AN ANTIPODAL RATIONAL 5-DESIGN WITH 37 POINTS, A 3-ADIC OBSTRUCTION FOR THE DEGREE-FIVE HILBERT–KAMKE EQUATIONS, AND THE NONEXISTENCE OF TWO-ORBIT WEIGHTED 9-DESIGNS ON $\mathbb{S}^3$

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Abstract. A Chebyshev-type quadrature of degree m, or an m-design, for a probability measure $w(t)\,dt$ on an interval I is a set of distinct points of I over which the unweighted average of every polynomial of degree at most m equals its integral. For the Chebyshev measure $(1-t^2)^{-1/2}dt/\pi$ on $(-1,1)$, Mishima, Lu, Sawa and Uchida determined the spectrum of antipodal rational 5-designs with an even number of points completely, proved that none with an odd number $2N+1 \leq 33$ of points exists, exhibited one with 35 points, and asked for one with 37 points as the smallest open case. We answer that question: the 37 rational numbers $\{0\} \cup \{\pm m/144\}$, where m runs over $\{3, 11, 18, 25, 39, 81, 91, 99, 111, 117, 122, 123, 125, 127, 133, 137, 141, 143\}$, form an antipodal 5-design. We also record a congruence obstruction on the denominators of such designs which the purely 2-adic machinery of the source does not see, and which together with that machinery cut the search space by a factor of about $3 \cdot 10^4$: writing the degree-five system in the homogeneous form $\sum m_i^2 = c_2h^2$, $\sum m_i^4 = c_4h^4$, the identity $x^4 = x^2$ on $\mathbb{Z}/3\mathbb{Z}$ forces $3 \mid h$ whenever $3 \nmid c_4 - c_2$, and then a mod 9 count forces the number of m_i prime to 3 to be divisible by 9. For the 37-point system this says that every common denominator of the points is divisible by 12, and that over the least one, 9 or 18 of the 18 positive numerators are prime to 3; the new configuration satisfies both, at the smaller of the two allowed counts.

A second, independent part of the paper settles a question about weighted spherical designs raised by Tanino, Tamaru, Hirao and Sawa. They construct designs on $\mathbb{S}^{n-1}$ as unions of hyperoctahedral orbits of *generalized corner vectors* $v_{a,s}$, and report for $n = 4$ that they could find no 11-design, and not even a 9-design, with two proper orbits. We show that none exists: for every pair of orbit types $s_1, s_2 \in \{0, 1, 2, 3\}$, all $a_1, a_2 > 0$ and all positive weights, the four B_4-invariant harmonic conditions of degree at most 9 are inconsistent. Their own Theorem 5.7 already excludes three of the five cases of their classification, for every $n \geq 3$; the content that is new here is the two remaining cases at $n = 4$. The degree is sharp — the corresponding two-orbit 7-design system is solvable — and Schur's four-orbit 11-design satisfies the very same four equations, so "two orbits" cannot be raised to "four". Two errata in the printed formulas of that source are recorded along the way.

Every theorem and proposition below is formally verified in Lean 4, and we state precisely what lies outside that guarantee.

1. Introduction

Designs, rationality, and Hilbert–Kamke equations. Let w be a probability density on an interval $I \subseteq \mathbb{R}$ with finite moments

$$a_k = \int_I t^k\, w(t)\, dt, \qquad k = 0, 1, 2, \ldots$$

An integration formula

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_I f(t)\, w(t)\, dt, \qquad x_1, \ldots, x_n \in I, \tag{1}$$

valid for every polynomial f of degree at most m, is a *Chebyshev-type quadrature of degree m*; when the x_i are pairwise distinct the configuration is called an *m-design*, and it is a *rational* design when every $x_i \in \mathbb{Q}$. Expanding (1) in the monomial basis, an m-design is exactly a solution of the

Diophantine system

$$x_1^k + x_2^k + \cdots + x_n^k = n\, a_k, \qquad k = 1, \ldots, m, \tag{2}$$

with pairwise distinct entries. Following [9] we call (2) a system of *Hilbert–Kamke equations*; the name is taken there from Cui, Xia and Xiang, who call (2) a Hilbert–Kamke type system [3, §5]. Systems of this kind for the classical orthogonal polynomial measures were treated systematically by Sawa and Uchida [12]. Rational designs go back to Hausdorff [4], who used them to simplify Hilbert's solution of Waring's problem [6], and an explicit construction of rational designs for the Gegenbauer measures is asked for by Bannai, Bannai, Ito and Tanaka [2, p. 208, Problem 2] — an attribution we take from [9, §1], the book itself being one we could not consult.

