

RESTRICTED RECIPROCAL RADO NUMBERS

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ABSTRACT. For integers $k \geq 2$, $r \geq 2$ and $1 \leq \ell \leq k$, let $f_r(k; \ell)$ denote the least n such that every r -colouring of $\{1, \dots, n\}$ admits a monochromatic solution of

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} = \frac{1}{x_{k+1}}$$

in which the number of distinct integers among $x_1, \dots, x_{k+1}$ is exactly $\ell + 1$. Dropping the distinctness condition gives the reciprocal Rado number $f_r(k)$, finite by Brown–Rödl and Lefmann and computed in small cases by Myers–Parrish and by Gaiser–Ramezanpour; the level ℓ was introduced by Gaiser for the linear equation $x_1 + \dots + x_k = x_{k+1}$. The two have not been combined, and the combination is posed here for the first time. We prove that $f_r(k; 1)$ exists for no $k \geq 2$ and no $r \geq 2$, by the explicit two-colouring $v \mapsto \lfloor \log_k v \rfloor \bmod 2$; the point is that the level-1 solutions of the reciprocal equation are exactly the pairs $\{a, ka\}$, which is literally the same set as for the linear equation. We determine the six values $f_2(2; 2) = 120$, $f_2(3; 2) = 84$, $f_2(3; 3) = 126$, $f_2(4; 2) = 108$, $f_2(4; 3) = 90$ and $f_2(4; 4) = 180$, each by an avoiding colouring at $n - 1$ checked against a provably complete solution list together with a refutation at n . In particular $f_2(k; \ell)$ is *not* monotone in ℓ — $f_2(4; 2) = 108 > 90 = f_2(4; 3)$ — whereas for each fixed level the linear analogue $S_2(k; \ell)$ agrees, for all large k , with a polynomial strictly increasing in ℓ ; so the transferred parameter does not inherit the shape of its original. We also prove $f_r(k; \ell) \geq f_r(k)$ for every level, with no search, and show the inequality is strict where it has been computed. Every theorem, proposition and lemma below is machine-checked in Lean 4.

1. INTRODUCTION

The equation. A *Rado number* of an equation E is the least n such that every colouring of $\{1, \dots, n\}$ with a fixed number of colours admits a monochromatic solution of E , when such an n exists; Rado’s theorem [11] decides for which linear E it always does. Replacing the variables of a partition-regular linear equation by their reciprocals is not covered by that theorem, and the basic case

$$(1) \quad \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} = \frac{1}{x_{k+1}}$$

is settled by theorems of Brown and Rödl [1] and, independently, of Lefmann [6], to the effect that Rado’s theorem survives replacing the variables by their reciprocals: for all $k \geq 2$ and $r \geq 1$ every r -colouring of a sufficiently long initial segment of the positive integers has a monochromatic solution of (1). Write $f_r(k)$ for the least such length, the *reciprocal Rado number*. Throughout, the variables in (1) are *not* required to be distinct; this is the convention of every source cited here.

Exact values are scarce. Myers and Parrish [9] computed $f_2(2) = 60$, $f_2(3) = 40$, $f_2(4) = 48$ and $f_3(2) = 3276$. Gaiser and Ramezanpour [3] proved the lower bounds $f_r(2) \geq 4^r/2$ for all $r \geq 1$ and $f_r(k) \geq (2^r - 1)k^r$ for all $k \geq 3$ and $r \geq 1$ (their Theorem 1.1), showed the second is sharp at $r = 2$ by determining $f_2(k) = 3k^2$ whenever $k = 3 \cdot 2^m$ with m a positive integer (their Theorem 1.2), and computed $f_2(k)$ for $5 \leq k \leq 25$ and $f_3(k)$ for $k = 3, 4, 5$. Their paper closes with four problems (its Problems 5.1–5.4): improve the lower bound for $f_3(k)$; improve it for $f_2(k)$ at odd prime powers; find a quadratic upper bound for $f_2(k)$ for all $k \geq 4$; and determine the least positive value of $a - b_1 - \dots - b_k$ over $a, b_1, \dots, b_k \in \{1/1, 1/2, \dots, 1/n\}$. None of them restricts how many of the x_i are equal.

