^3$, the weights $t_P = \dim_{\mathbb{F}_2}(U \cap P)$ and $w_H = \dim_{\mathbb{F}_2}(U \cap H)$; we write n_t for the number of points of weight t and a_w for the number of hyperplanes of weight w . Thus U is scattered iff $t_P \leq 1$ for all P , and the code of an $[n, 3]_{64/2}$ system U has minimum distance $d = n - \max_H w_H$. The scattered bound is $3m/2 = 9$.

In the formal development a field element $\sum_{i < 6} c_i \alpha^i$ is the integer $\sum c_i 2^i$, a vector (v_1, v_2, v_3) is the integer $v_1 + 64v_2 + 4096v_3$, and an $\mathbb{F}_2$ -subspace is the list of its elements; we use this encoding only in Appendix A.

3. A FINITE CRITERION FOR 2-SPANNABILITY

Let $U \subseteq \mathbb{F}_{64}^3$ be an $\mathbb{F}_2$ -subspace and, for hyperplanes H_1, H_2 , put $A = U \cap H_1$ and $B = U \cap H_2$; these are subgroups of U and $A + B = \{x + y : x \in A, y \in B\}$.

Lemma 3.1. $|A + B| \leq |A| \cdot |B|$, with $|A + B| = |A||B|/|A \cap B|$. In particular, if $A + B = U$ then $|A| \cdot |B| \geq |U|$.

Proof. $A + B$ is the image of $A \times B$ under addition, and the fibre over $x + y$ is the coset $(x, y) + \{(z, z) : z \in A \cap B\}$, of size $|A \cap B|$. $\square$

Lemma 3.2 (the criterion). *Let $U \subseteq \mathbb{F}_{64}^3$ be an $\mathbb{F}_2$ -subspace, let $w \geq 1$ with $(w - 1)w < |U|$, and suppose that $|U \cap H| \leq w$ for every hyperplane H . If $(U \cap H_1) + (U \cap H_2) \neq U$ for every two hyperplanes H_1, H_2 with $|U \cap H_1| = |U \cap H_2| = w$, then U is not 2-spannable.*

Proof. Suppose $(U \cap H_1) + (U \cap H_2) = U$. By Lemma 3.1, $|U \cap H_1| \cdot |U \cap H_2| \geq |U| > (w - 1)w$, so neither section has at most $w - 1$ elements; both have exactly w , contradicting the hypothesis on such pairs. $\square$

Lemma 3.2 is formalised as `not_two_spannable`, with the hypotheses given as decidable checks (a bound w on all 4161 sections, a list W of the hyperplanes attaining it, and the pair test over $W \times W$) and the conclusion in the form “for no a, b among the 4161 representatives is every $u \in U$ a sum $x + y$ with $x \in U \cap H_a$, $y \in U \cap H_b$ ”. Its proof does not use compiled evaluation. The converse direction is a certificate:

Lemma 3.3 (certificate). *If $a, b \neq 0$ and for every $u \in U$ there is $x \in U \cap H_a$ with $u - x \in H_b$, then $U = (U \cap H_a) + (U \cap H_b)$, so U is 2-spannable.*

This is `two_spannable_of_cert`; the hypothesis is the decidable check `coversBy`. Two remarks on the criterion. First, for $|U| = 2^n$ and $w = 2^{n-d}$ the condition $(w - 1)w < |U|$ holds as soon as $2(n - d) < n$, i.e. $d > n/2$, and then no pair can span at all; this recovers [2, Theorem 2.5], if $2d > n$ then C is rank-metric intersecting, whose proof there is Grassmann’s formula applied to the two rank supports. For an $[8, 3, d]_{64/2}$ system with $d \geq 4$ one has $w = 16$ and $15 \cdot 16 = 240 < 256$, so 2-spannability of a candidate for the open cell is decided by the pairs of hyperplanes of weight 4 alone; this is the form in which the cell is attacked in Sections 6–8. Second, in the independent reproduction described in Section 10 we used the exact form of Lemma 3.1: U is 2-spannable iff $|U \cap H_1| \cdot |U \cap H_2| = |U| \cdot |U \cap H_1 \cap H_2|$ for some pair.

The model itself is certified as follows.

Proposition 3.4 (faithfulness of the model). *In the formal development:*

- (i) the 64×64 multiplication table of $\mathbb{F}_{64}$ agrees with schoolbook shift-and-reduce multiplication modulo $\alpha^6 + \alpha^4 + \alpha^3 + \alpha + 1$ on all 4096 pairs;
- (ii) $\alpha^{63} = 1$ and $\alpha^j \neq 1$ for $0 < j < 63$, so α is primitive;
- (iii) the list of hyperplane representatives has 4161 distinct members, each normalised; exactly 4161 of the 2^{18} vectors of $\mathbb{F}_{64}^3$ are normalised; every nonzero vector has a normalised scalar multiple; and no normalised vector has a second normalised multiple. Hence $a \mapsto H_a$ runs over every $\mathbb{F}_{64}$ -hyperplane of $\mathbb{F}_{64}^3$ exactly once.

This is routine, but everything below quantifies over that list, so it is stated. (Formal: `fmul_eq_fmMulDef, alpha_primitive, pts_complete`.)

4. THE KNOWN LENGTHS OF THE ROW $k = 3$, $m = 6$ AT $q = 2$

Let G be the generator matrix of [1, Example 4.3],

$$G = \begin{pmatrix} 1 & 0 & 0 & \alpha^{54} & \alpha^{18} & \alpha^{26} & \alpha^{32} & \alpha^{22} & \alpha^{19} \\ 0 & 1 & 0 & \alpha^{59} & \alpha^{12} & \alpha^{50} & \alpha^{49} & \alpha^{57} & \alpha^5 \\ 0 & 0 & 1 & \alpha^{29} & \alpha^{56} & \alpha^{34} & \alpha^{61} & \alpha^{44} & \alpha^{54} \end{pmatrix},$$

and let $U_9 \subseteq \mathbb{F}_{64}^3$ be the $\mathbb{F}_2$ -span of its nine columns.