A measure is *symmetric* when $w(-t) = w(t)$, so that the odd moments vanish; an m-design is *antipodal* when its point set is invariant under $x \mapsto -x$. Two symmetric measures concern us:

$$\text{the } \textit{Chebyshev measure} \quad \frac{dt}{\pi\sqrt{1-t^2}} \ \text{ on } (-1,1), \qquad\qquad \text{the } \textit{Hermite measure} \quad \frac{e^{-t^2}dt}{\sqrt{\pi}} \ \text{ on } \mathbb{R}.$$

For an antipodal configuration all odd equations of (2) are automatic, and for $m = 5$ only the equations $k = 2$ and $k = 4$ survive. This is the system that Sections 2 to 5 are about. The restriction to antipodal configurations in odd degree is the one adopted in [9], which attributes the motivation to Misawa, Munemasa and Sawa [10]. Hilbert–Kamke systems in many variables are also the subject of an analytic line of work through the circle method, where the local conditions are handled by singular series rather than by explicit small moduli; see Wooley [15] for recent developments there.

The source and its open cell. Mishima, Lu, Sawa and Uchida [9] study exactly this degree-five case. Among their results, their Theorem 1.4 determines the even spectrum for the Chebyshev measure completely: an antipodal 5-design with $2N$ rational points exists if and only if $N = 3k$ with $k \geq 1$, $N = 3k+1$ with $k \geq 6$, or $N = 3k+2$ with $k \geq 3$. The odd case is left in a different state. In their concluding section they prove (their Theorem 8.1) that for $1 \leq N \leq 16$ there is no antipodal 5-design with $2N+1$ rational points, exhibit one with 35 points, and derive (their Theorem 8.2) the existence of an antipodal 5-design with $2N+1$ rational points whenever $N \equiv 5 \pmod 6$ and $N \notin \{5, 11\}$. The first cell left undecided is $2N + 1 = 37$, and they pose it as

Problem 1.1 ([9, Problem 8.3])**.** Find an antipodal 5-design with 37 rational points, as the smallest open case.

Results. We answer Problem 1.1 affirmatively. Put

$$M = \{3,\ 11,\ 18,\ 25,\ 39,\ 81,\ 91,\ 99,\ 111,\ 117,\ 122,\ 123,\ 125,\ 127,\ 133,\ 137,\ 141,\ 143\}, \tag{3}$$

a set of 18 integers, and

$$X_{37} = \{0\} \cup \left\{ \pm\frac{m}{144} \ : \ m \in M \right\} \subset (-1,1) \cap \mathbb{Q}. \tag{4}$$

Theorem 4.1 states that X_{37} has 37 elements and is an antipodal 5-design for the Chebyshev measure. Since $\gcd(144, 3, 11) = 1$, no prime divides 144 and every element of M, so 144 is the exact denominator of this configuration; we do *not* claim that it is the least denominator attained by a 37-point design, and Remark 4.3 records what is and is not known about that.

The configuration was found by a computer search, and the search was made feasible by a congruence obstruction that we also record. Clearing denominators in the degree-five antipodal system by writing the positive points as $x_i = m_i/(2h)$ turns it into the homogeneous integer system

$$\sum_{i=1}^{N} m_i^2 = c_2 h^2, \qquad \sum_{i=1}^{N} m_i^4 = c_4 h^4, \tag{5}$$

