SQUARES TILED BY PAIRWISE UNEQUAL RECTANGLES: THE SEVEN-PIECE COLUMN

KOREA SUPERINTELLIGENCE LABS

Abstract. For integers $n, m \geq 1$ let $W^*(n,m)$ be the minimum total length of the interior segments of a partition of the square $[0,n]^2$ into m axis-aligned rectangles with integer corners and pairwise distinct (unordered) dimension pairs, when such a partition exists. Lago Gómez, who introduced this quantity as the two-dimensional core of a problem about tilings of a cube, proved $W^*(n,m) = n + 2(m-2)$ throughout the middle regime $T_{m-2} \leq n < T_{m-1}$ for $4 \leq m \leq 6$, conjectured it for all $m \geq 4$, and asked for the case $m = 7$, where the middle regime is the finite range $15 \leq n \leq 20$. We settle that range: $W^*(n,7) = n + 10$ for $15 \leq n \leq 20$, the upper bounds by explicit tilings and the matching lower bounds by two independent exhaustive searches. The same two searches determine the whole deep regime of the same column, $W^*(n,7)$ for $5 \leq n \leq 14$, where the value climbs in steps of $+1$, drops once at $n = 10$, and reaches its middle-regime value three steps early, at $n = 12$. Independently of any search we prove three structural facts about this tiling class: the identity $W(T) + 2n = \sum_i (w_i + h_i)$ for *every* tiling; the piece-count ceiling $4m \leq n^2 + 6$, which settles every non-existence entry of the table of W^* for $3 \leq n \leq 5$ and gives the maximum piece counts $3, 5, 7$ at $n = 3, 4, 5$; and a layer-cake bound on the total semiperimeter, from which $W^*(4,5) = 10$ and $W^*(5,7) = 19$ follow with both halves proved. The last of these is the first exact value in the open column $m = 7$ that does not rest on a search. Every theorem, proposition, lemma and corollary below is machine-checked in Lean 4, except where a heading marks part of it as resting in addition on exhaustive computation.

1. Introduction

The quantity. Cut the square $[0,n]^2$ into m axis-aligned rectangles with integer corners, no two of them congruent, and measure the total length of the cuts. The minimum,

$$W^*(n,m),$$

is the object of this paper. It was introduced by Lago Gómez [1] as the two-dimensional core of a three-dimensional problem: the minimum total surface area of k boxes with integer sides and pairwise distinct dimension multisets whose union is the cube $[0,n]^3$. His reduction theorem [1, Theorem 3] states that within an explicit range the three-dimensional minimum interface is $n^2 + W^*(n, k-1)$, so the whole cube problem rests on the plane quantity above.

Two features make W^* delicate. First, it need not be monotone in n (Remark 6.4), and it is not always defined: for small n there are simply not enough pairwise non-congruent rectangles to go round. Second, the non-congruence constraint is a genuine constraint — it strictly raises the optimum on a nontrivial range of (n,m) — and outside that range it does nothing at all. Writing $T_j = j(j+1)/2$ for the triangular numbers, the picture proved in [1] is:

- *high regime*, $n \geq T_{m-1}$. The *comb* — one $(n-1) \times n$ block together with $m-1$ sticks $1 \times w_i$ of distinct widths summing to n — gives $W = n + m - 2$, and this matches the unrestricted lower bound $W \geq n + m - 2$, valid whenever $n \geq m - 2$ for every partition of the square into m rectangles with integer corners, congruent pieces allowed [1, Lemma 12]. So $W^*(n,m) = n + m - 2$ there, and non-congruence costs nothing. The threshold is exactly triangular because $m - 1$ distinct positive widths sum to at least T_{m-1}.
- *middle regime*, $T_{m-2} \leq n < T_{m-1}$. Here the comb's sticks can no longer all be one cell wide, and the value jumps by $m - 2$. Lago Gómez proves

$$W^*(n,m) = n + 2(m-2) \tag{1}$$

for $m = 4$ [1, Proposition 17], $m = 5$ [1, Proposition 18] and, by an exhaustive certificate, $m = 6$ [1, Proposition 19], and conjectures it in general [1, Conjecture 20].

- *deep regime*, $n < T_{m-2}$. No formula is proved. Below the threshold [1] records only the cells its middle-regime propositions happen to reach — $W^*(n, 5) = n + 6$ at $n = 4, 5$ and $W^*(n, 6) = n + 8$ at $n = 8, 9$ — together with $W^*(5, 6) = 14$ and the non-existence of $W^*(3, 4)$ in its Remark 16. The rest is Open Problem (2).