- Theorem 4.1** (Example 4.3 of [1], certified). (i) $|U_9| = 512$, i.e. the nine columns are $\mathbb{F}_2$ -independent, and the largest hyperplane section of U_9 has $16 = 2^4$ elements; hence U_9 spans $\mathbb{F}_{64}^3$, and the code generated by G is a nondegenerate code with parameters $[9, 3, 5]_{64/2}$.
- (ii) U_9 is scattered: its 511 nonzero vectors span 511 distinct points of $\text{PG}(2, 64)$. Since $\dim_{\mathbb{F}_2} U_9 = 9 = 3m/2$, it is maximum scattered.
- (iii) U_9 is not 2-spannable. By Theorem 2.2 the code is rank-metric intersecting.

Proof. (i) and (ii) are exhaustive computations over the 512 elements and the 4161 hyperplanes (`U9_card`, `U9_maxInter`, `U9_scattered`). (iii) is Lemma 3.2 with $w = 17$: every section has at most $16 < 17$ elements, the list of hyperplanes attaining 17 is empty, and $16 \cdot 17 = 272 < 512$ (`U9_intersecting`). Equivalently, $d = 5 > 9/2$. $\square$

Everything in Theorem 4.1 is asserted in [1, Example 4.3], whose argument is Corollary 3.5 there: $U_9^{\perp'}$ is maximum scattered. We checked that too, outside the formal development: with σ the standard form, $U_9^{\perp'}$ has $\mathbb{F}_2$ -dimension 9 and is scattered. One process note is worth recording, because it is the reason to certify published examples at all: a first transcription of G with the eighth column read as $(\alpha^{22}, \alpha^{57}, \alpha^5)^T$ instead of $(\alpha^{22}, \alpha^{57}, \alpha^{44})^T$ produced a subspace that is neither scattered nor non-2-spannable; the check caught it at once.

For $5 \leq n \leq 7$, [1, Theorem 2.5] gives existence without a printed $q = 2$ witness. Let U_5, U_6, U_7 be the $\mathbb{F}_2$ -spans of the columns of

$$G_5 = \begin{pmatrix} \alpha^{25} & \alpha^{33} & \alpha^{25} & \alpha^{52} & \alpha \\ \alpha^{36} & \alpha^{50} & \alpha^{35} & \alpha^{40} & \alpha^{57} \\ \alpha^{15} & \alpha^5 & \alpha^{53} & \alpha^{33} & 1 \end{pmatrix}, \quad G_6 = \begin{pmatrix} \alpha^{11} & \alpha^{59} & \alpha^{51} & \alpha^{44} & \alpha^{20} & \alpha^{22} \\ \alpha^{10} & \alpha^{50} & \alpha^{38} & \alpha^{58} & \alpha^{54} & \alpha^5 \\ \alpha^{19} & \alpha^{36} & \alpha^{31} & \alpha^2 & \alpha^{27} & \alpha^{29} \end{pmatrix},$$

$$G_7 = \begin{pmatrix} \alpha^{22} & \alpha^{25} & \alpha^{37} & \alpha^{24} & \alpha^{17} & \alpha^{38} & \alpha^3 \\ 0 & \alpha^{15} & \alpha^{12} & \alpha^6 & \alpha^{27} & \alpha^{14} & \alpha^{37} \\ \alpha^{13} & 0 & \alpha^6 & \alpha^3 & \alpha^{34} & \alpha & \alpha^8 \end{pmatrix}.$$

Theorem 4.2 (explicit witnesses for $n = 5, 6, 7$). $|U_n| = 2^n$ for $n = 5, 6, 7$; the largest hyperplane sections have 4, 8 and 8 elements respectively, so G_5, G_6, G_7 generate nondegenerate $[5, 3, 3]_{64/2}$, $[6, 3, 3]_{64/2}$ and $[7, 3, 4]_{64/2}$ codes; each U_n is scattered; and none of U_5, U_6, U_7 is 2-spannable. Hence rank-metric intersecting codes with these three parameter sets exist.

Proof. Exhaustive computations as in Theorem 4.1 (`U5_card`, `U6_card`, `U7_card`, `small_maxInter`, `small_scattered`). Non-2-spannability is Lemma 3.2: for U_5 with $w = 5$ ($4 \cdot 5 < 32$, no section attains 5) and

for U_7 with $w = 9$ ($8 \cdot 9 < 128$, no section attains 9), i.e. $d > n/2$ in both cases; for U_6 with $w = 8$ ($7 \cdot 8 = 56 < 64$), where exactly seven hyperplanes attain 8, namely H_a for

$$a \in \{(1, \alpha^9, \alpha^{53}), (1, \alpha^{35}, \alpha^{40}), (1, \alpha^{62}, \alpha^{22}), (1, \alpha^{61}, \alpha^5), (1, \alpha^{45}, \alpha^{34}), (1, \alpha^{55}, \alpha^{19}), (1, \alpha^{56}, \alpha^7)\},$$

and the 49 ordered pairs among them are checked not to span U_6 (`U5_intersecting`, `U6_intersecting`, `U7_intersecting`). $\square$

The three subspaces were found by a local search and are, as far as we know, the first printed $q = 2$ witnesses for these parameters; the existence statement is not new. The case $n = 6$ is the one where the pair test does real work ($d = 3 = n/2$), and it contrasts with $m = 5$: over $\mathbb{F}_{q^5}$ no $[6, 3, 3]_{q^5/q}$ rank-metric intersecting code exists for any q ([1, Theorem 5.4], which for $q = 2$ was [2, Theorem 4.9], an exhaustive Magma computation).

5. EVERY RANK-METRIC PUNCTURING OF EXAMPLE 4.3 IS 2-SPANNABLE

[1, Remark 4.5] reports a search for $[8, 3]_{64/2}$ codes “obtained by puncturing the code in Example 4.3” in the rank-metric sense. We make that precise as follows. Puncturing a nondegenerate $[n, k]_{q^m/q}$ code C at a coordinate deletes that column from a generator matrix; since the equivalences $C \mapsto CA$, $A \in \text{GL}(n, q)$, act on the columns of G by invertible $\mathbb{F}_q$ -linear recombination, any $\mathbb{F}_q$ -basis of the q -system U of C is the column set of a generator matrix of an equivalent code. Hence the $\mathbb{F}_q$ -spans of the columns of the punctured generator matrices, as the equivalence varies, are exactly the $\mathbb{F}_q$ -hyperplanes of U , and we call the codes obtained this way the *rank-metric puncturings* of C .

