RESTRICTED GENERALIZED SCHUR NUMBERS: EXACT VALUES AT LEVELS THREE AND FOUR, AND THE THRESHOLD OF OPEN QUESTION 6.4

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Abstract. For integers $r \geq 1$, $k \geq 2$ and $\ell \geq 2$, the restricted generalized Schur number $S_r(k;\ell)$ is the least n such that every r-colouring of $\{1,\dots,n\}$ admits a monochromatic solution of $x_1 + \cdots + x_k = x_{k+1}$ in which exactly $\ell+1$ distinct integers occur. Gaiser introduced these numbers and proved that for each fixed $\ell \geq 2$ one has $S_2(k;\ell) = F(k,\ell) := k^2 + \big[\frac{(\ell+1)(\ell-2)}{2} + 2\big]k + \ell(\ell-2)$ for all sufficiently large k, without determining how large; his Open Question 6.4 asks for the least such threshold K. We compute twenty-one values of $S_r(k;\ell)$ exactly, ten of them new: $S_2(k;3)$ for $4 \leq k \leq 9$ and $S_2(k;4)$ for $5 \leq k \leq 8$. Making the arithmetic of Gaiser's Section 5 explicit, we show that his own sufficient condition for the upper bound holds at level 3 exactly when $k \geq 10$, and at level 4 exactly when $k \geq 20$. Since our values cover every k from 3 to 10 at level 3, this answers Open Question 6.4 there: the least K is 3, the smallest value the definition permits. At level 4 the same reading leaves the eleven values $9 \leq k \leq 19$ open. In the opposite direction we prove the identity $2F(k,k)+(k-1)(k-2) = k^3+4k^2-5k+2$, whose right-hand side is twice a published lower bound for the weak Schur number $S_2(k;k)$; consequently $K > \ell$ for every $\ell \geq 5$, and the closed form cannot begin at $k = \ell$ except at levels 3 and 4. All statements are machine-checked in Lean 4.

1. Introduction

Schur's theorem says that for every r there is an n such that every r-colouring of $\{1,\dots,n\}$ contains a monochromatic solution of $x + y = z$. The *generalized Schur number* $S_r(k)$ replaces the equation by $x_1+\cdots+x_k = x_{k+1}$, and the *weak* Schur number asks in addition that $x_1,\dots,x_k$ be pairwise distinct. Between these two extremes there is a whole scale, and it is the scale that Gaiser [1] isolated.

For a solution $\mathcal{S} = (x_1,\dots,x_k,x_{k+1})$ of

$$x_1 + x_2 + \cdots + x_k = x_{k+1} \tag{1}$$

in positive integers, write $c(\mathcal{S})$ for the number of distinct integers among $x_1,\dots,x_{k+1}$. The *restricted generalized Schur number* $S_r(k;\ell)$ is the least n such that every r-colouring of $\{1,\dots,n\}$ admits a monochromatic solution $\mathcal{S}$ of (1) with $c(\mathcal{S}) = \ell+1$. Taking $\ell = k$ recovers the r-colour weak Schur number, $S_r(k;k) = \mathrm{WS}_k(r)$, and small ℓ interpolates down towards the unrestricted problem. Gaiser also observes that these are disjunctive Rado numbers: the condition $c(\mathcal{S}) = \ell+1$ is a disjunction of equalities and disequalities among the variables.

Gaiser's main theorem is that for every fixed $\ell \geq 2$ there is a K with

$$S_2(k;\ell) = F(k,\ell) := k^2 + \Big[\frac{(\ell+1)(\ell-2)}{2} + 2\Big]k + \ell(\ell-2) \qquad \text{for all } k \geq K, \tag{2}$$

and at $\ell = 2$ he determines K: $S_2(k;2) = k^2+2k$ for $k \geq 3$, with the single exception $S_2(2;2) = 9 \neq 8 = F(2,2)$. For $\ell \geq 3$ the value of K is left undetermined — the proof introduces it as "some large enough K, which will be determined later" and then argues only that a certain quantity linear in k is eventually dominated by the quadratic $F(k,\ell)$. This is the content of his closing question.

Question 1.1 ([1, Open Question 6.4])**.** Let $\ell \geq 3$. What is the smallest $K \in \mathbb{N}$ such that $S_2(k;\ell) = F(k,\ell)$ for all $k \geq K$?

Two features of the problem make it hard to attack from tables. First, Gaiser's paper contains no table of values at all; the only numerical anchors in the literature are the diagonal cells $\ell = k$, which are weak Schur numbers and are tabulated by Ahmed, Boza, Revuelta and Sanz [2] and carried by the OEIS as A365190 and A365191 [5]. Second, $\ell \le k$ is forced by the definition (a solution of (1) has at most $k+1$ distinct entries), so $K \ge \ell$ always, and the question is exactly whether the closed form already holds at the bottom of the admissible range.

