CERTIFIED VALUES OF RECIPROCAL RADO NUMBERS, AND A MINIMUM OVER UNIT FRACTIONS

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Abstract. For $k \geq 2$ and $r \geq 1$ the reciprocal Rado number $f_r(k)$ is the least n such that every r-colouring of $\{1, \ldots, n\}$ admits a monochromatic solution of $1/x_1 + \cdots + 1/x_k = 1/x_{k+1}$, the x_i not necessarily distinct. Gaiser and Ramezanpour recently proved the sharp bound $f_2(k) \geq 3k^2$ for $k \geq 3$, tabulated 28 values of $f_r(k)$, twenty-four of them computed there for the first time by SAT solving, and posed four problems. We contribute two things. First, a decision procedure for $f_r(k)$ in which every step is machine-checked: a backtracking enumeration of the solution set of the equation, proved both complete and sound; a conjunctive normal form encoding whose fidelity — every avoiding colouring satisfies the formula — is proved rather than asserted; and refutations replayed inside a proof assistant from LRAT certificates. It delivers $f_2(2) = 60$, $f_2(3) = 40$, $f_2(4) = 48$, $f_2(5) = 80$, $f_3(3) = 585$ and $f_3(2) = 3276$ as two-sided theorems carrying certificates, and we are aware of no earlier machine-checkable certificate for a reciprocal Rado number. Second, we determine the first twenty-five values of the quantity of Problem 5.4 of Gaiser and Ramezanpour: the least positive value of $a - b_1 - \cdots - b_k$ with $a, b_1, \ldots, b_k$ unit fractions of denominator at most n. The answer is *not* always a unit fraction — at $k = 2$, $n = 16$ it is $5/1456$ — so the shape of the $k = 1$ case does not persist, and it is not monotone in k. Every theorem below is formally verified in Lean 4.

1. Introduction

Fix integers $k \geq 2$ and $r \geq 1$ and consider the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \;=\; \frac{1}{x_{k+1}}, \tag{1}$$

whose variables range over positive integers and *need not be distinct*. A colouring $\chi\colon \{1, \ldots, n\} \to \{0, \ldots, r-1\}$ is *avoiding* if no solution of (1) inside $\{1, \ldots, n\}$ has all of its $k+1$ entries the same colour. The *reciprocal Rado number* $f_r(k)$ is the least n for which no avoiding r-colouring of $\{1, \ldots, n\}$ exists. That such an n exists at all rests on Theorem 2.2 of Brown and Rödl [1]: if a system of homogeneous equations has a monochromatic solution under every finite colouring of the positive integers, then so does the system obtained by replacing every variable by its reciprocal. Rado's theorem supplies the hypothesis for $x_1 + \cdots + x_k = x_{k+1}$, and the proof is finitary — it transports an r-colouring χ of $\{1, \ldots, S\}$ to the colouring $x \mapsto \chi(S/x)$ of $\{1, \ldots, T\}$, where T is a Rado number of that linear equation and S is the least common multiple of $1, \ldots, T$ — so $f_r(k)$ is finite. Brown and Rödl are also quantitative at $r = 2$: their Theorem 2.3 gives $f_2(k) \leq k^6 - (k^2 - k)^2$. They close with the remark that Lefmann [6] had independently obtained results including their Theorem 2.2, and [4] and [8] both attribute the finiteness to the two papers independently. Quantitatively, Gaiser [3] has shown $f_2(k) = O(k^3)$ and $f_3(k) = O(k^{43})$; that paper computes no value of $f_r(k)$ of its own, and the five exact ones it quotes — $f_2(2) = 60$, $f_2(3) = 40$, $f_2(4) = 48$, $f_2(5) = 39$ and $f_3(2) = 3276$ — are Myers and Parrish's.

The dropping of the distinctness requirement is not a technicality. It makes $1/1 = k \cdot (1/k)$ a solution of (1), so in every avoiding colouring the integers 1 and k receive different colours, and in particular $f_r(k) \geq 2$ for all $k \geq 2$, $r \geq 1$. It is the convention of [4] and it is the one used throughout this paper.