The source does not define the phrase, but this is the notion of the rank-metric literature, and it is stated there in exactly this form. Bartoli, Giannoni, Marino and Neri define the *puncturing of C on A* , for $A \in \mathbb{F}_q^{n \times n'}$ of rank n' , to be $\pi(C, A) = \{cA : c \in C\}$ ([19, Definition 2.9]), and prove ([19, Theorem 2.10]) that *an $[n', k]_{q^m/q}$ code C' is a puncturing of C if and only if some $[n', k]_{q^m/q}$ system associated with C' is contained in U* ; for $n' = n - 1$ that is the description above. The same operation is the *right puncturing* $\mathcal{X}R = \{MR : M \in \mathcal{X}\}$ of a matrix code $\mathcal{X} \subseteq \mathbb{F}_q^{k \times m}$ by a full-rank R in Borello and Scotti [21, Section 2] — an extended abstract that shares an author with [1] — and is $\pi_g(C) = \{f \circ g : f \in C\}$ for injective $\mathbb{F}_q$ -linear g in Garnier, Lavauzelle, Nardi and Zappatore ([22, Definition 3.1]), who note that up to equivalence it depends only on $\text{im } g$ ([22, Remark 3.3]), that the matrix and vector-code forms agree ([22, Remark 3.5 and Lemma 3.10]), and that up to transposition the matrix form is the puncturing of [20], which we did not read. Deleting one coordinate of G outright is the special case $A = (I_{n-1} \mid 0)^T$; Theorem 5.1 covers it as the special case of the nine coordinate hyperplanes. One caveat: an $\mathbb{F}_q$ -hyperplane H of U is a q -system of the punctured code only when H still spans $\mathbb{F}_{q^m}^k$, which fails exactly when $d(C) = 1$. Here $d = 5$, so it does not arise.

For U_9 these are its $2^9 - 1 = 511$ $\mathbb{F}_2$ -hyperplanes, indexed by the nonzero $\mathbb{F}_2$ -linear functionals f on the coordinates of the nine columns of G ; each has 256 elements and, since no hyperplane of $\mathbb{F}_{64}^3$ contains more than 16 elements of U_9 (Theorem 4.1(i)), each spans $\mathbb{F}_{64}^3$, so each is an $[8, 3]_{64/2}$ system.

Theorem 5.1. (i) *The 511 subspaces $\ker f \subseteq U_9$, f a nonzero $\mathbb{F}_2$ -functional on the nine coordinates, are 511 distinct 8-dimensional $\mathbb{F}_2$ -subspaces of U_9 ; they are therefore all the $\mathbb{F}_2$ -hyperplanes of U_9 .*

(ii) *For each of them there are two hyperplanes H_a, H_b of $\mathbb{F}_{64}^3$ with $\ker f = (\ker f \cap H_a) + (\ker f \cap H_b)$. Consequently every rank-metric puncturing of the $[9, 3, 5]_{64/2}$ code of [1, Example 4.3] is 2-spannable, and none of them is rank-metric intersecting.*

Proof. (i) is a computation: each $\ker f$ is generated by eight explicit vectors, has 256 elements, lies in U_9 , and the 511 sorted element lists are pairwise distinct (`punct_are_the_hyperplanes`). (ii): a pair (a, b) of normalised vectors is recorded for each f , and Lemma 3.3 is verified for it from the definition (`punct_all_spannable`, `punct_two_spannable`). The 511 pairs were found by an independent search program and are re-verified in the formal development; in the reproduction of Section 10 the 511 subspaces were also found 2-spannable by the exact pair test, without the certificates. $\square$

The statement is the one the source reports; what is added is that it is now a theorem with explicit certificates rather than the outcome of a search. Every one of the 511 punctured systems has a hyperplane section of exactly 16 elements, i.e. every punctured code is an $[8, 3, 4]_{64/2}$ code, so $d \geq 4$ is not what fails.

6. A COUNTING IDENTITY, OUTSIDE THE FORMAL DEVELOPMENT

The results of this section are proved by hand and are *not* formalised; they organise the search of Section 8 and give a second proof in Section 7.

Proposition 6.1. *Let $U \subseteq \mathbb{F}_{64}^3$ be an $\mathbb{F}_2$ -subspace with $\dim_{\mathbb{F}_2} U = 8$ spanning $\mathbb{F}_{64}^3$, with a_w and n_t as in Section 2.3. Then $2 \leq w_H \leq 7$ for every hyperplane H , $t_P \leq 6$ for every point P , and*

$$(1) \quad a_4 + 7a_5 + 35a_6 + 155a_7 = 31 + 4n_2 + 28n_3 + 140n_4 + 620n_5 + 2604n_6.$$

Proof. A nonzero vector of $\mathbb{F}_{64}^3$ lies in exactly 65 hyperplanes, and two nonzero vectors lie in exactly one common hyperplane if they span distinct points and in 65 if they span the same point. Counting pairs (vector, hyperplane) and ordered triples (two distinct nonzero vectors, a hyperplane containing both) gives

$$\sum_H (2^{w_H} - 1) = 65 \cdot 255, \quad \sum_H (2^{w_H} - 1)(2^{w_H} - 2) = 255 \cdot 254 + 64 \sum_P (2^{t_P} - 1)(2^{t_P} - 2).$$

Since $\dim_{\mathbb{F}_2}(U \cap H) \geq 8 + 12 - 18 = 2$ and $U \not\subseteq H$, $w_H \in \{2, \dots, 7\}$; and $t_P \leq \dim_{\mathbb{F}_2} P = 6$. The polynomial $(x - 4)(x - 8) = (x - 1)(x - 2) - 9(x - 1) + 21$ vanishes at $x = 4, 8$, so

$$\begin{aligned} \sum_H (2^{w_H} - 4)(2^{w_H} - 8) &= 255 \cdot 254 - 9 \cdot 65 \cdot 255 + 21 \cdot 4161 + 64 \sum_P (2^{t_P} - 1)(2^{t_P} - 2) \\ &= 2976 + 64 \sum_P (2^{t_P} - 1)(2^{t_P} - 2), \end{aligned}$$

and dividing by 96 gives (1): $(2^w - 4)(2^w - 8)/96 = 1, 7, 35, 155$ for $w = 4, 5, 6, 7$, $2976/96 = 31$, and $\frac{2}{3}(2^t - 1)(2^t - 2) = 4, 28, 140, 620, 2604$ for $t = 2, \dots, 6$. $\square$