The level. Five weeks after [3], Gaiser [4] introduced a new parameter for the *linear* equation $x_1 + \dots + x_k = x_{k+1}$: for $\ell \leq k$, the number $S_r(k; \ell)$ is the least n , if it exists, such that every r -colouring of $\{1, \dots, n\}$ has a monochromatic solution in which the number of distinct integers is exactly $\ell + 1$. We call ℓ the *level* of a solution. Level k is the all-distinct case: $S_r(k; k) = S_r^*(k)$, the generalized Schur number for pairwise distinct variables, whose existence was proved by Irving [5]. Level 1 is the degenerate case $x_1 = \dots = x_k$. Two of that paper's results matter here. First, an existence dichotomy (its Theorem 2.4): for $k \geq 2$ and $\ell \leq k$, the number $S_r(k; \ell)$ exists if and only if $\ell \geq 2$ or $r = 1$; the negative half is a two-colouring built from the blocks $A_i = [k^{i-1}, k^i)$, using that $a \in A_i$ forces $ka \in A_{i+1}$. Second, a closed form (its Theorem 1.1): for each fixed $\ell \geq 2$ and all sufficiently large k ,

$$(2) \quad S_2(k; \ell) = k^2 + \left\lceil \frac{(\ell+1)(\ell-2)}{2} + 2 \right\rceil k + \ell(\ell-2),$$

which reads $k^2 + 2k$, $k^2 + 4k + 3$, $k^2 + 7k + 8$ at $\ell = 2, 3, 4$ and is strictly increasing in ℓ ; at $\ell = 2$ the same theorem gives $S_2(k; 2) = k^2 + 2k$ for all $k \geq 3$. Theorem numbers for [3] and [4] refer throughout to the versions cited in the bibliography.

The question of this paper, and the fact that nobody has asked it. The equation of [3] and the level of [4] have not been combined. We combine them.

Definition 1.1. For $k \geq 2$, $r \geq 2$ and $\ell \geq 1$, let $f_r(k; \ell)$ be the least n such that every r -colouring of $\{1, \dots, n\}$ admits a monochromatic solution of (1) inside $\{1, \dots, n\}$ whose number of distinct integers is exactly $\ell + 1$, if such an n exists; otherwise we say that $f_r(k; \ell)$ does not exist.

We state plainly that this question is posed here for the first time, and that no source asks for it: it is a transfer of a parameter from one paper of a two-paper sequence to the equation of the other. A self-posed question deserves a sharper account of the literature than a published one, so we record what was searched. On 2026-08-28 a full-text search of a local 85 GB arXiv corpus for *reciprocal Rado*, *restricted generalized Schur* and the titles of [1] and [2] returned exactly fourteen papers: nine on partition regularity or nonstandard methods citing [1], together with [2], [3], [4] and two further preprints of the same author. None of them restricts the number of distinct values in a unit-fraction Rado problem. The arXiv API returned one item for `all:"reciprocal Rado"` and one for `all:"unit fractions" AND all:"Rado number"`; the author listing `au:"Gaiser"` returned twenty-nine items, seven of them by the author of [2], [3] and [4], with nothing on the crossing and nothing posted after [4]. The nearest neighbour on the distinctness axis is [13], which announces that a linear system $Mx = 0$ satisfying the column condition and admitting an element-distinct solution has monochromatic element-distinct solutions under every finite colouring; that preprint was withdrawn by its authors, who state that its conclusions had been proved earlier by others. Either way it computes no Rado numbers and does not touch (1). Finally, the rows of the table below, the extended level-2 row of Section 7 and the unrestricted row return nothing in the OEIS [10]; the only sub-row with any match is the all-distinct diagonal 120, 126, 180, whose three hits (A112343, A356176, A392780) are elementary number-theoretic sequences unrelated to Ramsey theory. As a positive control, the two-colour weak Schur numbers 9, 24, 52, 101, 166, 253 are found on the nose (A365190). A reader repeating any of these searches after the date above should of course expect them to have moved.

Remark 1.2 (a notation collision). In [2] the symbol $f_r(k, \ell)$, *with a comma*, already denotes a different object: the least n forcing a monochromatic solution of $x_1^{-1/\ell} + \dots + x_k^{-1/\ell} = y^{-1/\ell}$, so that $f_r(k, 1) = f_r(k)$, for which that paper proves upper bounds of the shape $6^\ell k^\ell (k+1)^\ell (k+2)^\ell$ and no exact values. Our $f_r(k; \ell)$, *with a semicolon*, follows the notation $S_r(k; \ell)$ of [4] and is unrelated to it. The two differ by one character, and a reader of [2] should be warned. We would not object to renaming ours $g_r(k; \ell)$.