The first open column is $m = 7$, and the paper's Open Problem (1) names it precisely:

> "Conjecture 20 for $m \geq 7$: the middle regime of $m = 6$ fell to an exhaustive certificate (Proposition 19); for $m = 7$ the range $T_5 = 15 \leq n \leq 20$ is equally finite and only computationally heavier."

Its Open Problem (2) asks for "the deep regimes (W^* below T_{m-2}), where the staircase steps of $+1$ appear".

What is settled here. We answer both, at $m = 7$.

(1) **The middle regime.** $W^*(n, 7) = n + 10$ for every n with $15 \leq n \leq 20$ (Theorem 6.1), so (1) holds at $m = 7$. The upper bounds are explicit tilings, formally verified; the matching lower bounds come from two independent exhaustive searches that agree at every cell.

(2) **The deep regime.** $W^*(n, 7)$ for $5 \leq n \leq 14$ is

$$19,\ 19,\ 20,\ 21,\ 22,\ 21,\ 22,\ 22,\ 23,\ 24,$$

i.e. $n + 13$ at $n = 8, 9$, then $n + 11$ at $n = 10, 11$, then already the middle-regime value $n + 10$ from $n = 12$ on (Theorem 6.2). The value therefore switches on three steps below the threshold $T_5 = 15$; the source reports an overshoot of two steps at $m = 6$. Below $n = 5$ no tiling exists at all, and that part is a theorem rather than a search (Corollary 4.2).

(3) **Structure, independent of any search.** Section 3 proves the area identity $\sum_i w_i h_i = n^2$ and the wall–semiperimeter identity $W(T) + 2n = \sum_i (w_i + h_i)$ for every tiling, the latter being the arithmetic form that [1, Remark 15] obtains by a strip count, proved here from the geometric definition of W, by a different argument. Section 4 proves the piece-count ceiling $4m \leq n^2 + 6$ and reads it backwards to settle every non-existence entry of the table of W^* for $3 \leq n \leq 5$, together with the maximum piece counts $3, 5, 7$ at $n = 3, 4, 5$. Section 5 proves a layer-cake lower bound on the total semiperimeter, and with it the two exact values $W^*(4, 5) = 10$ and $W^*(5, 7) = 19$ — both halves, no search. The second is the first exact value in the open column $m = 7$ obtained without an exhaustion.

Standing conventions. Every statement labelled Theorem, Proposition, Lemma or Corollary below is machine-checked (Section 8), with one exception: a heading that calls part of a statement *computational* means that that part additionally rests on an exhaustive search carried out outside the proof assistant, and the proof sketch then says exactly which finite search was run. In every such case the corresponding upper bound is separately a theorem. Remarks marked *(outside the formal development)* record computations that are neither theorems nor certified by the verification of Section 8.

Related work. The class of objects here — partitions of a square into pairwise non-congruent integer-sided rectangles — is exactly the class of the *Mondrian art problem*, whose literature optimises a different functional of the same tilings, namely the *defect*, the difference between the largest and the smallest piece area [2, 3]. We use one result from there: the maximum piece count at $n = 3$ (Section 4). The classical dissection literature [4, 5] constrains the pieces to be squares and counts them rather than measuring the cuts, and the "unequal rectangles" of the packing literature [6] are packed into a square rather than partitioning it, so gaps are allowed; neither quantity is W^*.

2. Preliminaries

Fix $n \geq 1$ and identify $[0,n]^2$ with the $n \times n$ array of unit cells, the cell (i,j) being $[i,i+1] \times [j,j+1]$ for $0 \leq i,j < n$.

Definition 2.1. A *piece* is an axis-aligned rectangle $R = [x, x+w] \times [y, y+h]$ with integer corners and $w, h \geq 1$; we call $w \times h$ its *dimensions*, wh its *area*, $w+h$ its *semiperimeter* and the unordered pair $\{w,h\}$ its *dimension pair*. A finite list $T = (R_1, \ldots, R_m)$ of pieces contained in $[0,n]^2$ is a *tiling* of the $n \times n$ square if every unit cell lies in exactly one R_i. It is *admissible* if in addition the dimension pairs $\{w_i, h_i\}$ are pairwise distinct, i.e. the pieces are pairwise non-congruent.

That a dimension pair is *unordered* is the reading forced by the source's own use of the constraint, and it is not a harmless convention: a 1×2 piece and a 2×1 piece may not both occur. See Proposition 7.2.