with $(c_2, c_4) = (2N+1,\ 3(2N+1))$, $(2N,\ 6N)$ and $(2N,\ 12N)$ for the Chebyshev-odd, Chebyshev-even and Hermite-even configurations respectively. The nonexistence machinery of [9] (their Lemmas 6.6–6.8) reduces (5) modulo 4 and modulo 16 and counts the odd m_i; no odd prime occurs anywhere in it. Theorems 3.2 and 3.4 supply the 3-adic half of the same idea. The mechanism is that $x^4 = x^2$ holds identically on $\mathbb{Z}/3\mathbb{Z}$, so the two equations of (5) collapse onto each other modulo 3; the conclusion is that $3 \mid h$ whenever $3 \nmid c_4 - c_2$, and that then the number of m_i prime to 3 is divisible by 9. Specialised to $(c_2, c_4) = (37, 111)$ this gives Theorem 3.5: every integer solution of the 37-point system has $6 \mid h$ and $9 \mid \#\{i : 3 \nmid m_i\}$. Corollary 3.6 reads that back on the design side — every common denominator of the 37 points is divisible by 12, and over the least one exactly 9 or 18 of the 18 numerators are prime to 3 — at the cost of two elementary steps that are not part of the formal development and that Section 8 names. The configuration (4) satisfies both: its denominator is $2h = 144$, and exactly 9 of the 18 elements of M are prime to 3, the smaller of the two allowed counts.

The same collapse applies to the four point configurations printed in [9], which we re-certify in Section 5 and which satisfy the new constraints in every case. The comparison is instructive: for the 22-point, 38-point and 28-point configurations the half-denominator h equals 45, 63 and 15, all odd, and indeed the 2-adic hypothesis $4 \nmid c_4 - c_2$ fails for all three; for the two odd Chebyshev cases, 35 and 37 points, it holds, and $h = 546$ and $h = 72$ are both even. The 3-adic hypothesis $3 \nmid c_4 - c_2$ holds in all five cases, and 3 divides 546, 45, 63, 15 and 72.

Designs on the sphere from generalized corner vectors. Section 6 settles a second existence question of the same kind, for cubature on the sphere rather than on an interval, and there the answer is negative. Write B_n for the hyperoctahedral group of signed permutation matrices acting on $\mathbb{R}^n$ and x^{B_n} for the orbit of x. For $a > 0$ and $0 \leq s \leq n-1$ the *generalized corner vector* of [14, Definition 2.19] is

$$v_{a,s} = \frac{1}{\sqrt{a^2 + s}}\Big(a\, e_1 + e_2 + \cdots + e_{s+1}\Big) \in \mathbb{S}^{n-1}, \tag{6}$$

which for $a = 1$ is the classical corner vector $v_{s+1} = (e_1 + \cdots + e_{s+1})/\sqrt{s+1}$; the orbit $v_{a,s}^{B_n}$ is called *proper* when $a \neq 1$. A weighted t-design of the type studied in [14, §5] with two orbits is a pair of orbits together with positive weights W_1, W_2 such that

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(y)\, d\rho = W_1 \sum_{x \in v_{a_1,s_1}^{B_n}} f(x) + W_2 \sum_{x \in v_{a_2,s_2}^{B_n}} f(x) \qquad \text{for every } f \in \mathcal{P}_t(\mathbb{S}^{n-1}), \tag{7}$$

$\mathcal{P}_t$ denoting the polynomials of degree at most t restricted to the sphere. For $n = 4$ the authors of [14] exhibit an 11-design with three proper orbits and two 13-designs with five orbits, and then record, verbatim [14, §6]:

We could not have found 11*-designs, and even* 9*-designs with* 2 *proper orbits.*

Theorem 6.4 and Corollary 6.5 show that there is nothing to find: on $\mathbb{S}^3$ no weighted 9-design of the form (7) exists, for any $a_1, a_2 > 0$ and any $s_1, s_2 \in \{0, 1, 2, 3\}$, proper or not. The 11-design half of the sentence follows, an 11-design being in particular a 9-design. The route is a reduction — the authors' own — of (7) at $t = 9$, $n = 4$ to a system of four polynomial equations in $A_i = a_i^2$ with positive coefficients w_1, w_2, followed by a case analysis over the ten unordered pairs (s_1, s_2). Eight of the ten die at a single linear functional of the four equations; the remaining two need an elimination, and both end at a cubic or a quadratic with no positive root.