Results. Levels above k are empty and level 1 turns out to be the degenerate one here as well, so the interesting range is $2 \leq \ell \leq k$. For $r = 2$ we determine every value with $k \leq 4$:

$k \setminus \ell$	1	2	3	4	$f_2(k)$
2	none	120	—	—	60
3	none	84	126	—	40
4	none	108	90	180	48

The last column is the unrestricted number of [9, 3] and serves as a control: it was recomputed here from the definition of $f_r(k)$ before any entry of the table was claimed.

The $\ell = 1$ column is a theorem and not a blank (Theorem 3.3): $f_r(k; 1)$ exists for no $k \geq 2$ and no $r \geq 2$. The reason is stronger than an analogy with [4]. The level-1 solutions of (1) are exactly the pairs $\{a, ka\}$ (Proposition 3.1 and the paragraph after it), which is *the same set of pairs* as for the linear equation, so the construction of [4] applies verbatim; we write it as the explicit colouring $v \mapsto \lfloor \log_k v \rfloor \bmod 2$.

The six values are Theorem 4.1. The $k = 2$ row is complete, since $k = 2$ admits only $\ell \in \{1, 2\}$. The headline is Theorem 5.1: the table is not monotone in ℓ , since $f_2(4; 2) = 108 > 90 = f_2(4; 3)$, while (2) is strictly increasing in ℓ . It is not decreasing either — the $k = 4$ row reads 108, 90, 180. So the transferred parameter does not inherit the shape of the original, which is the substance of the crossing: the table has to be computed and cannot be guessed from (2).

Section 6 proves $f_r(k; \ell) \geq f_r(k)$ for every level, with no search, and shows the two avoidance conditions genuinely differ. Section 7 records five further values obtained outside the formally verified development, and Section 8 lists what we regard as the natural next questions — first among them whether $f_r(k; \ell)$ is finite at all for $\ell \geq 2$, which nothing here proves. Section 9 describes the verification.

2. DEFINITIONS, AND THE SHAPE OF A SOLUTION

Fix an integer $k \geq 2$ and write $[n] = \{1, 2, \dots, n\}$.

Definition 2.1. A *solution* of (1) in $[n]$ is a tuple $x = (x_1, \dots, x_{k+1}) \in [n]^{k+1}$ with $\sum_{i=1}^k x_i^{-1} = x_{k+1}^{-1}$ in $\mathbb{Q}$. Its *level* is

$$\text{lev}(x) = |\{x_1, x_2, \dots, x_{k+1}\}| - 1.$$

Given $c : [n] \rightarrow [r]$, the solution x is *monochromatic* if $c(x_i) = c(x_j)$ for all i, j . We say c *avoids level ℓ* if no solution of level ℓ in $[n]$ is monochromatic, and that c *avoids* if no solution at all is monochromatic.

Tuples are ordered and their entries may repeat; the level counts distinct *values*, not coordinates. Restricting an avoiding colouring of $[n]$ to $[m]$ with $m \leq n$ changes neither the equation nor the level, so the set of n for which an avoiding colouring exists is downward closed, and Definition 1.1 may equivalently be read in the form the formal statements take:

$$(3) \quad f_r(k; \ell) = n \iff \begin{cases} \text{some } r\text{-colouring of } [n-1] \text{ avoids level } \ell, \text{ and} \\ \text{no } r\text{-colouring of } [n] \text{ avoids level } \ell. \end{cases}$$

The unrestricted number $f_r(k)$ is defined by the same convention with the level condition deleted.

The following lemma is the reason the definition is well posed, and is the mirror image of the corresponding remark for the linear equation, where the right-hand side is always *larger* than every summand.

Lemma 2.2. *Let $k \geq 2$ and let x be a solution of (1) in $[n]$. Then $x_{k+1} < x_i$ for every $i \leq k$.*

Proof. All the terms x_j^{-1} are positive and there are $k \geq 2$ of them, so for each $i \leq k$ we have $x_{k+1}^{-1} = \sum_{j \leq k} x_j^{-1} > x_i^{-1}$; since x_{k+1} and x_i are positive integers, $x_{k+1} < x_i$. $\square$

Corollary 2.3. *For $k \geq 2$ the right-hand side of a solution is a value none of $x_1, \dots, x_k$ takes, so $\text{lev}(x) = |\{x_1, \dots, x_k\}|$ and $1 \leq \text{lev}(x) \leq k$. Consequently $f_r(k; \ell)$ is of interest only for $1 \leq \ell \leq k$, and “exactly $\ell + 1$ distinct integers among all $k + 1$ ” is the same condition as “exactly ℓ distinct integers on the left”.*

We state the definition over all $k + 1$ coordinates, as [4] does, rather than in the convenient form of Corollary 2.3, so that it can be compared with the source sentence directly. Corollary 2.3 is not itself formalised: it is the immediate arithmetic consequence of Lemma 2.2, which is, and the formal development invokes Lemma 2.2 directly wherever this counting is needed.