1.1. **The source, and what it leaves open.** Gaiser and Ramezanpour [4] prove the following lower bounds (their Theorem 1.1): for all $r \geq 1$,

$$f_r(2) \ \geq \ 4^r/2, \qquad \text{and} \qquad f_r(k) \ \geq \ (2^r - 1)\, k^r \quad (k \geq 3). \tag{2}$$

For 2-colourings they also prove a version with general positive coefficients (their Theorem 1.4). There the equation is $a_1/x_1 + \cdots + a_k/x_k = 1/x_{k+1}$ with $a_1, \ldots, a_k$ positive integers, $f(a_1, \ldots, a_k)$ is the least n such that every 2-colouring of $\{1, \ldots, n\}$ has a monochromatic solution of it, and the bounds are $f(a_1, a_2) \geq 2\min\{a_1, a_2\}(a_1 + a_2)^2$ and, for $k \geq 3$, $f(a_1, \ldots, a_k) \geq \big(2\min_i a_i + 1\big)(a_1 + \cdots + a_k)^2$. At $a_1 = \cdots = a_k = 1$ the second reads $f_2(k) \geq 3k^2$, which is the case $r = 2$ of the second bound in (2). That case is *sharp*: their Theorem 1.2 shows $f_2(k) = 3k^2$ whenever $k = 3 \cdot 2^m$ with $m \geq 1$, and $f_2(k) \geq 3k^2 + 1$ whenever k is an odd prime power. Their Conjecture 1.3 is that the first alternative is the generic one.

Conjecture 1.1 ([4, Conjecture 1.3])**.** If $k \geq 4$ is not an odd prime power, then $f_2(k) = 3k^2$.

The evidence is a table of 28 values — twelve for even k with $2 \leq k \leq 24$, twelve for odd k with $3 \leq k \leq 25$, and four values of $f_3(k)$ for $2 \leq k \leq 5$. Four of them, $f_2(2)$, $f_2(3)$, $f_2(4)$ and $f_3(2)$, are attributed there to Myers and Parrish [8]; the remaining twenty-four are computed in [4] by SAT solving. Writing the first two tables of [4] against $3k^2$:

k	2	3	4	5	6	7	8	9	10	11	12	13
$f_2(k)$	60	40	48	80	108	150	192	252	300	372	432	513
$3k^2$	12	27	48	75	108	147	192	243	300	363	432	507

k	14	15	16	17	18	19	20	21	22	23	24	25
$f_2(k)$	588	675	768	880	972	1092	1200	1323	1452	1608	1728	1887
$3k^2$	588	675	768	867	972	1083	1200	1323	1452	1587	1728	1875

and the two rows agree at exactly those $k \in \{4, \ldots, 25\}$ that are not odd prime powers, which is Conjecture 1.1 on its range of evidence.

Remark 1.2 (an erratum in [4]). This remark concerns the text of [4] and not any of its theorems; it is recorded here because it is the sentence that says which k are exceptional. Announcing the computations in its introduction, [4] writes: "The key feature of these computational results is that, for all $k \in \{3, 4, \ldots, 25\}$, we have $f_2(k) = 3k^2$ if and only if k is an odd prime power." The negation has been dropped. The two displays above, which are that paper's own tables, say the opposite; so do its Theorem 3.1 (an odd prime power forces $f_2(k) \geq 3k^2 + 1$), its Conjecture 1.3 (Conjecture 1.1 above), which the offending sentence introduces with "Hence", and the sentence following its second table, which has the negation in place: "Tables 1 and 2 suggest that if k is an odd prime power, then $f_2(k) \geq 3k^2 + 1$; otherwise, $f_2(k) = 3k^2$ for $k \geq 4$." Every value computed below agrees with the corrected reading. Nothing in this paper depends on the sentence. Both quotations are from v1, the only version of [4] at the time of writing.

Section 5 of [4] poses four problems. Problems 5.1 and 5.2 ask for the lower bounds (2) to be improved — for $f_3(k)$, and for $f_2(k)$ at odd prime powers, respectively; Problem 5.3 asks for a quadratic upper bound for $f_2(k)$ when $k \geq 4$. The fourth is the one that asks for numbers, and prints none.

Problem 1.3 ([4, Problem 5.4])**.** Let $U(n) = \{1/1, 1/2, \ldots, 1/n\}$ and $k \geq 2$. Determine

$$\min\{\, a - b_1 - b_2 - \cdots - b_k > 0 \ : \ a, b_1, b_2, \ldots, b_k \in U(n) \,\}.$$

Problem 1.3 is the general-k form of Lemma 2.1 of [4], which reads: for positive integers s and t, if $a, b \in \{s, s+1, \ldots, s+t\}$ with $a < b$, then $1/a - 1/b \geq 1/((s+t-1)(s+t))$. The hypothesis is on s and t, and there is no n in it; taking $s = 1$ and $t = n - 1$ bounds the $k = 1$ case of Problem 1.3 below by $1/((n-1)n)$, and $a = n - 1$, $b = n$ attains that bound, so the case $k = 1$ is settled with the answer $1/((n-1)n)$ for $n \geq 2$. It is this special case that [4] names in the sentence introducing Problem 5.4, and the lemma itself is what the proof of (2) invokes, once in each of its two halves.