Definition 2.2. The *internal wall length* $W(T)$ of a tiling T is the number of unit edges lying strictly inside the square whose two adjacent cells belong to different pieces. Equivalently it is the one-dimensional measure of the union of the piece boundaries minus the boundary of the square: the total length of the interior cuts, each counted once.

Definition 2.3. $W^*(n,m) = \min\{W(T) : T \text{ an admissible tiling of the } n \times n \text{ square with } m \text{ pieces}\}$, defined exactly when such a T exists. We write $W^*(n,m) \leq w$ and $W^*(n,m) \geq w$ for the two halves, and $W^*(n,m) = w$ for their conjunction; the two halves are of quite different character, the first being witnessed by a single tiling and the second quantifying over all of them.

Throughout, $\Sigma(T) = \sum_{i=1}^{m} (w_i + h_i)$ denotes the *total semiperimeter* of a tiling and $T_j = j(j+1)/2$.

The definition of W above is geometric, in terms of edges of the grid. The source [1, Remark 15] works instead with the arithmetic form $W = \Sigma(T) - 2n$, obtained by counting, in each of the n vertical strips, the segments into which the pieces meeting that strip cut it. Section 3 proves that the two agree for every tiling; until then we keep them apart, so that a misreading of either definition cannot propagate.

3. Two counting identities

Both identities in this section quantify over all tilings and use no distinctness.

Lemma 3.1 (area)**.** *Let $T = (R_1, \ldots, R_m)$ be a tiling of the $n \times n$ square. Then*

$$\sum_{i=1}^{m} w_i h_i \;=\; n^2.$$

Proof sketch. Double counting over the incidence relation "the cell p lies in the piece R_i". Summing the number of pieces containing p over the n^2 cells gives n^2, because the tiling covers each cell exactly once; summing the number of cells contained in R_i over the pieces gives $\sum_i w_i h_i$, because a piece contained in the square meets exactly $w_i h_i$ cells. The formal proof is an induction on the list of pieces together with an interval count; no search is involved. □

The identity is used without comment in [1], and its equal-area case is used in the same way in the Mondrian literature [3]; we record it because everything below rests on it.

Theorem 3.2 (wall and semiperimeter)**.** *For every tiling T of the $n \times n$ square,*

$$W(T) + 2n \;=\; \Sigma(T) \;=\; \sum_{i=1}^{m} (w_i + h_i).$$

In particular the geometric and the arithmetic readings of "internal wall length" agree, and minimising the wall is minimising the total semiperimeter.

Proof sketch. Split $W(T) = W_h(T) + W_v(T)$ according to whether the interior edge separates two vertically or two horizontally adjacent cells. There are exactly $n(n-1)$ interior edges of each kind. An interior edge between two vertically adjacent cells fails to be part of the wall exactly when a single piece contains both cells, and a piece of dimensions $w \times h$ contains exactly $w(h-1)$ vertically adjacent pairs of cells. Hence

$$W_h(T) + \sum_i w_i(h_i - 1) \;=\; n(n-1),$$

which by Lemma 3.1 collapses to $W_h(T) = \sum_i w_i - n$. The other half is the mirror image, and adding the two gives the statement.

This is not the proof in [1, Remark 15], which counts maximal segments inside a strip. Counting runs of pieces along a strip means reasoning about the index of the piece owning a cell; counting the complement, as above, needs only one fact about that index, namely that two cells have the same owner exactly when some single piece contains both, which follows from the covering condition by a single induction. □

Corollary 3.3. *Let n, m, w be given and suppose every admissible tiling of the $n \times n$ square with m pieces satisfies $\Sigma(T) \geq w + 2n$. Then $W^*(n,m) \geq w$.*

Corollary 3.3 is what makes the lower-bound half tractable: after it, a lower bound on W^* is a statement about the multiset of dimension pairs alone, with the geometry of the tiling eliminated. Sections 4 and 5 exploit this; the two searches of Section 6 exploit it as well, one of them using it as its organising principle.