Proposition 2.4. *Let $k \geq 2$, $r \geq 1$ and $\ell \geq 1$. If $f_r(k; \ell) = n$ then $n \geq 2$.*

Proof. The only tuple in $[1]^{k+1}$ is the constant tuple 1, and it is not a solution because $k \cdot 1 \neq 1$. So every colouring of $[1]$ avoids at every level, and by (3) no threshold can equal 1. $\square$

Proposition 2.4 is stated because without it the value 1 would satisfy (3) vacuously in the degenerate cases, and every statement below would be compatible with a mis-stated definition.

3. LEVEL ONE: NO THRESHOLD, FOR ANY k AND ANY r

Proposition 3.1. *Let $k \geq 2$ and let x be a solution of (1) in $[n]$ of level 1. Then*

$$x_1 = x_2 = \dots = x_k = k \cdot x_{k+1},$$

so the values of x are exactly the two integers x_{k+1} and $k x_{k+1}$.

Proof. By Lemma 2.2 the value x_{k+1} is distinct from every x_i with $i \leq k$. Level 1 means that $\{x_1, \dots, x_{k+1}\}$ has exactly two elements, so all of $x_1, \dots, x_k$ equal one value m , and the equation reads $k/m = 1/x_{k+1}$, i.e. $m = k x_{k+1}$. $\square$

The converse is immediate: whenever $ka \leq n$ the tuple $(ka, \dots, ka, a)$ is a level-1 solution, because $k \cdot (ka)^{-1} = a^{-1}$ and $ka \neq a$. So the level-1 solutions in $[n]$ are exactly the pairs $\{a, ka\}$ with $ka \leq n$. Only the instance $a = 1$ of the converse is used below, and it is what Proposition 3.4 records.

Remark 3.2. The level-1 solutions of the linear equation $x_1 + \dots + x_k = x_{k+1}$ are, by the same argument with the inequality reversed, the tuples $x_1 = \dots = x_k = a$, $x_{k+1} = ka$. As sets $\{a, ka\}$ the two families coincide. A colouring therefore avoids level 1 for (1) if and only if it avoids level 1 for the linear equation, and the non-existence half of the dichotomy of [4] transfers with no work at all. This coincidence is what makes Theorem 3.3 free rather than a new construction; it is special to level 1, and Sections 4–5 show that nothing like it holds at higher levels.

For $v \geq 1$ let $\lfloor \log_k v \rfloor$ denote the unique $j \geq 0$ with $k^j \leq v < k^{j+1}$.

Theorem 3.3. *Let $k \geq 2$ and $r \geq 2$. For every n , the r -colouring*

$$c_k(v) = 1 + (\lfloor \log_k v \rfloor \bmod 2) \in \{1, 2\} \subseteq [r]$$

of $[n]$ avoids level 1. Consequently $f_r(k; 1)$ does not exist: no natural number satisfies (3) for the level 1.

Proof. Let x be a level-1 solution in $[n]$. By Proposition 3.1, $x_1 = k x_{k+1}$. Writing $m = x_{k+1} \geq 1$ and $j = \lfloor \log_k m \rfloor$, we have $k^j \leq m < k^{j+1}$ and hence $k^{j+1} \leq km < k^{j+2}$, so $\lfloor \log_k(km) \rfloor = j + 1$. Thus $\lfloor \log_k x_1 \rfloor$ and $\lfloor \log_k x_{k+1} \rfloor$ have opposite parities, so $c_k(x_1) \neq c_k(x_{k+1})$ and x is not monochromatic. Since this holds for every n , every n admits an avoiding colouring and the second clause of (3) never holds. $\square$

The colouring is uniform in n , uses only two of the r colours, and involves no search.