Corollary 6.2. *Let U be an $[8, 3]_{64/2}$ system which is not 2-spannable. Then $w_H \leq 4$ for every hyperplane and $t_P \leq 3$ for every point; the number of hyperplanes of weight 4 is exactly $a_4 = 31 + 4n_2 + 28n_3 \geq 31$, with equality iff U is scattered; any two of these hyperplanes meet in a point of positive weight; and no point lies on all of them.*

Proof. By Theorem 2.2 and [1, Corollary 3.3], $d \geq n - m + 2 = 4$, i.e. $w_H \leq 4$; so $a_5 = a_6 = a_7 = 0$. The 65 hyperplanes through a point P cover $\mathbb{F}_{64}^3$, so $256 = |U| \leq 2^{t_P} + 65(16 - 2^{t_P})$, which forces $2^{t_P} \leq 12$, i.e. $t_P \leq 3$; hence $n_4 = n_5 = n_6 = 0$ and (1) reads $a_4 = 31 + 4n_2 + 28n_3$. If two hyperplanes H_1, H_2 of weight 4 met each other in a point of weight 0, their sections A, B would have $|A||B|/|A \cap B| = 256 = |U|$, so $A + B = U$ by Lemma 3.1. Finally, a point P of weight t carries at most s hyperplanes of weight 4 where $s(16 - 2^t) \leq 256 - 2^t$ (their sections meet pairwise exactly in $U \cap P$), i.e. $s \leq 18, 21, 31$ for $t = 1, 2, 3$; since $a_4 \geq 31 > 21$ when all $t_P \leq 2$ and $a_4 \geq 59$ when some $t_P = 3$, the hyperplanes of weight 4 are never concurrent. $\square$

Dually, $V = U^{\perp'}$ has $\mathbb{F}_2$ -dimension $10 > 9 = 3m/2$ and so cannot be scattered, which is precisely why $n = 9$ is easy (take V maximum scattered, [1, Corollary 3.5]) and $n = 8$ is not. Identity (1) was checked in the reproduction of Section 10 on 3000 random 8-dimensional spanning subspaces and on two explicit subspaces carrying a point of weight 5 and of weight 6, with no failure. Its last two terms are easy to overlook: a random 8-dimensional subspace has no point of weight above 3, so no sample of random subspaces — the research record behind this paper used 400 000 of them — can distinguish (1) from the same identity with n_5 and n_6 deleted. On the two explicit subspaces the deleted form is false, by 715 against 95 and by 2635 against 31; the first of them is the $\mathbb{F}_2$ -span of five independent vectors of $\langle e_1 \rangle_{\mathbb{F}_{64}}$ together with $e_2, e_3, e_2 + \alpha e_3$, and the second is the span of $\langle e_1 \rangle_{\mathbb{F}_{64}}$ with e_2 and e_3 . Corollary 6.2 is unaffected either way, since $t_P \leq 3$ under its hypothesis.

7. NO $\mathbb{F}_4$ -LINEAR q -SYSTEM

$\mathbb{F}_{64}$ has the subfields $\mathbb{F}_2, \mathbb{F}_4, \mathbb{F}_8, \mathbb{F}_{64}$. An $\mathbb{F}_8$ -linear (resp. $\mathbb{F}_{64}$ -linear) $\mathbb{F}_2$ -subspace has $\mathbb{F}_2$ -dimension divisible by 3 (resp. 6), and 8 is divisible by neither, so $\mathbb{F}_4$ is the only proper subfield over which an 8-dimensional $\mathbb{F}_2$ -subspace of $\mathbb{F}_{64}^3$ can be linear. Let $\omega = \alpha^{21}$, so that $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, and call $F = \langle e_1, e_2, e_3 \rangle_{\mathbb{F}_4}$, the 64 vectors with all coordinates in $\mathbb{F}_4$, the *frame*. $\mathbb{F}_4$ -linearity and 2-spannability of a subspace are both invariant under $\text{GL}(3, 64)$: its elements are $\mathbb{F}_{64}$ -linear, hence $\mathbb{F}_4$ -linear, and they permute the hyperplanes. Consequently the statement “an $[8, 3]_{64/2}$ code has an $\mathbb{F}_4$ -linear associated q -system” does not depend on the choice of the system.

Lemma 7.1 (reduction). *Every $\mathbb{F}_4$ -linear $\mathbb{F}_2$ -subspace $U \subseteq \mathbb{F}_{64}^3$ with $\dim_{\mathbb{F}_2} U = 8$ spanning $\mathbb{F}_{64}^3$ is equivalent under $\text{GL}(3, 64)$ to $F + \mathbb{F}_4 u = \langle e_1, e_2, e_3, u \rangle_{\mathbb{F}_4}$ for some $u \notin F$.*

Proof. U spans $\mathbb{F}_{64}^3$, so it contains three $\mathbb{F}_{64}$ -independent vectors; the element of $\text{GL}(3, 64)$ taking them to e_1, e_2, e_3 maps U to an $\mathbb{F}_4$ -linear subspace of $\mathbb{F}_4$ -dimension 4 containing F , which is $F + \mathbb{F}_4 u$ for any u in it outside F . $\square$

There are $(4^6 - 1)/3 = 1365$ such subspaces, one for each point of the quotient $\mathbb{F}_{64}^3/F \cong \mathbb{F}_4^6$ over $\mathbb{F}_4$.

Theorem 7.2. (i) $\omega^3 = 1, \omega \neq 1$, and $\{0, 1, \omega, \omega^2\}$ has four elements, so it is the subfield $\mathbb{F}_4 \subseteq \mathbb{F}_{64}$.
(ii) A list of 1365 vectors $u \notin F$ is recorded such that each $F + \mathbb{F}_4 u$ has 256 elements and every vector of $\mathbb{F}_{64}^3 \setminus F$ lies in exactly one of the 1365 subspaces $F + \mathbb{F}_4 u$ (and no vector of F is counted). Hence the list is complete: the 1365 subspaces are all the $\mathbb{F}_4$ -subspaces of $\mathbb{F}_2$ -dimension 8 containing F .
(iii) For each of the 1365 subspaces there are two hyperplanes H_a, H_b of $\mathbb{F}_{64}^3$ with $(F + \mathbb{F}_4 u) = ((F + \mathbb{F}_4 u) \cap H_a) + ((F + \mathbb{F}_4 u) \cap H_b)$.

Consequently every $\mathbb{F}_4$ -linear $[8, 3]_{64/2}$ system is 2-spannable, and no $[8, 3]_{64/2}$ rank-metric intersecting code has an $\mathbb{F}_4$ -linear q -system.