What this paper does. We compute $S_r(k;\ell)$ exactly in twenty-one cases, listed in Table 1. Ten of these values appear to be new: $S_2(k;3)$ for $4 \le k \le 9$ and $S_2(k;4)$ for $5 \le k \le 8$. The remaining eleven agree with published data — the whole $\ell = 2$ row is Gaiser's own theorem including its exception at $k = 2$, four diagonal cells are weak Schur numbers from [2], and the two three-colour cells overlap a recent computation of $S_3(k;2)$ [4] — so they serve as a control on the method rather than as results.

We then make Gaiser's Section 5 effective. That section proves the upper bound $S_2(k;\ell) \le F(k,\ell)$ by a case analysis in which every case is a chain of explicit identities among explicit integers, and the argument is legitimate at a given k exactly when those integers lie in $\{1,\dots,F(k,\ell)\}$ and the multiplicity counts it uses are non-negative. Gaiser collects both requirements himself, in a blanket hypothesis $k \ge 3\ell - 1$ and a displayed inequality bounding three case maxima by $F(k,\ell)$. Writing $G_\ell(k)$ for that conjunction, we prove

$$G_3(k) \iff k \ge 10, \qquad G_4(k) \iff k \ge 20,$$

as equivalences, so the thresholds are exact for the condition and not merely sufficient. At level 3 our computed cells cover $3 \le k \le 10$, which closes the gap and answers Question 1.1:

Theorem 1.2 (Theorem 4.3, informally)**.** *Granting Gaiser's Section 5 upper bound in the effective form his own hypotheses give, one has $S_2(k;3) = k^2 + 4k + 3$ for* every $k \ge 3$*. Thus at $\ell = 3$ the least K of Question 1.1 is* 3*, the smallest value the constraint $\ell \le k$ permits.*

At level 4 the same reading gives $K \le 20$ and our cells reach $k = 8$, leaving the eleven values $9 \le k \le 19$ undecided; we explain in Section 4 why closing them is out of reach of the present method even in principle, since the blanket hypothesis $k \ge 3\ell - 1 = 11$ alone forces a computation at $k = 10$, $n = F(10,4) = 178$, whose propositional encoding has 13 158 196 clauses.

For $\ell \ge 5$ the answer goes the other way, and it is a theorem rather than a data point. On the diagonal the closed form collapses to the cubic $F(k,k) = (k^3 + 3k^2 - 2k)/2$, and we prove the identity

$$2F(k,k) + (k-1)(k-2) = k^3 + 4k^2 - 5k + 2,$$

whose right-hand side is exactly twice the lower bound $\mathrm{WS}_k(2) \ge (k^3+4k^2-5k+2)/2$ that [2, Lemma 2.1] establishes for $k \ge 5$. So $F(k,k)$ falls short of a published lower bound for the very quantity it is supposed to equal, by exactly $(k-1)(k-2)/2$, and therefore $S_2(\ell;\ell) \ne F(\ell,\ell)$ for every $\ell \ge 5$: the least K of Question 1.1 exceeds ℓ at every level from 5 upwards. The case $\ell = 5$ is also visible directly in Table 1, where $S_2(5;5) = 101$ against $F(5,5) = 95$. Levels 3 and 4 are thus the only two levels at which $K = \ell$ is possible at all, and they are exactly the two levels at which our computations observe it.

Section 2 fixes notation and records the normal form we compute with. Section 3 states the values. Section 4 treats Question 1.1. Section 5 describes the search and the checks that constrain it, and Section 6 says what was verified and how.

2. Preliminaries

Throughout, $r \ge 1$ is the number of colours, $k \ge 2$ the number of summands in (1), and $\ell \ge 2$ the *level*. An r-colouring of $\{1,\dots,n\}$ is a map $\chi\colon \{1,\dots,n\} \to \{0,\dots,r-1\}$, and a solution of (1) is *monochromatic* for χ if all of $x_1,\dots,x_{k+1}$ lie in $\{1,\dots,n\}$ and receive the same colour.

Definition 2.1. $S_r(k;\ell)$ is the least $n \ge 1$ such that every r-colouring of $\{1,\dots,n\}$ admits a monochromatic solution $\mathcal{S}$ of (1) with $c(\mathcal{S}) = \ell + 1$.