Proposition 3.4. *For $n \geq k$ the tuple $(k, k, \dots, k, 1)$ is a level-1 solution of (1) in $[n]$.*

Proof. $k \cdot (1/k) = 1/1$, and the tuple has the two distinct entries k and 1. $\square$

So Theorem 3.3 is not a statement about an empty set: level-1 solutions exist in abundance, and are merely 2-colourable at every length. Together with Corollary 2.3 this leaves $2 \leq \ell \leq k$ as the interesting range, exactly as in [4]. We emphasise that we prove only the negative half of the dichotomy; whether $f_r(k; \ell)$ is finite for $\ell \geq 2$ is open here (Question 8.1).

4. SIX EXACT VALUES

Theorem 4.1. *With $r = 2$ colours,*

$$\begin{aligned} f_2(2; 2) &= 120, & f_2(3; 2) &= 84, & f_2(3; 3) &= 126, \\ f_2(4; 2) &= 108, & f_2(4; 3) &= 90, & f_2(4; 4) &= 180. \end{aligned}$$

In each case the displayed number is the unique natural number satisfying (3) for the corresponding pair (k, ℓ) .

Each value is two statements, and each is settled by a finite computation; we describe both halves and say exactly what the computations were.

The solution list. Both halves are decided against a finite list rather than against the quantifier over $[n]^{k+1}$. For k, ℓ, n let $L(k, \ell, n)$ be the list of pairs $(a, (b_1 \leq b_2 \leq \dots \leq b_k))$ with all entries in $[n]$, $\sum_i b_i^{-1} = a^{-1}$, and exactly $\ell + 1$ distinct integers among $a, b_1, \dots, b_k$.

Proposition 4.2. *$L(k, \ell, n)$ is sound and complete: every listed pair is a level- ℓ solution of (1) in $[n]$, and every level- ℓ solution tuple x in $[n]$ occurs in the list, as the pair consisting of x_{k+1} together with $x_1, \dots, x_k$ sorted.*

Proof sketch. $L(k, \ell, n)$ is obtained by filtering, by the level condition, a list of the solutions of (1) in $[n]$ that is itself sound and complete — the standard unit-fraction backtracking search, in which a remaining target s with j terms still to choose constrains the next denominator b to $\lceil 1/s \rceil \leq b \leq \min\{n, \lfloor j/s \rfloor\}$, the entries being taken in nondecreasing order. Filtering preserves both directions, so the only new point is that the level of the tuple x and the level of the pair it is listed as agree. The tuple's list of values in coordinate order and the list “ x_{k+1} followed by the sorted $x_1, \dots, x_k$ ” are permutations of each other, and deduplication is invariant under permutation, so the two counts of distinct values coincide. $\square$

The lower half. For each pair (k, ℓ) in Theorem 4.1 we exhibit an explicit 2-colouring of $[n - 1]$, given as a bit string of length $n - 1$, and verify by one pass over $L(k, \ell, n - 1)$ that no listed solution is monochromatic. By the completeness half of Proposition 4.2 this establishes the first clause of (3). The colourings were found by a satisfiability solver and are *not* trusted: each is re-checked against the list.

The upper half. For the second clause we use a propositional encoding. Let $\Phi(k, \ell, r, n)$ be the formula in the variables $X_{v,j}$, $v \in [n]$, $0 \leq j < r$, with the clauses

$$\bigvee_{j < r} X_{v,j} \quad (v \in [n]), \quad \bigvee_{v \in S} \neg X_{v,j} \quad (S \text{ the set of integers of a pair in } L(k, \ell, n), j < r).$$

Proposition 4.3. *If an r -colouring of $[n]$ avoids level ℓ , then the assignment $X_{v,j} = 1$ iff $c(v) = j$ satisfies $\Phi(k, \ell, r, n)$. Hence if $\Phi(k, \ell, r, n)$ is unsatisfiable, no r -colouring of $[n]$ avoids level ℓ .*

Proof. The first family of clauses holds because c is total. For the second, let S come from a pair in $L(k, \ell, n)$ and let $j < r$. By the soundness half of Proposition 4.2 the pair names a genuine level- ℓ solution; if every $v \in S$ had $c(v) = j$ that solution would be monochromatic, contradicting avoidance. So some $v \in S$ has $c(v) \neq j$ and the clause is satisfied. $\square$

Only this direction is needed, and only this direction is proved: a satisfying assignment need not yield an avoiding colouring, which is why no at-most-one-colour clauses appear. The asymmetry is safe — clauses the encoding omits can only make a refutation harder to find, never unsound.