1.2. **Results.** Two things are new here.

A certified decision procedure (Sections 2.1 and 4). Deciding a single value of $f_r(k)$ means settling two statements of very different character: the existence of an avoiding colouring of $\{1,\dots,n-1\}$, which one witness settles, and its non-existence at n, which quantifies over r^n colourings. We give a pipeline in which both halves are machine-checked end to end. Its unusual ingredient is forced by the nonlinearity of (1): before a propositional formula can be written down at all, one needs the finite set of solutions of (1) inside $\{1,\dots,n\}$, and one needs it *proved* to be exactly that set. Theorem 2.3 supplies it. Theorem 4.1 is the encoding's fidelity statement, and Theorem 4.3 the six values it certifies. We are aware of no earlier machine-checkable certificate for a reciprocal Rado number; Section 4.3 says which searches this claim rests on.

The first values of Problem 1.3 (Section 5). Writing $g_k(n)$ for the minimum in question, we determine $g_2(n)$ for $3 \le n \le 17$ and $g_3(n)$ for $4 \le n \le 13$ (Theorem 5.3), twenty-five values in all. Three features of the table are worth naming, since a conjecture would have to survive them. The answer is not always a unit fraction: $g_2(16) = 5/1456$, so no $1/N$ can be it (Theorem 5.4), and the shape of the $k = 1$ answer $1/((n-1)n)$ therefore does not persist. The answer is not monotone in k: $g_3(12) > g_2(12)$ while $g_3(11) < g_2(11)$ and $g_3(13) < g_2(13)$. And it is flat over long stretches whose jumps do not follow $(n-1)n$.

Along the way we record a formal proof of the case $r = 2$ of (2), uniform in k and using no search (Theorem 3.1), together with the observation that the colouring witnessing it fails at exactly $3k^2$ (Proposition 3.2); and we confirm, from both sides and independently of either published computation, the correction that [4, §3] makes to a value of [8] (Proposition 3.3 and Theorem 4.3).

Every numbered theorem, proposition and corollary below has been formally verified; Section 6 says precisely what that means and where the finite searches enter. The remarks are of two kinds, and each says which it is: some are comments on the verified statements, and the rest — Remarks 1.2, 4.6, 4.7 and 5.6 — report observations and computations made outside the formal development.

2. Preliminaries

Throughout, $[n] = \{1, 2, \dots, n\}$ and all sums of reciprocals are computed in $\mathbb{Q}$; no denominators are cleared anywhere in a statement.

Definition 2.1. Let $k \ge 1$, $r \ge 1$, $n \ge 0$. A tuple $x \in [n]^{k+1}$ is a *solution* if $\sum_{i=1}^{k} 1/x_i = 1/x_{k+1}$, and it is *monochromatic* for $\chi\colon [n] \to \{0,\dots,r-1\}$ if $\chi(x_1) = \cdots = \chi(x_{k+1})$. The colouring χ *avoids* if no solution is monochromatic for it, and $[n]$ is *feasible* if some r-colouring of it avoids. Finally $f_r(k) = n$ means that $[n-1]$ is feasible and $[n]$ is not.

Restricting an avoiding colouring of $[n]$ to $[m]$ with $m \le n$ again avoids, because a solution inside $[m]$ is one inside $[n]$ with the same colours. Hence feasibility is downward closed and at most one n satisfies Definition 2.1, so every value asserted below is asserted exactly and not up to an off-by-one. Note also that both halves of Definition 2.1 are statements about *all* tuples of $[n]^{k+1}$, of which there are n^{k+1}: at the largest cell treated here, $k = 2$ and $n = 3276$, that is $3.5 \cdot 10^{10}$ tuples per colouring, and 3^{3276} colourings. Everything in Sections 2.1 and 4 exists to replace those two quantifiers by finite objects.

2.1. **The solution set, and an enumeration proved to compute it.** A colouring cannot see a tuple, only the set of integers it uses; and the order of $x_1,\dots,x_k$ is immaterial. We therefore work with sorted pairs.

Definition 2.2. $\mathcal{S}(k,n)$ is the set of pairs $(a; b_1,\dots,b_k)$ with $a, b_1,\dots,b_k \in [n]$, $b_1 \le b_2 \le \cdots \le b_k$, and $1/b_1 + \cdots + 1/b_k = 1/a$.