Proof. (i)–(iii) are computations (`omega4_order_three`, `f4_partition`, `f4_all_spannable`); (iii) is Lemma 3.3 verified from the definition for each recorded pair (`f4_two_spannable`). The consequence follows from Lemma 7.1, the invariance of 2-spannability under $\text{GL}(3, 64)$, and Theorem 2.2. The reduction is the one step of this proof that is not machine-checked; everything finite is. In the reproduction of Section 10 the 1365 subspaces were enumerated independently (as the $\mathbb{F}_4$ -points of $\mathbb{F}_{64}^3/F$), found to coincide with the recorded list, and each was found 2-spannable by the exact pair test without using the certificates. $\square$

Remark 7.3 (an elementary proof). The exclusion, though not the certificates, follows from a dimension count. Let U be $\mathbb{F}_4$ -linear with $\dim_{\mathbb{F}_4} U = 4$, spanning $\mathbb{F}_{64}^3$, and let $u_1, \dots, u_4$ be an $\mathbb{F}_4$ -basis. Since $\dim_{\mathbb{F}_{64}} \mathbb{F}_{64}^3 = 3 < 4$ there is a nontrivial relation $\sum_i \lambda_i u_i = 0$ with $\lambda_i \in \mathbb{F}_{64}$. Writing each λ_i in an $\mathbb{F}_4$ -basis $\beta_1, \beta_2, \beta_3$ of $\mathbb{F}_{64}$, $\lambda_i = \sum_j \lambda_{ij} \beta_j$, the 3×4 matrix (λ_{ij}) over $\mathbb{F}_4$ has a nonzero kernel vector $f \in \mathbb{F}_4^4$, so $\sum_i \lambda_i f_i = 0$. Let $T = \{\sum c_i u_i : \sum f_i c_i = 0\}$, an $\mathbb{F}_4$ -hyperplane of U ($\mathbb{F}_2$ -dimension 6). Extending f to the $\mathbb{F}_{64}$ -linear functional $\tilde{f}$ on $\mathbb{F}_{64} \otimes_{\mathbb{F}_4} U$, the kernel of $\tilde{f}$ is $\mathbb{F}_{64} \otimes T$, and it contains $\kappa = \sum \lambda_i \otimes u_i$, which spans the kernel of the surjection $\mathbb{F}_{64} \otimes_{\mathbb{F}_4} U \rightarrow \mathbb{F}_{64}^3$; so this surjection is not injective on $\mathbb{F}_{64} \otimes T$, i.e. T is $\mathbb{F}_{64}$ -dependent and lies in some hyperplane H . Thus $w_H \geq 6$ and the code has $d \leq 8 - 6 = 2 < 3 = k$, contradicting [2, Theorem 4.2] ($=$ [1, Theorem 2.6]) for an intersecting code. In fact all 1365 subspaces of Theorem 7.2 have a section of exactly 64 elements, i.e. all are $[8, 3, 2]_{64/2}$ systems. So Theorem 7.2 is a machine-checked certificate of a routine fact; what it adds beyond the exclusion is a spanning pair of hyperplanes for each of the 1365 systems and a kernel-checked proof that they exhaust the classes. The same count applies whenever an $\mathbb{F}_{q^s}$ -linear system U has $\dim_{\mathbb{F}_{q^s}} U = r$ with $r > (r - k)m/s$, and gives $d \leq s$ there; the case used here is $r = k + 1$, where that condition reads $n > m$.

Remark 7.4 (a second proof by counting). Corollary 6.2 also excludes the case. If U is $\mathbb{F}_4$ -linear and not 2-spannable, every $U \cap P$ is an $\mathbb{F}_4$ -space, so every t_P is even and, by the corollary, $t_P \in \{0, 2\}$; then $3n_2 = 255$, $n_2 = 85$ and $a_4 = 31 + 340 = 371$. A hyperplane H of weight 4 meets U in an $\mathbb{F}_4$ -plane,

which carries $(16 - 1)/3 = 5$ points of L_U , each of weight 2 and hence on at most 21 hyperplanes of weight 4; every other hyperplane of weight 4 must meet H in one of those five points, so there are at most $5 \cdot 20 = 100 < 370$ of them. Neither this argument nor Remark 7.3 is formalised.

8. THE SEARCH FOR $n = 8$ AND A CERTIFIED NEAR-MISS

By Corollary 6.2, an $[8, 3]_{64/2}$ system U with $d \geq 4$ is not 2-spannable iff its hyperplanes of weight 4 pairwise meet in points of positive weight; call a pair of such hyperplanes *bad* when they meet in a point of weight 0, so that U solves the cell iff it has $d \geq 4$ and no bad pair. A random 8-dimensional subspace has about 437 bad pairs among $\binom{31}{2} = 465$ (the linear set has 255 of the 4161 points, so a random pair of its maximal hyperplanes meets inside it with probability about 6%). Let U_8 be the $\mathbb{F}_2$ -span of the columns of

$$G_8 = \begin{pmatrix} \alpha^{15} & 0 & \alpha^{58} & \alpha^{35} & \alpha^{12} & \alpha^{55} & \alpha^{37} & \alpha^{57} \\ \alpha^{47} & \alpha^{44} & \alpha^{11} & \alpha^6 & \alpha^{15} & \alpha^9 & \alpha^{56} & \alpha^{22} \\ \alpha^{33} & \alpha^2 & \alpha^{53} & \alpha^{15} & \alpha^{29} & \alpha^{16} & 1 & \alpha^{62} \end{pmatrix}.$$

Proposition 8.1 (the near-miss). *$|U_8| = 256$; U_8 is scattered; its largest hyperplane section has 16 elements, so G_8 generates a nondegenerate $[8, 3, 4]_{64/2}$ code; exactly 31 hyperplanes attain the section size 16 (the value forced by Corollary 6.2 for a scattered system); and exactly 108 of the 465 pairs of these hyperplanes span U_8 . In particular U_8 is 2-spannable — the pair $a = (1, \alpha^{31}, 0)$, $b = (1, \alpha^{57}, \alpha^{35})$ is certified to span it — and the code of G_8 is not rank-metric intersecting.*

Proof. Computation over the 256 elements and 4161 hyperplanes (`U8_near_miss`: the recorded list of 31 hyperplanes is checked to contain every hyperplane of section size 16 and to have 31 members, and the number of spanning pairs among them is computed with the `coversBy` test); the certificate is Lemma 3.3 (`U8_two_spannable`). $\square$

U_8 is the best candidate the following search produced. It is a computation about one explicit subspace; what it says about the cell is only that a configuration exists that is 108 bad pairs away from a solution where a random one is about 437 away.