What was run. The following table records the search actually performed for each value. The threshold n was first located outside the formal development by bisection, using the monotonicity noted after Definition 2.1; both halves were then re-established as above.

value	n	clauses of Φ	refutation (bytes)	solver time
$f_2(2; 2) = 120$	120	276	1 604	0.01 s
$f_2(3; 2) = 84$	84	266	2 560	< 0.01 s
$f_2(3; 3) = 126$	126	834	50 788	0.01 s
$f_2(4; 2) = 108$	108	422	918	< 0.01 s
$f_2(4; 3) = 90$	90	1050	12 045	< 0.01 s
$f_2(4; 4) = 180$	180	6138	315 646	0.06 s

The refutations are resolution-style certificates produced by a conflict-driven solver and replayed, clause by clause, by a checker whose correctness is itself proved; the certificate files are of the sizes shown. The whole table costs under five minutes of processor time outside the formal development, and the solver is nowhere near its limit: the binding cost is the enumeration of solutions, not the search over colourings. See Section 9.

Remark 4.4. The $k = 2$ row is complete: by Corollary 2.3 only $\ell \in \{1, 2\}$ occurs, and $\ell = 1$ has no value by Theorem 3.3. So $f_2(2; \cdot)$ is fully determined, and $f_2(2; 2) = 120$ is exactly twice the unrestricted $f_2(2) = 60$. The doubling does not persist: $84/40 = 2.1$ and $108/48 = 2.25$.

5. THE TABLE IS NOT MONOTONE IN THE LEVEL

The polynomial in (2) is strictly increasing in ℓ at every fixed $k \geq 1$, and by [4] it is $S_2(k; \ell)$ once k is large enough for the level in question: raising the level makes the restriction harder to satisfy and pushes the threshold up. That is the behaviour one would transfer to (1) by analogy. It is false.

Theorem 5.1. *There are k and levels $2 \leq \ell < \ell' \leq k$ with $f_2(k; \ell') < f_2(k; \ell)$. Explicitly,*

$$f_2(4; 2) = 108 > 90 = f_2(4; 3).$$

Proof. Immediate from the values $f_2(4; 2) = 108$ and $f_2(4; 3) = 90$ of Theorem 4.1. $\square$

Theorem 5.2. *Let n satisfy (3) for $(k, \ell, r) = (4, 2, 2)$ and let n' satisfy it for $(4, 3, 2)$. Then $n' < n$.*

Proof. Each of the two thresholds is unique when it exists, by (3) and the downward closure noted after Definition 2.1: if $m < m'$ both satisfied it, then $[m' - 1]$ would carry an avoiding colouring whose restriction to $[m]$ avoids, contradicting the second clause at m . So $n = 108$ and $n' = 90$ by Theorem 4.1. $\square$

Theorem 5.2 is the form in which the failure of monotonicity is a statement about the function rather than about two particular numerals. The failure is not one-sided either: the $k = 4$ row reads 108, 90, 180, so $f_2(4; \cdot)$ is neither non-decreasing nor non-increasing. We have no explanation for the dip at $\ell = 3$, and finding one is Question 8.2.

One consequence is practical. A reader who assumes monotonicity, knows $f_2(4; 2) = 108$, and wants $f_2(4; 3)$ will guess a value at least 108; the correct answer is 90. The following is therefore worth stating on its own.

Proposition 5.3. $f_2(4; 3) \neq 108$.

Proof. By Theorem 4.1 no 2-colouring of [90] avoids level 3 at $k = 4$; a fortiori none of [108] does, so 108 fails the first clause of (3). $\square$

6. COMPARISON WITH THE UNRESTRICTED NUMBER

Proposition 6.1. *Let $k \geq 2$, $r \geq 2$, $\ell \geq 1$. If $f_r(k)$ and $f_r(k; \ell)$ both exist then*

$$f_r(k; \ell) \geq f_r(k).$$

No search is involved.