4. The piece-count ceiling, and where W^* is undefined

Theorem 4.1 (piece-count ceiling)**.** *Every admissible tiling of the $n \times n$ square with m pieces satisfies*

$$4m \;\leq\; n^2 + 6.$$

Proof sketch. A piece of area at most 3 has a side of length 1 — otherwise both sides are at least 2 and the area is at least 4 — so its dimension pair is $\{1, \text{area}\}$, and distinct dimension pairs force distinct areas among such pieces. There are only k possible areas in $\{1, \ldots, k\}$, so at most k pieces have area $\leq k$, for each $k \leq 3$. The truncated layer cake

$$\sum_i w_i h_i \;\geq\; \sum_{k=0}^{3} \#\{i : w_i h_i \geq k+1\} \;\geq\; m + (m-1) + (m-2) + (m-3) \;=\; 4m - 6$$

then meets Lemma 3.1. No search; the formal proof is a list induction. □

The three hypotheses all do work. Without distinctness the ceiling is false: the nine unit squares tile the 3×3 square with $4m = 36 > 15 = n^2 + 6$. Without the requirement that pieces be non-degenerate and contained in the square it is false as well, since degenerate zero-width pieces are free and carry distinct dimension pairs. And the constant 6 cannot be replaced by anything below 4: the five-piece tiling of the 4×4 square has $4m = 20 = n^2 + 4$. All three of these are themselves verified statements.

Read backwards, the ceiling is a non-existence theorem.

Corollary 4.2. *If $n^2 + 6 < 4m$ then no admissible tiling of the $n \times n$ square with m pieces exists, so $W^*(n,m)$ is undefined. In particular $W^*(3,m)$ is undefined for every $m \geq 4$, $W^*(4,m)$ for every $m \geq 6$, and $W^*(5,m)$ for every $m \geq 8$.*

These are the non-existence entries of the table of $W^*(n,m)$ for $3 \leq n \leq 5$, and they cover $W^*(3,4)$, which is [1, Remark 16], here reproved from the definitions, together with $W^*(3,7)$ and $W^*(4,7)$, the two empty cells at the top of the column studied in Section 6. We have not located $W^*(4,m)$ for $m \geq 6$ or $W^*(5,m)$ for $m \geq 8$ in the literature, but each is a two-line area count and we claim nothing for them beyond that they are now theorems.

The bound is sharp at the small end. Write $P(n)$ for the largest number of pairwise non-congruent integer-sided rectangles that tile the $n \times n$ square — the right-hand end of the row $W^*(n, \cdot)$. (We avoid the letter M, which in the Mondrian literature denotes the minimum defect.)

Theorem 4.3. *$P(3) = 3$, $P(4) = 5$ and $P(5) = 7$.*

Proof sketch. The upper halves are Theorem 4.1, which forbids $m = 4$ at $n = 3$, $m = 6$ at $n = 4$ and $m = 8$ at $n = 5$. The lower halves are three explicit tilings, each checked by kernel evaluation: the pieces $3 \times 2 \mid 1 \times 1 \mid 2 \times 1$ at $n = 3$, and the five- and seven-piece tilings of Section 5 at $n = 4$ and $n = 5$. □

$P(3) = 3$ is known: Dalfó, Fiol and López record that the 3×3 square admits no partition into $k \geq 4$ non-congruent rectangles [2], a case they call trivial and leave without argument. $P(4) = 5$ and $P(5) = 7$ we did not find in print or in the OEIS, but they are small finite checks in exactly that published style, and we claim them only as verified, not as new. Beyond $n = 5$ the ceiling is no longer sharp — at $n = 6$ it permits $m \leq 10$ — and we prove nothing there.

5. Lower bounds from the semiperimeter

Corollary 3.3 turns a lower bound on W^* into a lower bound on $\Sigma(T)$, and the same layer-cake count that bounded the piece count bounds the semiperimeter. Let

$$D(k) \;=\; \#\big\{\{a,b\} : 1 \leq a \leq b,\; a+b \leq k\big\} \;=\; 0,\,0,\,1,\,2,\,4,\,6 \qquad (k = 0, 1, \ldots, 5),$$

the counts of dimension pairs of small semiperimeter, and set

$$B(m) \;=\; \sum_{k=0}^{5} \max\big(m - D(k),\, 0\big) \;=\; m + m + (m-1) + (m-2) + (m-4) + (m-6),$$

the subtractions truncated at 0. Thus $B(5) = 18$, $B(7) = 29$ and $B(9) = 41$.

Theorem 5.1 (semiperimeter bound)**.** *Every admissible tiling T of the $n \times n$ square with m pieces satisfies $\Sigma(T) \geq B(m)$; consequently*

$$W^*(n,m) \;\geq\; B(m) - 2n.$$

Proof sketch. A piece inside the square has both sides ≥ 1, so its dimension pair is one of the $D(k)$ pairs of semiperimeter at most k; as the pairs are distinct, at most $D(k)$ pieces have semiperimeter $\leq k$. The layer cake

$$\Sigma(T) \;\geq\; \sum_{k=0}^{5} \#\{i : w_i + h_i \geq k+1\}$$

then gives the first claim, and Corollary 3.3 the second. Again no search: a pigeonhole on dimension pairs and a list induction. □

Distinctness is essential and not decorative here: the nine unit squares tiling the 3×3 square have $\Sigma = 18$, far below $B(9) = 41$.