Two elementary remarks constrain the range of the parameters. A solution of (1) with all x_i positive has $x_{k+1} > x_i$ for every $i \le k$, so the right-hand side is always a value not occurring on the left; hence $2 \le c(\mathcal{S}) \le k+1$, and $c(\mathcal{S}) = \ell + 1$ says exactly that the multiset $\{x_1,\dots,x_k\}$ has ℓ distinct elements.

r	ℓ	k	$S_r(k;\ell)$	$F(k,\ell)$	clauses
2	2	2	9	8	41
2	2	3	15	15	103
2	2	4	24	24	240
2	2	5	35	35	461
2	2	6	48	48	806
2	3	3	24	24	572
2	3	4	35	35	2 669
2	3	5	48	48	8 212
2	3	6	63	63	20 349
2	3	7	80	80	43 986
2	3	8	99	99	85 785
2	3	9	120	120	155 408
2	3	10	143	143	264 521
2	4	4	52	52	17 608
2	4	5	68	68	105 454
2	4	6	86	86	391 794
2	4	7	106	106	1 143 610
2	4	8	128	128	2 858 958
2	5	5	101	95	1 014 035
3	2	2	24	—	420
3	2	3	58	—	2 437

Table 1. The twenty-one values established here, with the number of clauses in the propositional encoding refuted at $n = S_r(k;\ell)$. New here: $S_2(k;3)$ for $4 \le k \le 9$ and $S_2(k;4)$ for $5 \le k \le 8$. The column $F(k,\ell)$ is Gaiser's closed form (2), which is defined for $r = 2$ only.

In particular $S_r(k;\ell)$ is defined only for $2 \le \ell \le k$, so any threshold K answering Question 1.1 satisfies $K \ge \ell$. Gaiser's Theorem 2.4 settles existence: $S_r(k;\ell)$ is finite if and only if $\ell \ge 2$ or $r = 1$.

The formulation of Definition 2.1 quantifies over the $k+1$ variables of the equation, which is the right form to check against [1] and the wrong form to search with: it costs n^k. Everything computational below is done in Gaiser's normal form instead.

Proposition 2.2 (Gaiser [1, Lemma 2.1])**.** *Let $C \subseteq \mathbb{N}$ and let $k \ge 2$, $\ell \ge 1$. The following are equivalent.*

(1) *There are $x_1, \dots, x_k \in C$, all positive, with $x_1 + \cdots + x_k \in C$ and exactly ℓ distinct values among $x_1, \dots, x_k$.*
(2) *There are $u_1 < u_2 < \cdots < u_\ell$ in C, all positive, and integers $a_1, \dots, a_\ell \ge 1$ with $a_1 + \cdots + a_\ell = k$, such that $a_1u_1 + \cdots + a_\ell u_\ell \in C$.*

Both directions are used below, and by opposite halves of the work: (ii)⇒(i) is what makes an upper bound legitimate — a constraint may be imposed on a colouring only because the tuple that generated it really is a solution — while (i)⇒(ii) is what makes a lower bound legitimate, since a solution the normal-form enumeration missed would make a witness colouring look admissible when it is not. We write $u = (u_1, \dots, u_\ell)$ for the tuple of distinct values and $a = (a_1, \dots, a_\ell)$ for its multiplicities, and call (u, a) a *normal-form solution* of level ℓ with sum k; its *members* are $u_1, \dots, u_\ell$ and $\sum_i a_iu_i$.

Finally, $F(k,\ell)$ is as in (2). The three levels used below are

$$F(k,2) = k^2 + 2k, \qquad F(k,3) = k^2 + 4k + 3, \qquad F(k,4) = k^2 + 7k + 8,$$

and on the diagonal $F(k,k) = \frac{1}{2}(k^3 + 3k^2 - 2k)$.

3. Exact values

Theorem 3.1. *For every k with $3 \le k \le 10$,*

$$S_2(k;3) = k^2 + 4k + 3,$$

that is, $S_2(k;3) = 24, 35, 48, 63, 80, 99, 120, 143$ *for* $k = 3, \dots, 10$.

Of these eight values, $S_2(3;3) = 24$ is the weak Schur number $\mathrm{WS}_3(2)$ recorded in [2, Table 3]; $S_2(10;3) = 143$ is a consequence of Gaiser's theorem once the threshold of Section 4 is known, and was computed here as a control on that threshold rather than as a new value. The six values at $4 \le k \le 9$ are new.

Theorem 3.2. *For every k with $4 \le k \le 8$,*

$$S_2(k;4) = k^2 + 7k + 8,$$

that is, $S_2(k;4) = 52, 68, 86, 106, 128$ *for* $k = 4, \dots, 8$.

Here $S_2(4;4) = 52$ is the weak Schur number $\mathrm{WS}_4(2)$ recorded in [2, Table 3]; the four values at $5 \le k \le 8$ are new. In both theorems the range of k begins at the smallest value the definition admits, namely $k = \ell$.

The next two propositions record the cells that agree with published data. None of them was read off a table: each was computed by the procedure of Section 5 and only then compared, so the agreement is a test of the procedure.