Remark 8.2 (the search, its outcome, and the price of the exhaustive direction). The following are measurements from the research record, outside the formal development. The protocol was fixed before the search: two processes of 45 minutes each, and if no witness appeared, the outcome would be reported as a search and non-existence would not be claimed. The checker evaluates a candidate by accumulating all 4161 section sizes from the 255 nonzero vectors and their 65 incident dual points and then testing the pairs of maximal hyperplanes; it ran at about 23 800 candidates per second on one core, and was cross-validated on the witnesses of Section 4 and on 200 random subspaces against two other implementations of 2-spannability (the exact pair test of Lemma 3.1 over all $\binom{4161}{2}$ pairs, and an explicit construction of $(U \cap H_1) + (U \cap H_2)$). An annealed local search over all 8-dimensional subspaces evaluated 54 256 902 candidates (best: 108 bad pairs, the subspace U_8); a puncturing search descended to 1093 distinct maximum scattered 9-dimensional subspaces and checked all 511 hyperplanes of each, 558 523 punctures in all (best: 117 bad pairs) — over a thousand times the single subspace of [1, Remark 4.5]. In total $5.48 \cdot 10^7$ candidates, none with $d \geq 4$ and no bad pair.

The exhaustive direction is priced as follows. The number of 8-dimensional $\mathbb{F}_2$ -subspaces of $\mathbb{F}_{64}^3$ is the Gaussian binomial $\binom{18}{8}_2 \approx 4.17 \cdot 10^{24}$, and the group $\text{GL}(3, 64) \rtimes \text{Gal}(\mathbb{F}_{64}/\mathbb{F}_2)$ acting on them has order $\approx 1.06 \cdot 10^{17}$, so there are at least $\approx 3.9 \cdot 10^7$ orbits; *checking* one representative per orbit would take about half a processor-hour at the measured rate. The obstruction is enumeration: the strongest canonical form available to us, “ $U \supseteq \langle e_1, e_2, e_3 \rangle_{\mathbb{F}_2}$ ”, leaves $\binom{15}{5}_2 \approx 3.77 \cdot 10^{15}$ candidates, on which the residual symmetry group has order only $|\text{GL}(3, 2)| \cdot 6 = 1008$: even at one check per residual orbit that is $3.7 \cdot 10^{12}$ evaluations, $1.6 \cdot 10^8$ seconds, of the order of five processor-years at the measured rate — and a thousand times that if the residual symmetry cannot be used. What would settle the cell at $q = 2$ is a canonical-form algorithm for $\mathbb{F}_2$ -subspaces of $\mathbb{F}_{64}^3$ under the full group, not more processor time.

9. CONTROLS

A certificate-based check can be satisfied by a misplaced quantifier while proving nothing, so the checks are exercised on cases where the answer is known.

Proposition 9.1 (negative controls). (i) (*Too short.*) Let U_4 be the $\mathbb{F}_2$ -span of the first four columns of G , a 4-dimensional system with $n = 4 < 2k - 1$. Then $U_4 = (U_4 \cap H_a) + (U_4 \cap H_b)$ for $a = (1, 0, 0)$, $b = (1, \alpha^{56}, 1)$: it is 2-spannable, as [1, Theorem 2.4] requires.

(ii) (*Too long.*) Let U_{10} be the $\mathbb{F}_2$ -span of the nine columns of G together with $(1 + \alpha, 0, 0)$, a 10-dimensional system with $n = 10 > 2m - 3$. Then $U_{10} = (U_{10} \cap H_a) + (U_{10} \cap H_b)$ for $a = (1, \alpha, 0)$, $b = (1, \alpha^{42}, 1)$.

(iii) (*A corrupted witness is rejected.*) Replacing the ninth column of G by $(\alpha^{55}, \alpha^5, \alpha^{54})^\top$ — the first coordinate $\alpha^{19} = \alpha + \alpha^2 + \alpha^3 + \alpha^5$ changed to $\alpha^{55} = \alpha^2 + \alpha^3 + \alpha^5$ — gives a 9-dimensional $\mathbb{F}_2$ -subspace U'_9 with $U'_9 = (U'_9 \cap H_a) + (U'_9 \cap H_b)$ for $a = (1, \alpha^2, 0)$, $b = (1, \alpha^7, \alpha^{62})$. So the certificate of Theorem 4.1(iii) is not an artefact of the shape of the check.

Proof. Each item is Lemma 3.3 verified from the definition for the stated pair, together with the element count 16, 1024, 512 and, in (iii), a check that the original ninth column is the vector being replaced (U4_spannable, U4_two_spannable, U10_spannable, U10_two_spannable, U9bad_spannable, U9bad_two_spannable). $\square$

(A different perturbation of the same coordinate, to $\alpha^{48} = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^5$, happens to give another scattered, non-2-spannable $[9, 3, 5]_{64/2}$ system; being close to a witness is not enough, which is the point of the control. This aside was checked in the reproduction only.)

Proposition 9.2 (non-vacuity). *The bound 16 on the hyperplane sections of U_9 is attained (the check with bound 15 fails), so $d = 5$ in Theorem 4.1 is exact; likewise the bound 8 for U_6 is attained; the list of seven hyperplanes used for U_6 in Theorem 4.2 has seven members, contains every hyperplane of section size 8, and its pair test succeeds, so that test does real work; and the list of hyperplane representatives has all 4161 members, so the quantifiers of Lemma 3.2 are not empty.*

Proof. checks_not_vacuous. $\square$

10. VERIFICATION

Every statement above marked by a formal name is a theorem of a Lean 4 development (Lean v4.32.0) that imports nothing from Mathlib: the objects are natural numbers, lists and bit operations, and the algebra is carried by two hand-proved list lemmas (a Nodup list contained in another is no longer; the $\mathbb{F}_2$ -span of a list is closed under addition) together with Lemmas 3.2 and 3.3, whose proofs are ordinary and use only the axioms `propext`, `Classical.choice` and `Quot.sound`. Every finite check (Propositions 3.4, 8.1, 9.1, 9.2, Theorems 4.1, 4.2, 5.1, 7.2) is discharged by compiled evaluation (`native_decide`), and `#print axioms` on each such theorem lists exactly the standard axioms together with the evaluation axioms of its own checks (for the four `*_intersecting` theorems, four of their own and the one of the `*_card` theorem they invoke); no declaration depends on `sorryAx`. The certificate data (511 + 1365 triples) is carried inside the sources; the longest single check takes under half a minute. The theorem statements are printed verbatim in Appendix A. Repository reference to be added at submission.