The enumeration is the standard unit-fraction backtracking, with the two pruning inequalities that make it terminate. Suppose j entries remain to be chosen, all of them at least ℓ, and they must contribute $s \in \mathbb{Q}$ to the sum. The next entry c satisfies $1/c \le s$, since the other $j-1$ contributions are positive, and $s \le j/c$, since every remaining entry is at least c. So c ranges over $\lceil 1/s \rceil \le c \le \lfloor j/s \rfloor$, intersected with $[\ell, n]$. Let $R(n, j, \ell, s)$ be the list produced by branching over exactly that range and recursing on $(n, j-1, c, s-1/c)$, with $R(n, 0, \ell, s)$ the one-element list $\{\,()\,\}$ if $s = 0$ and empty otherwise.

Theorem 2.3. *For all n, j, $\ell \ge 1$, $s \in \mathbb{Q}$ and every list L of positive integers,*

$$L \in R(n, j, \ell, s) \iff \big(|L| = j,\ L \text{ is nondecreasing with all entries in } [\ell, n],\ \textstyle\sum_{c \in L} 1/c = s\big).$$

Consequently $\mathcal{S}(k, n) = \{(a; L) : a \in [n],\ L \in R(n, k, 1, 1/a)\}$: the enumeration is both complete*, in that nothing that could be extended to a solution is pruned, and* sound*, in that nothing it lists fails the equation.*

Proof sketch. Induction on j, both directions at once; the base case is the definition of $R(n, 0, \ell, s)$ and the step is the branching range above, whose two endpoints are exactly the two displayed inequalities. Soundness needs the upper endpoint to be an upper bound and completeness needs the lower endpoint to be a lower bound, so a single equivalence carries both. No search is involved and the proof is uniform in all four parameters. □

Both directions are used, and for different halves of a threshold. Completeness is what turns "no member of $\mathcal{S}(k, n)$ is monochromatic under χ" into "χ avoids":

Corollary 2.4. *Let χ be an r-colouring of $[n]$. If no $(a; b_1, \dots, b_k) \in \mathcal{S}(k, n)$ has $\chi(a) = \chi(b_1) = \cdots = \chi(b_k)$, then χ avoids.*

Proof sketch. A monochromatic tuple x yields the pair $(x_{k+1}; \mathrm{sort}(x_1, \dots, x_k))$, which lies in $\mathcal{S}(k, n)$ by completeness — sorting preserves length, membership and the reciprocal sum — and is monochromatic because colours travel with membership. □

Soundness is what makes a clause built from a listed pair a legitimate constraint; that is Theorem 4.1. The sizes involved are small, and this is the structural fact the whole method runs on:

(k, n)	$(2, 60)$	$(3, 40)$	$(4, 48)$	$(5, 80)$	$(3, 585)$	$(2, 3276)$
$\lvert\mathcal{S}(k, n)\rvert$	61	88	321	3227	5982	5965

At $(k, n) = (2, 3276)$ the $3.5 \cdot 10^{10}$ tuples of $[n]^{k+1}$ collapse to 5965 pairs.

3. The block colouring, and the bound $f_2(k) \ge 3k^2$

For $k \ge 3$ let χ_k be the 2-colouring of $[n]$ given by

$$\chi_k(v) = \begin{cases} 1, & 3 \le v < 3k, \\ 0, & \text{otherwise,} \end{cases}$$

so that the colour classes are the block $[3, 3k-1]$ and its complement $\{1, 2\} \cup [3k, n]$. This is the construction of [4] at $r = 2$.

Theorem 3.1 ([4, Theorem 1.1(ii)], at $r = 2$)**.** *Let $k \ge 3$ and $n < 3k^2$. Then χ_k avoids on $[n]$. Consequently $f_2(k) \ge 3k^2$ for every $k \ge 3$.*

Proof sketch. This is the source's theorem; we sketch the proof because at $r = 2$ it is four lines, because Proposition 3.2 and Proposition 3.3 both rest on it, and because it is the only statement in this paper that holds uniformly in k. Let $1/a = 1/b_1 + \cdots + 1/b_k$ be monochromatic in $[n]$. Since the sum has $k \ge 2$ positive terms, $a < b_i$ for every i.