We are careful about what in this is new. Proposition 5.5 of [14] splits the existence of a two-orbit weighted 9-design into five cases (i)–(v), and its own Theorem 5.7 rules out (iii), (iv) and (v) for every $n \geq 3$; those are theirs, and are reproved here at $n = 4$ only incidentally. What [14] leaves open, and what Theorem 6.4 closes, is cases (i) and (ii) at $n = 4$; Remarks 6.6 and 6.7 give the detail, the second of them explaining why the expectation recorded in [14] after its Table 3 — that "Case (i) may not occur even for $n \geq 4$" — rests on a mislabelling of that table. Section 6 carries the controls too: Schur's four-orbit 11-design on $\mathbb{S}^3$ satisfies the very same four equations with positive weights (Theorem 6.8), and the two-orbit system for 7-designs *is* solvable (Proposition 6.10), so the degree 9 is sharp.

What is claimed, and on what evidence. Theorem 4.1 answers a question that the source states as open. Problem 1.1 is quoted verbatim from the version 4 e-print source of [9], retrieved on 2026-08-29, and stands word for word in the published version as well; the paper had by then appeared in print, and on the same date it had zero recorded forward citations in each of Semantic Scholar, OpenAlex and Crossref. A full-text sweep of a local arXiv corpus of 260 mathematics shards, complete through the announcement window of 2026-08-28, returns exactly three papers containing the string `2503.21151`: the source itself and two papers citing it, one in its bibliography only and one for a definition. The four numerator sequences printed in [9] are absent from the OEIS, with controls confirming that the empty response is a true zero.

Theorems 3.2, 3.4 and 3.5 are elementary, and we present them as *not found in the literature* rather than as difficult. The evidence is a modulus census rather than a phrase search: a topical set of 81 papers, collected by full-text hits on *Hilbert–Kamke*, *antipodal* 5*-design*, *Chebyshev-type quadrature* and *hat-box*, was scanned for every modulus occurring in a `\pmod`, `\bmod` or `mod` construction. The moduli that occur are 2 (65 occurrences), 4 (59), 16 (14), 32 (8), 6 (7), 8 (7), 21 (2) and 1 (1); no modulus 3, 9 or 12 occurs anywhere in the set, and the phrases *divisible by 3*, *divisible by 9* and *coprime to 3* return no hits in it. As a positive control the same script reports 13 occurrences of (mod 4) and 11 of (mod 16) in [9] itself, matching its Lemmas 6.6–6.8. The Hilbert–Kamke papers in that set belong to the circle-method line and use no explicit small modulus at all, handling local solvability through singular series. A reader who regards Theorems 3.2 and 3.4 as folklore will get no argument from us; what we do claim for them is the consequence for the named cell, Theorem 3.5, and the fact that they made the search of Section 4 run.

For Theorem 6.4 the evidence is likewise a record of searches, run on 2026-08-30. The source [14] then had no recorded forward citations in OpenAlex — zero for the e-print record and zero for the journal record, with an empty citing-works list — and none in Crossref, whose reference count for the DOI was zero; Semantic Scholar returned exactly two citing papers, one on the Prouhet–Tarry–Escott problem and one on tight t-fusion frames, neither of which contains a nonexistence statement for two-orbit designs. Positive controls fired on those services. A full-text sweep of a local arXiv corpus of 367 shards, incremental through the announcement window of 2026-08, finds the string `2501.11437` in exactly one shard and, inside that shard, only in the source's own header line, so no citing paper is present there; the phrases *generalized corner vector* and *corner-vector method* occur only inside the source itself, while as a positive control *spherical design* occurs in 221 shards. Searches of the arXiv metadata return only the source for *corner vector* together with *spherical design* and for *generalized corner vector*, and nothing at all for *proper orbit* together with *spherical design*, or for *spherical design* together with *nonexistence* in abstracts. The nearest published nonexistence statement is Bajnok's theorem [1, Proposition 15], quoted as [14, Theorem 2.18], which bounds the degree by 7 for unions of the orbits $v_k^{B_n}$ — that is, for $a_1 = a_2 = 1$ only.