Independently of the formal development, every number in this paper was recomputed for this paper by an exact program written from the definitions (a table-driven $\mathbb{F}_{64}$, $\mathbb{F}_2$ -spans as element lists, section sizes over all 4161 hyperplanes, the exact pair test of Lemma 3.1, explicit sumsets for certificates): the 4161 representatives, the parameters, scatteredness and non-2-spannability of U_5, U_6, U_7, U_9 and of $U_9^\perp$, the 511 puncturings and the 1365 $\mathbb{F}_4$ -subspaces (each found 2-spannable without the certificates, and every recorded certificate verified by sumset), the 31 hyperplanes and 108 spanning pairs of U_8 , the controls, identity (1) on random and on constructed subspaces, and the orbit and canonical-form counts of Remark 8.2.

The literature search behind Theorem 7.2 was: a full-text search of a local mirror of arXiv (890 739 papers, in mathematics, computer science and statistics, as of September 2026), in which the phrase

“rank-metric intersecting” occurs in exactly three papers, [2], [3] and [1], and “2-spannable” in exactly two, [2] and [1]; the arXiv abstract search, which returns the same papers; and OpenAlex, whose phrase search for “rank-metric intersecting” returns seven records — duplicate records of [2] and [1], both with zero recorded citations there, a 2025 doctoral dissertation of one of the authors of [2], and the two July 2026 preprints [22] and [23], in neither of which the word “intersecting” occurs outside the bibliography. One further work on rank-metric intersecting codes, not on arXiv, was reached through the bibliography of [22] — a bibliography the full-text mirror does not carry, being a separate file of that e-print, which is why the phrase search could not see the pointer: the extended abstract [21], which introduces *matrix* rank-metric intersecting codes through the slice spaces of the generator tensor. We read it: it generalises [2] in a different direction, and contains no occurrence of $m = 6$, q^6 , $[8, \cdot]$, $64/2$ or “spannable”. We also read [2] (no occurrence of $m = 6$, q^6 , $[8, \cdot]$ or $64/2$; its Question 4.11 is the general one) and [3] (no parameter results); the dissertation we did not read. So the search found nothing on the cell beyond [1] itself — but “not found” is not “new”, and in this case Remark 7.3 shows the result is elementary anyway.

APPENDIX A. THE FORMAL STATEMENTS

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). `Nat` is the type of natural numbers; field elements, vectors and subspaces are encoded as in Section 2.3; `^^^` is bitwise exclusive or, i.e. addition in $\mathbb{F}_{64}^3$; `dot a v` is $\langle a, v \rangle$; `pts` is the list of the 4161 normalised vectors; `spanOf gs` is the list of all $\mathbb{F}_2$ -combinations of the list `gs`; `inter U a` is $U \cap H_a$; `coversBy U A b` tests that every $u \in U$ is $x + y$ with $x \in A$ and $y \in H_b$; `lenOK U w` tests $|U \cap H_a| \leq w$ for all a ; `coversList U w W` tests that every a with $|U \cap H_a| = w$ is in W ; `bigPairsOK U W` tests that no pair from W spans U ; `maxInter U` is $\max_a |U \cap H_a|$; `normv v` is the normalised representative of $\langle v \rangle_{\mathbb{F}_{64}}$; `apow k` is α^k ; `mkv x y z` is (x, y, z) . The generator lists `ex43cols`, `gens5`, `gens6`, `gens7`, `gens8` are the columns of G, G_5, G_6, G_7, G_8 ; `U9, U5, ...` are their spans; `W6, W8` the lists of maximal hyperplanes; `punctU f` the puncturing $\ker f$; `punctCerts` and `f4Certs` the recorded triples (f, a, b) and (u, a, b) ; `f4U u` the subspace $F + \mathbb{F}_4 u$; `f4Cover` the array counting, for each vector, the recorded subspaces containing it (frame vectors excluded); `isFrame` membership in F ; `omega4` is α^{21} ; `spanningPairCount U W` the number of unordered pairs from W whose sections span U .

The criterion and the certificate (Lemmas 3.2, 3.3).

```
theorem not_two_spannable (U : List Nat) (w : Nat) (W : List Nat)
  (hnd : U.Nodup) (hcl : ∀ x ∈ U, ∀ y ∈ U, (x ^^^ y) ∈ U)
  (hlen : lenOK U w = true) (hWl : coversList U w W = true)
  (hsmall : (w - 1) * w < U.length)
  (hbig : bigPairsOK U W = true) :
  ∀ a ∈ pts, ∀ b ∈ pts, ¬ (∀ u ∈ U, ∃ x ∈ U, dot a x = 0 ∧ dot b (u ^^^ x) = 0)
```

```
theorem two_spannable_of_cert (U : List Nat) (a b : Nat)
  (hc : coversBy U (inter U a) b = true) :
  ∀ u ∈ U, ∃ x ∈ U, dot a x = 0 ∧ dot b (u ^^^ x) = 0
```

The model (Proposition 3.4).

```
theorem fmul_eq_fmulDef :
  ((List.range 64).all fun x => (List.range 64).all fun y =>
    fmul x y == fmulDef x y) = true

theorem alpha_primitive :
  apow 63 = 1 ∧ ((List.range 62).all fun k => apow (k + 1) != 1) = true

theorem pts_complete :
  pts.length = 4161 ∧ pts.Nodup ∧
  (pts.all fun v => isNorm v) = true ∧
```

```

((List.range 262144).filter fun v => isNorm v).length = 4161 ∧
((List.range 262144).all fun v =>
  v == 0 || isNorm (scal (finv (lead v)) v)) = true ∧
(pts.all fun v => (List.range 62).all fun k =>
  ! isNorm (scal (apow (k + 1)) v)) = true

```

Example 4.3 (Theorem 4.1).

```

theorem U9_card : U9.length = 512 ∧ U9.Nodup
theorem U9_maxInter : maxInter U9 = 16
theorem U9_scattered : ((U9.filter fun u => u != 0).map normv).Nodup
theorem U9_intersecting :
  ∀ a ∈ pts, ∀ b ∈ pts,
    ¬ (∀ u ∈ U9, ∃ x ∈ U9, dot a x = 0 ∧ dot b (u ^^^ x) = 0)