Proposition 3.3.
1. $S_2(k;2) = 9, 15, 24, 35, 48$ *for* $k = 2, 3, 4, 5, 6$. *In particular* $S_2(k;2) = k^2 + 2k$ *for* $3 \le k \le 6$, *and* $S_2(2;2) = 9 \ne 8 = F(2,2)$.
2. $S_2(2;2) = 9$, $S_2(3;3) = 24$, $S_2(4;4) = 52$, $S_2(5;5) = 101$ *and* $S_3(2;2) = 24$.

Part (i) is Gaiser's Theorem 1.1 at $\ell = 2$, exception included, so the failure of the closed form at $k = 2$ is here a verified fact rather than a quoted one. Part (ii) lists the diagonal cells, where $S_r(k;k) = \mathrm{WS}_k(r)$: the four two-colour values are the first four entries of A365190 and the last is the first entry of A365191, both sequences being sourced to [2]. Equivalently $S_3(2;2) = \mathrm{WS}(3) + 1$ with $\mathrm{WS}(3) = 23$, the classical weak Schur number recorded in A072842 and, in the complementary convention, in A118771 [5].

Proposition 3.4. $S_3(2;2) = 24$ *and* $S_3(3;2) = 58$, *while the expression* $k^3 + 3k^2 + k - 1$ *of* [1, Open Question 6.2] *takes the values* 21 *and* 56 *at* $k = 2$ *and* $k = 3$.

Gaiser's Proposition 6.1 proves only the lower bound $S_3(k;2) \ge k^3 + 3k^2 + k - 1$, and his Open Question 6.2 asks whether equality holds for all large k; Proposition 3.4 therefore contradicts neither, but it does say that the three-colouring behind Proposition 6.1 is not extremal at small k. These two values were subsequently found to overlap the computation of [4], which reports $S_3(k;2)$ for $3 \le k \le 11$ and observes that all nine values miss the expression by exactly $k - 1$, matching $k(k+1)(k+2) - 2$; at $k = 3$ that is 58, in agreement with Proposition 3.4. The two computations are independent, ours predating our discovery of [4], so each confirms the other. That record reports validating its encoding against sixteen previously published values before applying it to the open question, among them the 3-colour Schur numbers $S_3(k) = k^3 + 2k^2 - 2$ of [3]. We add nothing to that range and record the cells here only as further control.

4. Open Question 6.4

4.1. **Gaiser's Section 5, made effective.** The upper bound in (2) is proved in [1, §5] by a four-way case analysis on the lexicographically least monochromatic ℓ-subset $u_1 < \cdots < u_\ell$ of $\{1, \dots, 2\ell - 1\}$. Each case is a chain of explicit identities between explicit integers, and each is valid at a given k exactly when the integers it names lie in the interval $\{1, \dots, F(k,\ell)\}$ under consideration and the multiplicity counts it forms are non-negative. Gaiser states both requirements himself: the blanket hypothesis $K \geq 3\ell - 1$, imposed at the start of the section and used by every case for the counts, and the displayed inequality

$$F(k,\ell) \;\geq\; \max\bigl\{\, 6k + 4\ell^2 - 10\ell + 1,\;\; V + U + (k-\ell)u_j - v,\;\; 8k + 10 \,\bigr\},$$

one entry per case, whose middle term he bounds in his own final display by $(k-\ell)\ell + \ell\bigl[\ell(2\ell-1) - \frac{\ell(\ell-1)}{2}\bigr] + \ell(k-\ell)(2\ell-2) + \frac{(\ell-1)(\ell-2)}{2}$.

Definition 4.1. For integers k, ℓ put

$$\begin{aligned} A(k,\ell) &= 6k + 4\ell^2 - 10\ell + 1,\\ B(k,\ell) &= (k-\ell)\ell + \ell\Bigl[\ell(2\ell-1) - \frac{\ell(\ell-1)}{2}\Bigr] + \ell(k-\ell)(2\ell-2) + \frac{(\ell-1)(\ell-2)}{2},\\ C(k) &= 8k + 10, \end{aligned}$$

and let $G_\ell(k)$ be the conjunction

$$3\ell - 1 \leq k \quad\text{and}\quad A(k,\ell) \leq F(k,\ell) \quad\text{and}\quad B(k,\ell) \leq F(k,\ell) \quad\text{and}\quad C(k) \leq F(k,\ell).$$

All four quantities are read as integers, so no subtraction is truncated and the two divisions by 2 are exact at the values used. Because B is Gaiser's own bound for Case 3's largest integer rather than the exact maximum over the configurations of that case, G_ℓ is a sufficient condition that is possibly weaker than a case-by-case analysis would give; a threshold computed from it is therefore an upper bound for the true least k at which the Section 5 argument goes through. Case 1 imposes no condition of its own: its largest integer is $F(k,\ell)$ itself, and the intermediate quantity it uses is $F(k,\ell) - k - \frac{(\ell-1)(\ell-2)}{2} + 1 \leq F(k,\ell)$.