- If the colour is 1, then $b_i < 3k$ for all i, so $1/a > k/(3k) = 1/3$ and $a < 3$, contradicting $a \ge 3$.
- If the colour is 0 and no $b_i \le 2$, then $b_i \ge 3k$ for all i, so $1/a \le 1/3$; and $b_i < 3k^2$ for all i, so $1/a > k/(3k^2) = 1/(3k)$. Hence $3 \le a < 3k$, which is the *other* colour class.

- If the colour is 0 and some $b_i \le 2$, then $a < b_i \le 2$ forces $a = 1$, so the k reciprocals sum to 1. Writing t for the number of b_i that are at most 2, each such b_i equals 2 and the others are at least $3k$, whence $t/2 \le 1 \le t/2 + 1/3$ and $t = 2$. The remaining $k - 2 \ge 1$ terms then sum to 0 while each is positive.

The hypothesis $k \ge 3$ is used once, in the last bullet. No case involves a search; the argument is rational arithmetic uniform in k and n. □

The hypothesis $n < 3k^2$ cannot be weakened.

Proposition 3.2. *For every $k \ge 3$, χ_k does not avoid on $[3k^2]$.*

Proof sketch. At $n = 3k^2$ the identity $1/(3k) = k \cdot \big(1/(3k^2)\big)$ is available inside $[n]$, and both $3k$ and $3k^2$ lie outside the block $[3, 3k-1]$, so the solution it names is monochromatic in colour 0. The proof is uniform in k and uses no search. The solution itself was first found by exhaustive search: at $k = 3$ the unique monochromatic solution under χ_3 on $[27]$ is $1/9 = 3 \cdot (1/27)$, and at $k = 4$ the unique one under χ_4 on $[48]$ is $1/12 = 4 \cdot (1/48)$. Uniqueness at those two k rests on those two exhaustive searches and is not claimed for general k. □

A consequence of Theorem 3.1 settles, with no search whatsoever, a discrepancy in the published record. Computation 21 of [8], on page 12, gives the 2-colour reciprocal Rado numbers for three, four, five and six variables as 60, 40, 48 and 39; its last clause reads "With 5 variables it is 48, and with 6 variables it is 39." Six variables is $k = 5$, so it states $f_2(5) = 39$, and Section 3 of [4] observes that this is impossible, because (2) gives $f_2(5) \ge 3 \cdot 5^2 = 75$. The value was quoted again in [3].

Proposition 3.3. *$f_2(5) \ne m$ for every $m < 75$. In particular $f_2(5) \ne 39$.*

Proof sketch. Immediate from Theorem 3.1 at $k = 5$: χ_5 avoids on $[m]$ for every $m \le 74$, so no such m can be the least infeasible size. No colouring is searched for and no formula is solved. □

Theorem 4.3 below supplies the other half of the correction, $f_2(5) = 80$, so that neither half is taken on the authority of either published computation.

4. A certified decision procedure

4.1. **The encoding.** Fix k, r, n. The propositional variables are $x_{v,j}$ for $v \in [n]$ and $0 \le j < r$, read as "the integer v has colour j". The formula $\Phi_{k,r,n}$ is the conjunction of

- an *at-least-one* clause $\bigvee_{0 \le j < r} x_{v,j}$ for each $v \in [n]$; and
- a *solution* clause $\bigvee_{v \in V(p)} \neg x_{v,j}$ for each $p \in \mathcal{S}(k,n)$ and each colour j, where $V(p) \subseteq [n]$ is the set of distinct integers occurring in p.

There are no *at-most-one* clauses, and that omission is deliberate: only the direction "avoiding colouring ⇒ satisfying assignment" is ever used, and it holds whether or not a colouring is forced to be single-valued. Repeated entries are common — $1/1 = 1/3 + 1/3 + 1/3$ has $V = \{1, 3\}$ — so using the distinct integers keeps the clauses short.

Theorem 4.1 (encoding fidelity)**.** *Let χ be an avoiding r-colouring of $[n]$ and let A_χ be the assignment that makes $x_{v,j}$ true exactly when $\chi(v) = j$. Then A_χ satisfies $\Phi_{k,r,n}$.*

Proof sketch. An at-least-one clause is satisfied by the literal naming the colour v actually has. For a solution clause coming from $p \in \mathcal{S}(k,n)$ and colour j: by the soundness half of Theorem 2.3, p really is a solution of (1) inside $[n]$, so, χ being avoiding, its entries cannot all have colour j; any entry whose colour differs from j supplies a true literal. No search is involved. □

Corollary 4.2. *If $\Phi_{k,r,n}$ is unsatisfiable, then every r-colouring of $[n]$ has a monochromatic solution of* (1)*; that is, $[n]$ is infeasible.*

Corollary 4.2 is the only place where a SAT solver enters, and it enters in the direction that makes an omitted clause harmless: a clause left out can only make a refutation harder to find, never make one unsound. The converse — that a satisfying assignment yields an avoiding colouring — is never used and never proved, which is also why the completeness half of Theorem 2.3 plays no role here. It plays its role in Corollary 2.4, on the other side of the threshold.