```

The witnesses for $n = 5, 6, 7$ (Theorem 4.2).

```

theorem U5_card : U5.length = 32 ∧ U5.Nodup
theorem U6_card : U6.length = 64 ∧ U6.Nodup
theorem U7_card : U7.length = 128 ∧ U7.Nodup
theorem small_maxInter : maxInter U5 = 4 ∧ maxInter U6 = 8 ∧ maxInter U7 = 8
theorem small_scattered :
  ((U5.filter fun u => u != 0).map normv).Nodup ∧
  ((U6.filter fun u => u != 0).map normv).Nodup ∧
  ((U7.filter fun u => u != 0).map normv).Nodup
theorem U5_intersecting :
  ∀ a ∈ pts, ∀ b ∈ pts,
    ¬ (∀ u ∈ U5, ∃ x ∈ U5, dot a x = 0 ∧ dot b (u ^^^ x) = 0)
theorem U6_intersecting :
  ∀ a ∈ pts, ∀ b ∈ pts,
    ¬ (∀ u ∈ U6, ∃ x ∈ U6, dot a x = 0 ∧ dot b (u ^^^ x) = 0)
theorem U7_intersecting :
  ∀ a ∈ pts, ∀ b ∈ pts,
    ¬ (∀ u ∈ U7, ∃ x ∈ U7, dot a x = 0 ∧ dot b (u ^^^ x) = 0)

```

The puncturings (Theorem 5.1).

```

theorem punct_are_the_hyperplanes :
  punctCerts.length = 511 ∧
  (punctCerts.all fun t => (punctU t.1).length == 256) = true ∧
  (punctCerts.all fun t => (punctU t.1).all fun u => U9.contains u) = true ∧
  ((punctCerts.map fun t => (punctU t.1).mergeSort (fun x y => x ≤ y)).map
    (fun l => l.foldl (fun s x => s * 262147 + x) 1)).Nodup

theorem punct_all_spannable :
  (punctCerts.all fun t =>
    pts.contains t.2.1 && pts.contains t.2.2 &&
    coversBy (punctU t.1) (inter (punctU t.1) t.2.1) t.2.2) = true

theorem punct_two_spannable (t : Nat × Nat × Nat) (ht : t ∈ punctCerts) :
  ∀ u ∈ punctU t.1, ∃ x ∈ punctU t.1,
    dot t.2.1 x = 0 ∧ dot t.2.2 (u ^^^ x) = 0

```

The $\mathbb{F}_4$ -linear case (Theorem 7.2).

```

theorem omega4_order_three :
  fmul omega4 (fmul omega4 omega4) = 1 ∧ omega4 ≠ 1 ∧

```

```

((List.range 64).filter fun x => inF4 x).length = 4

theorem f4_partition :
  f4Certs.length = 1365 ∧
  (f4Certs.all fun t => (f4U t.1).length == 256) = true ∧
  ((List.range 262144).all fun v =>
    f4Cover[v]! == (if isFrame v then 0 else 1)) = true

theorem f4_all_spannable :
  (f4Certs.all fun t =>
    pts.contains t.2.1 && pts.contains t.2.2 &&
    coversBy (f4U t.1) (inter (f4U t.1) t.2.1) t.2.2) = true

theorem f4_two_spannable (t : Nat × Nat × Nat) (ht : t ∈ f4Certs) :
  ∀ u ∈ f4U t.1, ∃ x ∈ f4U t.1,
    dot t.2.1 x = 0 ∧ dot t.2.2 (u ^^^ x) = 0

```

The near-miss (Proposition 8.1).

```

theorem U8_near_miss :
  U8.length = 256 ∧ U8.Nodup ∧
  ((U8.filter fun u => u != 0).map normv).Nodup ∧
  maxInter U8 = 16 ∧ W8.length = 31 ∧ coversList U8 16 W8 = true ∧
  spanningPairCount U8 W8 = 108

```

```

theorem U8_two_spannable :
  ∀ u ∈ U8, ∃ x ∈ U8, dot 1601 x = 0 ∧ dot 70017 (u ^^^ x) = 0

```

Here $1601 = (1, \alpha^{31}, 0)$ and $70017 = (1, \alpha^{57}, \alpha^{35})$ in the encoding of Section 2.3.

Controls (Propositions 9.1, 9.2).

```

theorem U4_spannable :
  U4.length = 16 ∧ pts.contains 1 = true ∧ pts.contains 4289 = true ∧
  coversBy U4 (inter U4 1) 4289 = true

theorem U4_two_spannable :
  ∀ u ∈ U4, ∃ x ∈ U4, dot 1 x = 0 ∧ dot 4289 (u ^^^ x) = 0

theorem U10_spannable :
  U10.length = 1024 ∧ pts.contains 129 = true ∧ pts.contains 5057 = true ∧
  coversBy U10 (inter U10 129) 5057 = true

theorem U10_two_spannable :
  ∀ u ∈ U10, ∃ x ∈ U10, dot 129 x = 0 ∧ dot 5057 (u ^^^ x) = 0

theorem U9bad_spannable :
  U9bad.length = 512 ∧ (ex43cols.getD 8 0 = 92206) ∧
  coversBy U9bad (inter U9bad 257) 187777 = true

theorem U9bad_two_spannable :
  ∀ u ∈ U9bad, ∃ x ∈ U9bad, dot 257 x = 0 ∧ dot 187777 (u ^^^ x) = 0

theorem checks_not_vacuous :
  lenOK U9 15 = false ∧ lenOK U6 7 = false ∧ W6.length = 7 ∧
  pts.length = 4161 ∧
  coversList U6 8 W6 = true ∧ bigPairsOK U6 W6 = true

```

Here $1 = (1, 0, 0)$, $4289 = (1, \alpha^{56}, 1)$, $129 = (1, \alpha, 0)$, $5057 = (1, \alpha^{42}, 1)$, $257 = (1, \alpha^2, 0)$, $187777 = (1, \alpha^7, \alpha^{62})$, $92206 = (\alpha^{19}, \alpha^5, \alpha^{54})$ is the ninth column of G , and `u9bad` replaces it by $92204 = (\alpha^{55}, \alpha^5, \alpha^{54})$.

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