Lemma 4.2. *For every integer k, $\;G_3(k) \iff k \geq 10$.*

Proof. At $\ell = 3$ the three case maxima simplify to $A(k,3) = 6k+7$, $B(k,3) = 15k-8$ and $C(k) = 8k+10$, against $F(k,3) = k^2+4k+3$; the blanket hypothesis is $k \geq 8$. The binding constraint is Case 3: $k^2+4k+3 \geq 15k-8$ fails at $k = 9$ $(120 < 127)$ and holds at $k = 10$ $(143 \geq 142)$, and the quadratic $k^2 - 11k + 11$ is increasing beyond $k = 10$, so it holds for all $k \geq 10$. The other two constraints and the blanket hypothesis are implied at $k \geq 10$. Both directions are checked: the forward one by refuting each $k \leq 9$ individually, so the equivalence is exact rather than a sufficient condition. ☐

Theorem 4.3. *Assume*

$$(\dagger)\qquad S_2(k;3) = k^2 + 4k + 3 \textit{ for every } k \textit{ with } G_3(k).$$

Then $S_2(k;3) = k^2+4k+3$ for every $k \geq 3$. Consequently the least K of Question 1.1 at $\ell = 3$ is $K = 3$.

Proof. Let $k \geq 3$. If $k \leq 10$ the conclusion is Theorem 3.1. If $k \geq 11$ then $G_3(k)$ holds by Lemma 4.2 and the conclusion is $(\dagger)$. Since $\ell = 3$ forces $k \geq 3$, no smaller K is admissible. ☐

Hypothesis $(\dagger)$ is Gaiser's Theorem 1.1 restricted to the range in which his Section 5 proof supplies its own justification; it is carried as an explicit hypothesis rather than asserted, because the correctness of that proof is quoted here and not re-derived. What is established unconditionally is everything else: the eight cells at $3 \leq k \leq 10$, and the fact that G_3 begins at exactly $k = 10$, so that $k \in \{8, 9\}$ is the entire gap between the computed range and the range Gaiser's argument reaches. Both of the cells filling that gap, $S_2(8;3) = 99$ and $S_2(9;3) = 120$, are new.

The redundancy at $k = 10$ is deliberate. If Lemma 4.2 reads Section 5 correctly, then $S_2(10;3) = 143$ is a consequence of Gaiser's own theorem, so computing it independently tests the reading. It agrees.

Remark 4.4. Question 1.1 has a second half, asking the same for the relaxed variant $S_2(k;\tilde{\ell})$, defined by demanding $c(\mathcal{S}) \geq \ell + 1$ instead of $c(\mathcal{S}) = \ell + 1$. At $\ell = 3$ the answer follows from Theorem 4.3 with no further computation, using two statements of Gaiser's: $S_2(k;\tilde{\ell}) \leq S_2(k;\ell)$, immediate from the definitions since the relaxed variant asks for more solutions to be present, and the lower bound $S_2(k;\tilde{\ell}) \geq F(k,\ell)$ of his Corollary 6.3, which is asserted for all $k \geq 2$ and not only for large k. Sandwiching, $F(k,3) \leq S_2(k;\tilde{3}) \leq S_2(k;3) = F(k,3)$, so $S_2(k;\tilde{3}) = F(k,3)$ for every $k \geq 3$ and the least threshold is again 3. This remark is prose resting on the two quoted statements; the relaxed variant is not part of the formal development of Section 6.

4.2. **Level four, and why it does not close.**

Lemma 4.5. *For every integer k, $G_4(k) \iff k \geq 20$. Consequently, with Theorem 3.2, the values of $S_2(k;4)$ left undecided by the combination of Gaiser's Section 5 and the present computation are exactly those with $9 \leq k \leq 19$.*

Proof. At $\ell = 4$ the case maxima are $A(k,4) = 6k+25$, $B(k,4) = 28k-21$ and $C(k) = 8k+10$, against $F(k,4) = k^2 + 7k + 8$, with blanket hypothesis $k \geq 11$. Again Case 3 binds: $k^2 + 7k + 8 \geq 28k - 21$ fails at $k = 19$ ($502 < 511$) and holds at $k = 20$ ($548 \geq 539$), and thereafter. Each $k \leq 19$ is refuted individually. □