Proof. A solution of level ℓ is in particular a solution, so a colouring that avoids avoids at level ℓ . Suppose $f_r(k; \ell) < f_r(k)$. By (3) for $f_r(k)$ there is an avoiding r -colouring of $[f_r(k) - 1]$, and $f_r(k; \ell) \leq f_r(k) - 1$, so its restriction to $[f_r(k; \ell)]$ avoids and hence avoids level ℓ — contradicting the second clause of (3) for $f_r(k; \ell)$. $\square$

Proposition 6.2. *The two avoidance conditions are genuinely different, and the inequality of Proposition 6.1 can be strict. At $k = r = 2$ and $n = 60$ there is a 2-colouring of $[60]$ that avoids level 2, while no 2-colouring of $[60]$ avoids; and $f_2(2; 2) = 120 = 2 f_2(2)$.*

Proof. The first half of Theorem 4.1 for $(k, \ell) = (2, 2)$ gives an avoiding colouring of $[119]$ for level 2, whose restriction to $[60]$ avoids level 2. That no 2-colouring of $[60]$ avoids is the second clause of (3) for $f_2(2) = 60$ [9, 3], which was re-established here in the same two-sided way as Theorem 4.1. $\square$

Proposition 6.2 is the statement that keeps the subject of this paper from being a repackaging of the unrestricted one: level-2 avoidance at $k = 2$ is strictly weaker than avoidance, at a length where the difference is visible.

Proposition 6.3. *For every level ℓ , no $m < 60$ satisfies (3) for $f_2(2; \ell)$.*

Proof. Proposition 6.1 with $f_2(2) = 60$. $\square$

Remark 6.4. Combining Proposition 6.1 with the lower bound $f_2(k) \geq 3k^2$ for $k \geq 3$ of [3] — the case $r = 2$ of its Theorem 1.1(ii), which that paper shows to be sharp for infinitely many k by its Theorem 1.2 — gives, in prose and with no computation,

$$f_2(k; \ell) \geq 3k^2 \quad (k \geq 3, \ell \geq 1),$$

whenever the left-hand side exists. The computed values are well above it: 84 and 126 against 27 at $k = 3$, and 108, 90, 180 against 48 at $k = 4$. The gap suggests that the restricted problem is not merely the unrestricted one with a constant factor. This inequality quotes a published bound and is accordingly not part of the formal development; Proposition 6.1 is.

7. FURTHER VALUES, COMPUTED OUTSIDE THE FORMAL DEVELOPMENT

The values in this section were obtained by a level-pruned enumerator written in C together with a satisfiability solver, by bisection on n . *They are not part of the formally verified development*, and we label them as such throughout; they are recorded because they are the data any conjecture about the shape of the table would have to fit.

Remark 7.1 (computations outside the formal development). With $r = 2$,

$$f_2(5; 2) = 96, \quad f_2(5; 3) = 126, \quad f_2(6; 2) = 144, \quad f_2(7; 2) = 176, \quad f_2(8; 2) = 240.$$

The values 96 and 126 were located twice, by bisection and by a linear scan of n , with the same outcome. Consequently the level-2 row reads

$$f_2(k; 2) = 120, 84, 108, 96, 144, 176, 240 \quad (k = 2, \dots, 8),$$

which is not monotone in k either, and which the OEIS [10] does not contain, nor does it contain the tail 96, 144, 176, 240. For $5 \leq k \leq 8$ these values satisfy $f_2(k; 2) \leq 4k^2$, with equality at $k = 6$; whether that is a pattern or a coincidence of four data points is Question 8.2. Note also the coincidence $f_2(5; 3) = 126 = f_2(3; 3)$.

The reason these are not proved here is cost, and it is worth recording because it names what would have to improve. The level- ℓ list $L(k, \ell, n)$ is obtained by filtering the full solution list at k , so producing it costs the unrestricted enumeration at that k : measured, 33 seconds in C at $k = 5$, $n = 200$, and level pruning does not help, because the work is in reaching the leaves of the backtracking tree rather than in the leaves. Under the interpreter that evaluates such computations inside the proof assistant we measure a factor of about 25 on this code, which puts a single evaluation at about a quarter of an hour and one $k = 5$ value, which needs two of them, at about half an hour — against forty seconds for the whole $k = 4$ row. The remedy is a different enumeration: instead of enumerating tuples and filtering by level, enumerate the ℓ distinct values $u_1 < \dots < u_\ell$ and their multiplicities $a_1 + \dots + a_\ell = k$ subject to $a_1/u_1 + \dots + a_\ell/u_\ell = 1/u_{\ell+1}$, at cost n^ℓ rather than n^k . This is the normal form [4] uses for the linear equation, and it would make the whole level-2 row an n^2 computation independent of k .

8. OPEN QUESTIONS

Question 8.1. Does $f_r(k; \ell)$ exist for all $k \geq 2$, $r \geq 2$ and $2 \leq \ell \leq k$?