4.2. **Six certified values.**

Theorem 4.3.

$$f_2(2) = 60, \quad f_2(3) = 40, \quad f_2(4) = 48, \quad f_2(5) = 80, \quad f_3(3) = 585, \quad f_3(2) = 3276.$$

All six values are published: they are cells of the tables of [4], four of them attributed there to [8]. What is new is not the numbers but the form of the evidence, so we describe it in full.

Proof sketch. Each value is two independent finite computations, one per half of Definition 2.1.

Feasibility at $n-1$. An explicit array of colours for $[n-1]$ is exhibited — found by running the solver CaDiCaL [2] on $\Phi_{k,r,n-1}$, but its provenance is irrelevant to the proof — and the finite computation is: enumerate $\mathcal{S}(k, n-1)$ by Theorem 2.3 and check that no member of it is monochromatic under the array. This is a single Boolean, and Corollary 2.4 turns it into feasibility. The computation is a walk over $|\mathcal{S}(k, n-1)|$ pairs, a number of the order given at the end of Section 2.1.

Infeasibility at n. A refutation of $\Phi_{k,r,n}$, found by CaDiCaL and recorded as an LRAT certificate, is replayed by a verified LRAT checker; the finite computation is that the checker returns true on the pair (formula, certificate). Corollary 4.2 turns it into infeasibility. Crucially the formula the checker is run on is $\Phi_{k,r,n}$ as defined above, not a file: the DIMACS file handed to the solver was itself printed from that definition, so no second implementation has to be argued to agree with the first.

The two computations per cell have the following sizes.

cell	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$	$f_3(3)$	$f_3(2)$
n	60	40	48	80	585	3276
variables	120	80	96	160	1755	9828
clauses	182	216	690	6534	18531	21171
CNF (bytes)	1 974	2 900	11 629	146 292	385 784	370 973
LRAT (bytes)	267	1 017	677	1 177	2 737 978	14 253 744

The clause counts are $n + r\,|\mathcal{S}(k, n)|$, as they must be. □

Corollary 4.4. *Each of the six values is the unique n satisfying Definition 2.1 for its cell. In particular $f_2(3) \neq 48$, although 48 is the correct value of $f_2(4)$.*

Remark 4.5. At $k = 4$ the value $48 = 3 \cdot 4^2$ is sharp for Theorem 3.1, and the avoiding colouring of [47] that the solver returned is exactly χ_4: the block $\{3, \ldots, 11\}$ in one colour, $\{1, 2\} \cup \{12, \ldots, 47\}$ in the other. The search recovered, at one cell, the colouring that Theorem 3.1 proves avoids at every $k \geq 3$ with no search at all.

4.3. **On earlier certificates.** The claim attached to Theorem 4.3 is a negative one and we state what it rests on. Neither published computation of a reciprocal Rado number emits a machine-checkable refutation. Myers and Parrish [8] do print certificates, and define a full one as the list of all maximal avoiding colourings (their Appendix C); but for the two reciprocal numbers treated there, $f_2(2)$ and $f_3(2)$, their Appendix C.3 prints one avoiding colouring apiece — the lower-bound witness, with the two or three solutions that block its extension — and nothing at all for $f_2(3)$, $f_2(4)$ or $f_2(5)$; the words DRAT and LRAT do not occur in the paper. Gaiser and Ramezanpour [4] report that their computation was conducted in Python using the PySAT interfaces to CaDiCaL and Glucose, with the script available from the first author's website, and record no DRAT or LRAT output. The route used here for replaying certificates inside a proof assistant — running a verified LRAT checker on the certificate as compiled code, by reflection — is the one Szeider [10] packages as a standalone import tool, and his worked targets are the Schur number $S(4) = 44$ and the Ramsey number $R(4,4) = 18$. Schur's equation is linear, and that is where the nonlinearity of (1) costs something. For a linear equation $ax + by = cz$ a candidate triple is checked by one division, so a clause list can be written down directly; for (1) there is no such test, and the solution set has to be enumerated first — and, if the resulting clauses are to be legitimate constraints, the enumeration has to be proved correct. That is the role of Theorem 2.3, and it has no counterpart in the linear case.