Level 4 therefore does not close, and the obstruction is not the width of the gap alone. The blanket hypothesis $k \geq 3\ell - 1 = 11$ is imposed before any case is examined, so even an exact analysis of Case 3 — which would lower the Case 3 requirement, and does: running the maximum over the actual configurations rather than over Gaiser's bound gives $14k - 11$ at $\ell = 3$ and $26k - 29$ at $\ell = 4$, i.e. thresholds 9 and 17 — still leaves a cell at $k = 10$ to be decided by computation. That cell has $n = F(10,4) = 178$, and its propositional encoding (Section 5) has 13 158 196 clauses, about 4.6 times the largest instance handled here. We have not attempted it. It is worth stressing that the sharper Case 3 bound just quoted is deliberately *not* the one formalised: a threshold derived from Gaiser's own displayed algebra is a statement about his paper, whereas one derived from a re-analysis is a statement about the re-analysis.

4.3. **Levels five and above.**

Theorem 4.6. (1) $S_2(5;5) = 101$, *whereas* $F(5,5) = 95$.
(2) *For every integer $k \geq 2$,*

$$2F(k,k) + (k-1)(k-2) = k^3 + 4k^2 - 5k + 2.$$

Proof. Part (i) is one of the cells of Table 1, established by the two-sided procedure of Section 5, together with the evaluation $F(5,5) = 25 + 55 + 15 = 95$. For part (ii), $F(k,k) = k^2 + \left[\frac{(k+1)(k-2)}{2} + 2\right]k + k(k-2) = \frac{1}{2}(k^3 + 3k^2 - 2k)$, the halving being exact because $(k+1)(k-2)$ is even for every k; adding $(k-1)(k-2) = k^2 - 3k + 2$ to $k^3 + 3k^2 - 2k$ gives $k^3 + 4k^2 - 5k + 2$. In the formal development the identity is stated with $k = m + 2$ so that no subtraction is truncated, and is proved by a parity split on m to discharge the halving. □

Corollary 4.7. *For every $\ell \geq 5$, the least K of Question 1.1 satisfies $K > \ell$.*

Proof. Ahmed, Boza, Revuelta and Sanz [2, Lemma 2.1] prove $\mathrm{WS}_k(2) \geq (k^3 + 4k^2 - 5k + 2)/2$ for $k \geq 5$, by an explicit two-interval colouring. Since $S_2(k;k) = \mathrm{WS}_k(2)$, Theorem 4.6(ii) gives

$$S_2(k;k) - F(k,k) \;\geq\; \frac{k^3 + 4k^2 - 5k + 2}{2} - F(k,k) \;=\; \frac{(k-1)(k-2)}{2} \;>\; 0$$

for every $k \geq 5$. Taking $k = \ell$ shows that the closed form fails at the smallest admissible k, so no threshold equal to ℓ can answer Question 1.1 at that level. □

The lower bound of [2] is quoted, not reproved; the identity of Theorem 4.6(ii), which is the step that is not evident, is what has been verified. At $\ell = 5$ the conclusion is independent of the quotation, being visible in Theorem 4.6(i).

Remark 4.8. The size of the gap explains the shape of the older literature. At $k = 6, 7, 8, 9$ one computes $F(k, k) = 156$, 238, 344, 477, which is exactly the row of lower bounds ≥ 156, ≥ 238, ≥ 344, ≥ 477 recorded in [2, Table 3]; that is, the diagonal construction underlying (2) is the classical one, which the construction of [2, Lemma 2.1] superseded. Already at $k = 5$ the same table carries the exact value 101 in place of the old bound $F(5, 5) = 95$.

Remark 4.9. Two conventions in [1] deserve a word, since Remark 4.4 depends on the first. (a) The introduction defines the relaxed variant by $c(\mathcal{S}) \geq \ell + 1$, while Section 6 defines it by $c(\mathcal{S}) \geq \ell$. The second reading is untenable: every solution of (1) has $c(\mathcal{S}) \geq 2$, so under it $S_2(k; \tilde{2})$ would equal the ordinary generalized Schur number $S_2(k) = k^2 + k - 1$, contradicting the assertion $S_2(k; \tilde{2}) = k^2 + 2k$ made in Corollary 6.3 of the same section. Under the introduction's reading, on the other hand, our computations give $S_2(k; \tilde{2}) = 9, 15, 24, 35, 48$ for $k = 2, \ldots, 6$ and $S_2(k; \tilde{3}) = 24, 35, 48, 63$ for $k = 3, \ldots, 6$, reproducing every value Section 6 asserts and satisfying Corollary 6.3 with equality. We read Section 6's "$c(\mathcal{S}) \geq \ell$" as a typographical slip for "$c(\mathcal{S}) \geq \ell + 1$". (b) The concluding sentence of Section 5 reads "for all $k \in K$" and "for all $k \geq T$" where the preceding sentence correctly has $k \geq K$; both are slips for $k \geq K$, since K is a natural number and $T = k + \frac{(\ell+1)(\ell-2)}{2}$ exceeds k for every $\ell \geq 3$. Neither point affects the mathematics.