Theorem 5.2. *$W^*(4, 5) = 10$, both halves proved.*

Proof sketch. $B(5) - 2 \cdot 4 = 18 - 8 = 10$ gives the lower half by Theorem 5.1. The upper half is an explicit five-piece tiling of the 4×4 square with wall exactly 10. □

The value 10 is not new: it is the case $n = 4$ of [1, Proposition 18], $W^*(n, 5) = n + 6$ for $4 \leq n \leq 9$. We include it because it is a gate. It is the first cell at which this development proves a published value from both sides, so it tests the machinery of Theorem 5.1 against something already known before that machinery is used where nothing is known — which is the next statement.

Theorem 5.3. *$W^*(5, 7) = 19$, both halves proved.*

Proof sketch. $B(7) - 2 \cdot 5 = 29 - 10 = 19$ gives the lower half. The upper half is the tiling

$$2 \times 3 \mid 1 \times 5 \mid 1 \times 4 \mid 2 \times 2 \mid 1 \times 3 \mid 1 \times 2 \mid 1 \times 1$$

of the 5×5 square, whose internal wall length is exactly 19. Both bounds are tight: the witness has wall exactly 19, so $W^*(5,7) \geq 20$ is false and the lower bound is not an artifact of slack. □

This is the first exact value in the column $m = 7$ obtained without an exhaustive search, and we have not found it stated anywhere: the source never writes $W^*(5,7)$ — its Remark 16 mentions $n = 5$, $k = 7$ only to quote $W^*(5,6) = 14$ and the three-dimensional $I(5,7) = 39$ — and the Mondrian literature optimises the defect instead.

Away from these two cells the bound is true but slack, because it ignores the area identity of Lemma 3.1 entirely: at $(6,7)$ it gives only $W^* \geq 17$ against a computed 19. Minimising $\sum_i (a_i + b_i)$ subject to $\sum_i a_i b_i = n^2$ over sets of distinct pairs — that is, using both identities of Section 3 at once — is the natural next step and is not attempted here.

6. The column $m = 7$

The whole column, with – marking a pair at which no admissible tiling exists:

n	3	4	5	6	7	8	9	10	11	12	13	14
$W^*(n,7)$	–	–	19	19	20	21	22	21	22	22	23	24

n	15	16	17	18	19	20	21	22	23	24
$W^*(n,7)$	25	26	27	28	29	30	26	27	28	29

The block $15 \leq n \leq 20$ is Open Problem (1) of [1]; the cells to its right are the high regime, where the value $n+5$ is already known; everything to its left is the deep regime of Open Problem (2). The two dashes are theorems (Corollary 4.2), as is the entry at $n = 5$ (Theorem 5.3); at every other defined cell the displayed value is a proved upper bound and a computed lower bound, except in the high regime where the lower bound is [1, Lemma 12].

The optimal shape. For $12 \leq n \leq 20$ the following seven pieces tile the $n \times n$ square:

$$(n-2) \times n \;\mid\; 2 \times (n-8) \;\mid\; 2 \times 3 \;\mid\; 2 \times 2 \;\mid\; 1 \times 3 \;\mid\; 1 \times 2 \;\mid\; 1 \times 1. \tag{2}$$

The block occupies the leftmost $n-2$ columns. The remaining $2 \times n$ strip is cut horizontally into the $2 \times (n-8)$, 2×3 and 2×2 pieces, stacked from the top; the 2×3 region left at the bottom is cut vertically, its left column becoming the 1×3 stick and its right column the 1×2 and 1×1 sticks. The dimension pairs are pairwise distinct as soon as $n - 8 \geq 4$, i.e. $n \geq 12$. Areas sum to n^2, and the semiperimeters sum to

$$(2n-2) + (n-6) + 5 + 4 + 4 + 3 + 2 \;=\; 3n + 10,$$

so by Theorem 3.2 the internal wall length is exactly $(3n + 10) - 2n = n + 10$.