We searched the forward literature of [4] and [8], and keyword searches for “reciprocal Rado”, “Rado unit fractions” and “unit fraction Rado number”, on 2026-08-23. We have not searched MathSciNet, zbMATH or Google Scholar. Not finding a thing is weaker evidence than finding one, and the claim should be read accordingly.

Remark 4.6 (the frontier, outside the formal development). The wall is the enumeration, not the solver. CaDiCaL settles every cell above in under two seconds, including $f_3(2)$ at $n = 3276$. What runs out is $\mathcal{S}(k,n)$: at the next cell, $f_2(6) = 108$, enumerating $\mathcal{S}(6,107)$ costs roughly $9.6 \cdot 10^7$ backtracking nodes, about 565 seconds under the proof assistant's interpreter, which puts the cell out of reach for now. We did reproduce $f_2(6) = 108$ outside the formal development, by an independent C enumerator followed by CaDiCaL, in seconds; the same route did not finish at $k = 7$ within sixteen minutes. All timings in this remark were taken on a heavily shared machine and should be read with a tolerance of $\pm 25\%$; none of them supports any statement in this paper.

Remark 4.7 (independent reproduction). Before any of the six values was stated formally, each was recomputed from Definition 2.1 by a route sharing no code with the formal development: a C program enumerating the solution supports inside $[n]$ and then either running a two-colour depth-first search or emitting DIMACS for CaDiCaL. For $f_2(2)$ the search reported an avoiding colouring at 59 after 936 nodes and exhaustion at 60 after 151 274 298 nodes; for $f_2(4)$, 3 511 216 and 40 801 654 nodes. All seven cells attempted, including $f_2(6) = 108$, agreed with the published tables.

5. The least positive gap between unit fractions

Definition 5.1. For $k \geq 1$ and $n \geq 1$ put $U(n) = \{1/1, 1/2, \ldots, 1/n\}$ and

$$D_k(n) = \big\{\, a - b_1 - \cdots - b_k \ : \ a, b_1, \ldots, b_k \in U(n) \,\big\} \cap (0, \infty),$$

repetitions among the b_i and between a and the b_i being allowed. When $D_k(n) \neq \emptyset$ we write $g_k(n) = \min D_k(n)$, the quantity of Problem 1.3.

Proposition 5.2. *For given k and n, at most one rational q has all three of the following properties: $q = a - b_1 - \cdots - b_k$ for some $a, b_1, \ldots, b_k \in U(n)$; $q > 0$; and $q \leq a - b_1 - \cdots - b_k$ whenever the latter is positive. Hence $g_k(n)$ is pinned down by that description, and each value below is an exact answer rather than a one-sided bound.*

For $k = 1$ and $n \geq 2$ the answer is $g_1(n) = 1/((n-1)n)$, the special case of Lemma 2.1 of [4] recorded in Section 1. For $k \geq 2$ the source prints no value.

Theorem 5.3. *The following twenty-five values hold.*

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$g_2(n)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{42}$	$\frac{1}{120}$	$\frac{1}{120}$	$\frac{1}{140}$	$\frac{1}{140}$	$\frac{1}{180}$	$\frac{1}{180}$	$\frac{1}{280}$	$\frac{1}{280}$	$\frac{5}{1456}$	$\frac{1}{1428}$
$g_3(n)$	–	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{70}$	$\frac{1}{120}$	$\frac{1}{120}$	$\frac{1}{140}$	$\frac{5}{792}$	$\frac{5}{792}$	$\frac{2}{819}$				

Proof sketch. Each entry is settled by exhaustive evaluation. The defining property of Proposition 5.2 quantifies over $[n]^{k+1}$, so at $(k,n) = (3,13)$ it is $13^4 = 28\,561$ rational subtractions and at $(k,n) = (2,17)$ it is $17^3 = 4\,913$; each value costs one pass to exhibit an attaining tuple and one pass to verify minimality. There is no mathematical content beyond the arithmetic: the entire claim of the theorem is that these finite searches were run and returned these rationals. $\square$

Theorem 5.4. $g_2(16) = 5/1456$. *In particular $g_2(16) \neq 1/N$ for every positive integer N: the minimum in Problem 1.3 is not always a unit fraction.*

Proof sketch. The value is the entry of Theorem 5.3 at $k = 2$, $n = 16$, and it is attained by $1/7 - 1/13 - 1/16$. If $1/N = 5/1456$ then $5N = 1456$, which has no integral solution. The only computational ingredient is the single exhaustive search over $16^3 = 4096$ tuples. □

Since $g_1(n) = 1/((n-1)n)$ is a unit fraction for every $n \geq 2$, Theorem 5.4 says that the shape of the $k = 1$ answer does not survive to $k = 2$; the smallest n at which it fails is 16. At $k = 3$ it fails earlier, at $n = 11$, where the answer is 5/792.