5. How the values were obtained

Each value in Table 1 is established in two halves. The lower half, $S_r(k; \ell) > n - 1$, is witnessed by an explicit colouring of $\{1, \ldots, n-1\}$ with no monochromatic solution of level ℓ; checking it requires enumerating every candidate solution inside $\{1, \ldots, n-1\}$, so the enumeration must be *complete*. The upper half, $S_r(k; \ell) \leq n$, is a statement about all r^n colourings; it is reduced to the unsatisfiability of a propositional formula and certified, so the enumeration behind that formula must be *sound*. The two halves fail in opposite directions, which is what makes the pair informative: a solution the enumeration misses invalidates the witness, and a spurious solution it invents invalidates the refutation.

5.1. **The enumeration.**

Lemma 5.1. *Fix k, ℓ, n. Let $E(k, \ell, n)$ be the list obtained by running over all strictly increasing tuples $u = (u_1, \ldots, u_\ell)$ of positive integers with $u_1 + \cdots + u_\ell \leq n$, in lexicographic order, and for each of them over all compositions $a = (a_1, \ldots, a_\ell)$ of k into ℓ positive parts, in lexicographic order, emitting the pair $\left(u, \sum_i a_i u_i\right)$ whenever $\sum_i a_i u_i \leq n$. Then $E(k, \ell, n)$ contains a pair (u, v) if and only if there is a normal-form solution (u, a) of level ℓ with sum k all of whose members lie in $\{1, \ldots, n\}$ and with $v = \sum_i a_i u_i$.*

Proof sketch. Soundness is a structural induction on the two recursions. For completeness the only point needing an argument is that the budget $u_1 + \cdots + u_\ell \leq n$ discards nothing: if $a_i \geq 1$ for every i then $\sum_i u_i \leq \sum_i a_i u_i$, so a tuple whose plain sum already exceeds n carries no normal-form solution with all members in $\{1, \ldots, n\}$. Hence the budget is exact, not a heuristic. □

The budget is what makes the larger cells feasible. At $\ell = 5$, $n = 96$ the budgeted recursion visits 385 499 tuples where the unbudgeted one would visit $\binom{96}{5} = 61\,124\,064$, a reduction by a factor of about 160, and completeness costs exactly one extra inequality.

5.2. **The propositional encoding.** Fix k, ℓ, r, n with $\ell \geq 1$. Introduce a variable $X_{i,j}$ for each $i \in \{1, \ldots, n\}$ and each colour $j \in \{0, \ldots, r-1\}$, meaning "i has colour j", and let $\Phi(k, \ell, r, n)$ be the conjunction of

- $X_{i,0} \vee \cdots \vee X_{i,r-1}$ for each $i \in \{1, \ldots, n\}$, and
- $\neg X_{w_1,j} \vee \cdots \vee \neg X_{w_{\ell+1},j}$ for each pair $(u, v) \in E(k, \ell, n)$ with members $w_1, \ldots, w_{\ell+1}$ and each colour j.

Proposition 5.2. *If χ is an r-colouring of $\{1, \ldots, n\}$ admitting no monochromatic solution of* (1) *with $c(\mathcal{S}) = \ell + 1$, then the assignment $X_{i,j} := [\chi(i) = j]$ satisfies $\Phi(k, \ell, r, n)$. Consequently, if $\Phi(k, \ell, r, n)$ is unsatisfiable then $S_r(k; \ell) \leq n$.*

Proof sketch. The at-least-one-colour clauses hold because χ is total. For a solution clause, let $(u, v) \in E(k, \ell, n)$ and j be given. By Lemma 5.1 there is a composition a making (u, a) a normal-form solution with members $w_1, \ldots, w_{\ell+1}$ in $\{1, \ldots, n\}$, and by Proposition 2.2, direction (ii)⇒(i), expanding a_i copies of u_i produces a genuine solution of (1) with exactly $\ell + 1$ distinct integers. Were all of $w_1, \ldots, w_{\ell+1}$ coloured j, that solution would be monochromatic; so some member is not, and the clause is satisfied. The second sentence is the contrapositive. □

Only this direction is proved, and only this direction is needed. The converse — that a satisfying assignment yields an admissible colouring — is never used, which is why no at-most-one-colour clauses appear: adding a colouring to the satisfying assignments can only make a refutation harder to find, never make one unsound. The same asymmetry means that an omitted or duplicated solution clause cannot produce a false theorem, only a failed computation.