Compare the comb of the high regime: one $(n-1) \times n$ block plus six sticks $1 \times w_i$ of distinct widths summing to n, with semiperimeters $(2n-1) + (6+n) = 3n+5$ and wall $n+5$. Below $n = T_6 = 21$ six distinct widths cannot sum to n, and (2) shows that the cheapest repair costs exactly $5 = m-2$: the block's semiperimeter falls by one, from $2n-1$ to $2n-2$, while the six small pieces rise by six in total. That is the "thickening" mechanism that [1] identifies in the paragraph preceding its Proposition 19 — "each stick that must 'thicken' to height ≥ 2 costs exactly $+1$ in total semiperimeter" — predicts; the content of the next theorem is that it is not merely available but optimal.

Theorem 6.1 (the middle regime at $m = 7$; equality is computational)**.** *For every n with $15 \leq n \leq 20$:*

(a) $W^*(n,7) \leq n + 10$, *witnessed by the tiling* (2)*;*
(b) $W^*(n,7) = n + 10$.

Equivalently, $W^*(n,7) = 25, 26, 27, 28, 29, 30$ *at* $n = 15, \ldots, 20$*, and Conjecture 20 of* [1] *— equation* (1) *— holds at* $m = 7$*, settling Open Problem (1) of* [1] *for that column.*

Proof of (a), and sketch of (b). Part (a) is six independent finite checks, one per n: that the seven rectangles of (2) lie inside the square, cover every cell exactly once, have pairwise distinct unordered dimension pairs, and have internal wall length exactly $n+10$. Each is decided by kernel evaluation on the explicit coordinates, with no appeal to compiled code and no axioms whatsoever; the largest instance decides a cover of 400 cells and a wall over 760 interior edges.

Part (b) is where an exhaustive search enters, and the exact search is this. Two independent programs were written, sharing no code and no algorithm.

Method A enumerates tilings directly: it places pieces at the lexicographically first empty cell, which generates each tiling exactly once, and prunes on total semiperimeter with three lower bounds together with one symmetry reduction (transposition preserves both W and the multiset of unordered dimension pairs, so the piece at the corner cell may be assumed to have $w \le h$).

Method B enumerates instead the possible *sets* of dimension pairs. By Theorem 3.2 and Lemma 3.1, a tiling with wall W determines a set of m distinct pairs with $\sum a_i b_i = n^2$ and $\sum (a_i + b_i) = 2n + W$; the program enumerates all such sets and tests each for tileability by exact cover. It is roughly two orders of magnitude faster than Method A and can fail in different ways.

Both were run to completion on the whole range $5 \le n \le 20$ of the column and agree at every cell. For (b) specifically, both refute $W \le n+9$: Method B at the cap $\Sigma \le 2n + (n+9)$ examines 7 candidate dimension sets at $n = 15$ rising to 225 at $n = 20$, and 1.2×10^5 to 5.5×10^6 exact-cover nodes; Method A, doing the same refutation geometrically without the identity, takes 2.1×10^7 nodes at $n = 15$ and 4.4×10^8 at $n = 20$. Combined with (a), that is (b).

The shared point of failure of the two methods is Theorem 3.2, which both use to convert wall length into semiperimeter. It is a theorem here, and was in addition checked geometrically against the definition of W on all 660 463 tilings of every square up to 6×6 by up to 7 pieces, distinct or not, of which 33 436 are non-guillotine: zero discrepancies. □

An arithmetic short cut for (b) is tempting and is unsound; we record why, since it is the first thing one tries. At $n = 15$ the seven distinct pairs $\{14,14\}, \{3,4\}, \{2,3\}, \{1,4\}, \{2,2\}, \{1,2\}, \{1,1\}$ have area exactly 225 and semiperimeter exactly $54 = 3n + 9$, so the two identities of Section 3 alone admit $W = 24$. That set is not tileable — a 14×14 corner piece leaves a strip of width 1, which cannot hold a 3×4 — and only the geometry says so. The exact-cover half of Method B is therefore load-bearing, which is also why the lower half of Theorem 6.1 is not yet a theorem in the sense of part (a).

Theorem 6.2 (the deep regime at $m = 7$; equalities are computational)**.** *For $5 \le n \le 14$ the upper bounds*

$$W^*(n,7) \ \le\ 19,\ 19,\ 20,\ 21,\ 22,\ 21,\ 22,\ 22,\ 23,\ 24 \qquad (n = 5, 6, \ldots, 14)$$

hold, witnessed by explicit tilings, and each is an equality. Thus $W^*(n,7) = n + 13$ at $n = 8, 9$; $W^*(n,7) = n + 11$ at $n = 10, 11$; and $W^*(n,7) = n + 10$ already for $12 \le n \le 14$. Below $n = 5$ the quantity is undefined (Corollary 4.2).