This is the first thing to want, and nothing here proves it: the six values of Theorem 4.1 establish finiteness one case at a time, and Theorem 3.3 settles only the negative half. For the linear equation [4] obtains existence from Rado’s theorem for a single equation together with an upgrade that makes a monochromatic solution element-distinct, applied to a disjunction of equations. A reciprocal analogue would need partition regularity of $a_1/x_1 + \dots + a_\ell/x_\ell = 1/y$ for positive integer coefficients a_i — [1] and [6] give at least the case $a_1 = \dots = a_\ell = 1$ — together with an analogue of that distinctness upgrade for the reciprocal equation, which the linear one does not supply. If it goes through, the dichotomy for $f_r(k; \ell)$ would be the linear one, namely existence exactly when $\ell \geq 2$ or $r = 1$, whose negative half is Theorem 3.3.

Question 8.2. Does the level-2 row $f_2(k; 2)$ have a closed form, as $S_2(k; 2) = k^2 + 2k$ does for the linear equation? What causes the dip $f_2(4; 3) = 90 < 108 = f_2(4; 2)$, and is $k = 4$ the smallest k at which monotonicity in ℓ fails?

The data of Theorem 4.1 and Remark 7.1 neither exhibit a polynomial nor exclude one; the values $k = 9, 10$ at level 2 would settle whether $f_2(k; 2) \leq 4k^2$ survives.

Question 8.3. What is the behaviour of the all-distinct column $f_r(k; k)$, the reciprocal analogue of the all-distinct generalized Schur number $S_r^*(k)$ of [5], which [4] identifies with $S_r(k; k)$?

We know $f_2(2; 2) = 120$, $f_2(3; 3) = 126$, $f_2(4; 4) = 180$. The column grows, but from three points nothing can be said. Its linear counterpart is hard too: the two-colour weak Schur numbers are known only up to $k = 7$, where they read 9, 24, 52, 101, 166, 253 ([10], A365190).

Question 8.4. What is $f_3(2; 2)$?

If it exists then by Proposition 6.1 it is at least $f_3(2) = 3276$ [9]. At $k = 2$ the enumeration is cheap, so this is a question about the strength of the refutation and not about the enumeration — the first three-colour value anyone should attempt.

9. VERIFICATION

Every theorem, proposition and lemma of this paper has been formally verified in Lean 4 (toolchain v4.32.0) against *Mathlib* [8, 7], and the development contains no `sorry`; repository reference to be added at submission. Three kinds of material are outside that guarantee and are marked as such where they occur: Corollary 2.3, the immediate arithmetic consequence of Lemma 2.2; the inequality of Remark 6.4, which quotes a published bound; and the values of Section 7. The remarks and open questions are discussion. The search-free statements — Lemma 2.2, Proposition 2.4, the level-1 results (Propositions 3.1 and 3.4, Theorem 3.3), the soundness and completeness of the solution list

(Proposition 4.2), the encoding fidelity (Proposition 4.3) and the comparison results of Section 6 (Propositions 6.1, 6.2 and 6.3) — depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`; in particular the whole $\ell = 1$ column is proved uniformly in k , r and n , with no evaluation. What is delegated to compiled evaluation is exactly the finite searches behind the six values of Theorem 4.1, two per value: the check that the exhibited bit string of length $n - 1$ leaves no member of the complete list $L(k, \ell, n - 1)$ monochromatic, and the replay of a resolution certificate witnessing that $\Phi(k, \ell, 2, n)$ is unsatisfiable, the certificate being checked by the verified checker distributed with the prover, in the manner of [12]. The formula handed to the solver is not read from a file but printed from the same definition of $\Phi(k, \ell, r, n)$ that Proposition 4.3 is proved about, so there is no second generator to keep in agreement, and clause order — which the certificate format relies on to name clauses by position — is guaranteed by construction rather than by comparison. As a control on the definitions themselves, the published values $f_2(2) = 60$, $f_2(3) = 40$, $f_2(4) = 48$ and $f_3(2) = 3276$ of [9, 3] were reproduced from the definition of $f_r(k)$ by an independent enumerator in C before any restricted value was claimed, and the three resulting clause counts 182, 216 and 690 agreed exactly with those of an earlier independent formalisation of the unrestricted problem. The values of Remark 7.1 are outside all of this and are labelled as such.

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