For the lower half the direction of Proposition 2.2 used is (i)⇒(ii), together with the completeness half of Lemma 5.1: a candidate colouring χ of $\{1, \ldots, n-1\}$ is accepted only when, for every pair in $E(k, \ell, n-1)$ and every colour, some member of that pair is not given that colour.

5.3. **Controls.** Five things could be wrong while every theorem above still held vacuously, and each is tested.

Proposition 5.3. (1) The predicate is inhabited. $1 + 2 + 3 = 6$ *is a solution of level* 3*, and the pair it induces occurs in* $E(3, 3, 6)$*.*

(2) The level counts what it should. $1 + 1 + 4 = 6$ *is a solution of level* 2*, not* 3*; and* $S_2(3; 2) = 15 \neq 24 = S_2(3; 3)$*, so the two levels are different problems at the same* k*.*

(3) The encodings are not vacuously unsatisfiable. $\Phi(7, 3, 2, 79)$ *and* $\Phi(3, 2, 3, 57)$ *are satisfiable, one step below the corresponding thresholds.*

(4) No value is off by one. *For each cell the neighbouring values, and distant values in both directions, are refuted; each cell moreover comes with the statement that no other value can be the threshold.*

(5) Published expressions are tested, not fitted. $F(2, 2) = 8$ *is refuted as the value of* $S_2(2; 2)$*, and the expression of* [1, Open Question 6.2] *is refuted as the value of* $S_3(k; 2)$ *at* $k = 2$ *and* $k = 3$*.*

Remark 5.4. A remark on scale, since it is what limits the tables above. Across instances of 505 714, 1 143 610 and 2 858 958 clauses the memory used to check a refutation inside the proof assistant grows linearly, at 0.65, 0.68 and 0.67 KiB per clause respectively — flat over a factor of 5.7, because the cost is dominated by the representation of the formula itself. Under a 4 GiB budget this puts the ceiling at roughly six million clauses. Certificate size predicts nothing: the largest instance here has a 23 KB certificate against 57 KB for one 2.5 times smaller. The satisfiability solver is nowhere near its limit on any two-colour instance in this range, the hardest taking 3.55 seconds.

6. Verification

Every numbered theorem, proposition, lemma and corollary of this paper has been formally stated and proved in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; the development contains no `sorry`. The definition verified is Definition 2.1, stated over the variables of (1) exactly as written there, and Proposition 2.2 is proved in both directions so that the normal form used for computation is provably the same object. Each value of Table 1 is proved as the conjunction of a witnessed lower bound and a certified upper bound: the witness colouring is an explicit array, checked by evaluation against the complete enumeration of Lemma 5.1; the upper bound is an LRAT refutation of $\Phi(k, \ell, r, n)$, checked by Lean's verified certificate checker, with the formula rebuilt inside the proof assistant from Lemma 5.1 rather than read from a file. Since LRAT identifies input clauses by position, the clause order is load-bearing, and the agreement between the in-system formula and the generator that produced the certificates was checked by byte comparison of the emitted DIMACS at three different parameter triples rather than assumed. The reduction of Proposition 5.2 and the enumeration correctness of Lemma 5.1

are proved outright; the two evaluation steps per cell — certificate checking and witness checking — are performed by compiled evaluation and are the only additional trust assumption. The exceptions to the first sentence are Remark 4.4, Corollary 4.7 and Remark 4.9, which draw on statements quoted from [1] and [2] that are not themselves formalised; in each case the quoted statement is identified in the text, and the step that is not evident — the identity of Theorem 4.6(ii) — is verified. The reflected-certificate technique used here follows [6]. Repository reference to be added at submission.

References

[1] C. Gaiser, *Restricted generalized Schur numbers*, arXiv:2608.08789v1 (9 August 2026).

[2] T. Ahmed, L. Boza, M. P. Revuelta and M. I. Sanz, *Exact values and lower bounds on the n-color weak Schur numbers for $n = 2, 3$*, Ramanujan J. **62** (2023), 347–363; doi:10.1007/s11139-023-00760-y.

[3] L. Boza, J. M. Marín, M. P. Revuelta and M. I. Sanz, 3*-color Schur numbers*, Discrete Appl. Math. **263** (2019), 59–68; doi:10.1016/j.dam.2018.06.030.

[4] AI Daiki, *Exact values of the restricted generalized Schur numbers $S_3(k; 2)$ for $k = 3, \ldots, 11$: Gaiser's conjectured formula fails throughout the computed range*, Zenodo preprint (19 August 2026), doi:10.5281/zenodo.22012336.

[5] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A365190, A365191, A072842, A118771 (accessed 28 August 2026).

[6] S. Szeider, *LRAT-Catcher: importing SAT solver certificates into Lean4 by reflection*, arXiv:2607.00815 (1 July 2026).