Proof sketch. The upper bounds are ten explicit tilings, each checked exactly as in Theorem 6.1(a); at $n = 12, 13, 14$ the witness is (2). The equalities are the two exhaustive searches of Theorem 6.1, run over the same range and agreeing at every cell. The case $n = 5$ is more than a computation: it is Theorem 5.3, whose lower half is a theorem about every tiling. The non-existence at $n = 3, 4$ is Corollary 4.2, likewise a theorem. □

Two features of this staircase deserve comment, neither of them predicted by (1).

Remark 6.3. The middle-regime value $n + 10$ switches on at $n = 12$, three steps below the threshold $T_5 = 15$ at which (1) is asserted. An overshoot of the same kind occurs at $m = 6$, where [1, Proposition 19] records $W^*(n,6) = n + 8$ "for all $10 \le n \le 14$, and indeed for $8 \le n \le 14$" — two steps below its threshold $T_4 = 10$. So the overshoot is not an accident of $m = 7$, though its size is not constant: the triangular threshold is where (1) *starts being provable by the comb argument*, not where the value changes.

Remark 6.4. $W^*(\cdot, 7)$ is *not* monotone in n: reading the values of Theorem 6.2 in increasing n gives $19, 19, 20, 21, 22, 21, 22, 22, 23, 24$, which drops by one between $n = 9$ and $n = 10$, exactly where the deep-regime formula changes from $n + 13$ to $n + 11$. A conjecture asserting monotonicity in the deep regime would therefore be false.

Controls on the far side. The high regime begins at $T_6 = 21$, immediately past the open range, and there the value is known.

Proposition 6.5 (known value, reproduced)**.** *For* $21 \le n \le 24$ *the comb tiling of the* $n \times n$ *square by seven pieces — one* $(n-1) \times n$ *block and, stacked in the last column, six sticks of width* 1 *and heights* $1, 2, 3, 4, 5, n-15$ *— is admissible and has internal wall length exactly* $n+5$*, so* $W^*(n, 7) \le n+5$*; the matching lower bound is* [1, Lemma 12]*, applicable since* $n \ge 5$*, so* $W^*(n, 7) = n + 5$ *there.*

Nothing here is new — $n+m-2$ is the source's proved high-regime value — and the proposition is included as a control: the same construction and the same checking apparatus that produce the open cells land exactly on a published value one step past them, at the largest square this development reaches. The check at $n = 24$ decides a cover of 576 cells and a wall over 1104 interior edges by kernel evaluation alone.

Proposition 6.6 (reproduction of published values)**.** *Thirty cells at which* [1] *states a value of* $W^*(n, m)$ *with* $m \le 6$ *— two from Proposition 17, six from Proposition 18, seven from the exhaustive certificate of Proposition 19, thirteen carrying the high-regime comb value, and the two exceptional entries* $W^*(3, 4)$ *undefined and* $W^*(5, 6) = 14$ *of Remark 16 — are reproduced here from the definitions of Section 2, and every one agrees. At each of the twenty-nine defined cells the upper bound* $W^*(n, m) \le w$ *is a verified theorem with* w *the published value, hence tight against a proved theorem of* [1]*; the thirtieth is the non-existence entry* $W^*(3, 4)$*, which is Corollary 4.2.*

Proposition 6.6 is the reason the rest of the paper should be believed rather than merely read. The two search programs were written from the definitions quoted in Section 2 before any value in [1] was entered into them, and a misreading of "distinct dimension pair" or of "wall length" would have shown up at the first cell whose value is not given by the comb formula. The two exceptional entries are the sharpest part of the gate, since no formula produces them.

7. The role of the distinctness constraint

Proposition 7.1. *The* 15×15 *square admits a partition into seven rectangles — one* 14×15 *block together with six sticks of width* 1 *and heights* $3, 3, 3, 2, 2, 2$ *— with internal wall length exactly* 20*. It is not admissible, the stick heights being repeated. Its wall length and that of the seven-piece admissible tiling* (2) *at* $n = 15$ *differ by exactly* 5*.*

By [1, Lemma 12], which applies since $15 \ge 7 - 2$, every partition of the 15×15 square into seven rectangles has $W \ge n + m - 2 = 20$, so the partition above is optimal without the distinctness constraint. With the computed value $W^*(15, 7) = 25$ of Theorem 6.1, the surcharge for distinctness at $(15, 7)$ is therefore exactly $5 = m - 2$. We state Proposition 7.1 as the difference of two witnesses because that difference is a theorem, whereas naming it "the surcharge" additionally invokes the computational half of Theorem 6.1. This is the two-dimensional instance of [1, Remark 22], which says of the three-dimensional problem that distinctness "does not change the optimal formula; it delays its onset … and in the intermediate range it charges an exact surcharge".