Remark 5.5 (three features of the table). First, non-monotonicity in k: $g_3(12) = 5/792 > 1/180 = g_2(12)$, whereas $g_3(11) = 5/792 < 1/140 = g_2(11)$ and $g_3(13) = 2/819 < 1/180 = g_2(13)$. Increasing the number of subtracted terms neither always helps nor always hurts, which is unsurprising once one notes that exactly k terms must be used, so $D_2(n)$ and $D_3(n)$ are incomparable sets. Second, flatness: each of 1/6, 1/20, 1/120, 1/140, 1/180 and 1/280 is the value of g_2 at two consecutive n, and the positions of the jumps do not follow $(n-1)n$. Third, the values are not in the literature: the denominator sequence $6, 6, 20, 20, 42, 120, 120, 140, 140, 180$ of g_2 returns nothing from the On-Line Encyclopedia of Integer Sequences [9], queried on 2026-08-23 with $1, 4, 13, 44, 160$ as a positive control, which returns A045652 on the nose.

Remark 5.6 (computations outside the formal development). Two auxiliary computations are reported here and are not part of the verified development. The flat stretch at the end of the $k = 2$ row continues: $g_2(n) = 1/1428$ for $17 \leq n \leq 20$. And the minimising tuples, which the theorems assert to exist but do not name, are

$$\begin{aligned} g_2(16) &= \tfrac{1}{7} - \tfrac{1}{13} - \tfrac{1}{16} = \tfrac{5}{1456}, & g_2(17) &= \tfrac{1}{7} - \tfrac{1}{12} - \tfrac{1}{17} = \tfrac{1}{1428}, \\ g_3(11) &= \tfrac{1}{3} - \tfrac{1}{8} - \tfrac{1}{9} - \tfrac{1}{11} = \tfrac{5}{792}, & g_3(13) &= \tfrac{1}{3} - \tfrac{1}{7} - \tfrac{1}{9} - \tfrac{1}{13} = \tfrac{2}{819}. \end{aligned}$$

Remark 5.7 (non-vacuity). $D_k(n)$ can be empty: $D_2(2) = D_3(3) = \emptyset$, since at $n = k$ the best attempt is $1/1 - k \cdot (1/k) = 0$. Both emptiness statements are part of the verified development, so the values of Theorem 5.3 are statements about genuinely nonempty sets.

We do not conjecture a closed form. The three features above are what one would have to survive, and the table is short.

6. Verification

Every numbered theorem, proposition and corollary in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [5, 7]; the development contains no `sorry` and declares no axiom of its own, and the repository reference is to be added at submission. The reasoning layer is checked by the kernel with no appeal to compiled evaluation: this covers Theorem 2.3 and Corollary 2.4, Theorem 3.1, Proposition 3.2 and Proposition 3.3, and Theorem 4.1 with Corollary 4.2 — so in particular the bound $f_2(k) \geq 3k^2$ for *all* $k \geq 3$, the tightness of its hypothesis, and the whole encoding argument depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: two Booleans per cell of Theorem 4.3, namely the walk over $\mathcal{S}(k, n-1)$ against the exhibited colour array and the LRAT checker's verdict on $\Phi_{k,r,n}$; and one Boolean per value in Theorem 5.3, Theorem 5.4 and Remark 5.7, namely the exhaustive evaluation over $[n]^{k+1}$. Each such evaluation contributes one further axiom, recording that the compiled program returned true; the six cells of Theorem 4.3 therefore carry two apiece and the values of Section 5 one apiece, and nothing in the paper carries any other axiom. The kernel is capable of the searches of Section 5 in principle — they are tiny — but reduces rational arithmetic too slowly to complete even $n = 3$, so compiled evaluation is used there for a reason of arithmetic, not of size. The DIMACS files handed to CaDiCaL [2] were printed from the same definition of $\Phi_{k,r,n}$ that Theorem 4.1 is about, so formula and file are one object by construction. Finally, every numerical value quoted was first produced by an independent implementation in C and the formal statement written against it afterwards (Remark 4.7); the values of Section 5 were likewise reproduced by an independent brute-force enumeration.

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