Proposition 7.2 (the dimension pair is unordered)**.** *The four rectangles* 1×2*,* 2×1*,* 2×2*,* 1×1*, suitably placed, tile the* 3×3 *square; the tiling is not admissible.*

The point of Proposition 7.2 is that the reading of Definition 2.1 is forced and is testable. The source uses the unordered reading twice where the proof would otherwise collapse: in [1, Proposition 17], where three columns of a common height are distinct exactly when their widths are, and in [1, Proposition 18], where "a column cannot split in two (two identical $w \times 1$ pieces)". Under the ordered reading the tiling above would be admissible and every value in this paper would be a value of a different function. Four further controls of the same kind — an overlap, a gap, a piece protruding from the square, and the witness of Theorem 6.1 at $n = 15$ with one 1×1 piece displaced by a single row — are rejected as they should be.

Remark 7.3 (outside the formal development). The two programs of Theorem 6.1 run unchanged at $m = 8$, where the middle regime is $T_6 = 21 \leq n < T_7 = 28$. They report $W^*(n, 8) = n + 12$ at every n in that range, which is again the value (1) predicts, and they report $W^*(14, 8) = 27 < 28 = W^*(13, 8)$. Neither statement is a theorem in the sense of the rest of this paper: no tiling of a square by eight pieces has been formally checked, so not even the upper bounds are verified, and at $n \geq 21$ only the faster of the two programs was within budget (the case $n = 27$ cost 9.5×10^9 exact-cover nodes). We record the column as a computation and claim nothing more for it.

Remark 7.4. Combining Theorem 6.1(b) with the reduction theorem [1, Theorem 3] at $k = 8$ gives, for $15 \leq n \leq 20$, the three-dimensional values $I(n, 8) = n^2 + n + 10$ and $T(n, 8) = 8n^2 + 2n + 20$ — in the notation of [1], the minimum total area of the internal interfaces of a partition of $[0, n]^3$ into eight pairwise unequal boxes, and the corresponding minimum total surface area $6n^2 + 2I$. Both hypotheses of that theorem hold here: its range condition $4 \leq k \leq n + 2$, since $k = 8$ and $n \geq 15$, and its side condition $W^*(n, k-1) \leq 2n + k - 4$, which reads $n + 10 \leq 2n + 4$ and holds for $n \geq 6$. The two values inherit the computational status of Theorem 6.1(b) and are no part of the verified development.

8. Verification

With the single class of exceptions named at the end of this section, every statement above labelled Theorem, Proposition, Lemma or Corollary has been formally verified in Lean 4 (toolchain `v4.32.0`); repository reference to be added at submission. The development is free of `sorry` and imports no external library: every definition — pieces, covering, dimension pairs, the geometric wall length of Definition 2.2 — lives in Lean core, so that each finite check is a plain kernel evaluation rather than a call into compiled code. *Mathlib* is not used, and `native_decide` appears nowhere in the development; consequently no compiled evaluator is part of any trusted base here. The witness layer — the explicit tilings underlying Theorem 6.1(a), Theorem 6.2, Theorems 5.2, 5.3 and 4.3, Propositions 6.5, 6.6, 7.1 and 7.2 — reports *no axioms at all* under `#print axioms`, the individual checks being decidable evaluations on explicit coordinates. The general layer — Lemma 3.1, Theorem 3.2, Corollary 3.3, Theorem 4.1, Corollary 4.2, Theorem 4.3 and Theorem 5.1, and the lower halves of Theorems 5.2 and 5.3 that rest on them — depends on `propext`, `Classical.choice` and `Quot.sound` and on nothing else. What is *not* verified, and is flagged as computational in its heading, is exactly the lower half of Theorem 6.1(b) and of the equalities of Theorem 6.2: those rest on the two exhaustive C searches described in the proof of Theorem 6.1, which agree at every cell of the range and whose shared ingredient, the identity of Theorem 3.2, is itself a theorem here rather than an assumption. The whole formal development elaborates in a few minutes at well under two gigabytes of memory, and the identity of Theorem 3.2 additionally agrees with the 51 independent per-witness evaluations that preceded it.

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