We could not read the text of the book [2], nor non-arXiv venues absent from the three citation indexes, nor anything posted after 2026-08-30. For Section 6 in particular we did not consult MathSciNet, zbMATH or Google Scholar, nor the full text of Yamamoto, Hirao and Sawa [16] — the same authors, the same vector family and the same degree nine, but Euclidean designs on two concentric spheres in the ball, a different object, of which we read the abstract only — nor the full texts of Nozaki–Sawa [11], Sawa–Xu [13] and Heo–Xu [5].

Organisation. Section 2 fixes the definitions and records the moment values and the linearity statement that turns six power sums into a quadrature formula. Section 3 proves the congruence obstructions. Section 4 gives the 37-point design and describes the search. Section 5 re-certifies the four configurations printed in [9] and records the negative controls. Section 6 is independent of the preceding ones and treats the two-orbit designs on $\mathbb{S}^3$. Section 7 lists open questions and Section 8 describes the formal verification.

2. Preliminaries

Moment sequences. Throughout, the two moment sequences are given by their classical closed forms

$$a_k^{\mathrm{C}} = \begin{cases} \dbinom{k}{k/2} 2^{-k}, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases} \qquad a_k^{\mathrm{H}} = \begin{cases} \dfrac{k!}{(k/2)!\, 4^{k/2}}, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases} \tag{8}$$

which are the moments of the Chebyshev and Hermite measures respectively. The identification of (8) with the integrals $\int t^k w(t)\, dt$ is classical and is used only as motivation; see Section 8. The four values that enter the degree-five equations are the ones recorded in [9]: $a_2^{\mathrm{C}} = \frac{1}{2}$ and $a_4^{\mathrm{C}} = \frac{3}{8}$ in its §8, $a_2^{\mathrm{H}} = \frac{1}{2}$ and $a_4^{\mathrm{H}} = \frac{3}{4}$ in its §4.

Theorem 2.1. *$a_0^{\mathrm{C}} = 1$, $a_1^{\mathrm{C}} = a_3^{\mathrm{C}} = a_5^{\mathrm{C}} = 0$, $a_2^{\mathrm{C}} = \frac{1}{2}$, $a_4^{\mathrm{C}} = \frac{3}{8}$, and $a_0^{\mathrm{H}} = 1$, $a_2^{\mathrm{H}} = \frac{1}{2}$, $a_4^{\mathrm{H}} = \frac{3}{4}$.*

Proof. Evaluations of (8): $\binom{0}{0}/2^0 = 0!/(0! \cdot 1) = 1$, $\binom{2}{1}/2^2 = 1/2$, $\binom{4}{2}/2^4 = 6/16 = 3/8$, $2!/(1! \cdot 4) = 1/2$ and $4!/(2! \cdot 16) = 24/32 = 3/4$, and the odd cases are zero by definition. No search is involved. □

One further value is used below, $a_6^{\mathrm{C}} = \binom{6}{3}/2^6 = 20/64 = \frac{5}{16}$. It is the same one-line evaluation, but it is deliberately not part of Theorem 2.1: it enters only the degree-seven refutations, Propositions 4.5 and 5.5, and is verified inside their proofs.

Designs.

Definition 2.2. Let $a = (a_k)_{k \geq 0}$ be a moment sequence and let L be a finite list of rationals. We call L an *antipodal degree-five design* for a when

1. the entries of L are pairwise distinct;
2. L is closed under $x \mapsto -x$ (antipodality);
3. $\sum_{p \in L} p^k = |L|\, a_k$ for $k = 0, 1, \ldots, 5$.

Antipodality is folded into the definition because every configuration in this paper has it, and because it is what makes the odd equations of (2) automatic; in the terminology of [9] this is an antipodal 5-design whose points happen to be rational. Condition (iii) with $k = 0$ is the statement $|L| = |L|\, a_0$ and is vacuous for a probability measure; we keep it so that the design condition is literally the vanishing of the first six power-sum defects. The point of stating a design by its power sums is that this is a finite list of rational identities, while the defining property (1) quantifies over an infinite-dimensional space of polynomials. The two are equivalent by linearity.

Theorem 2.3. *Let L be an antipodal degree-five design for a and let $c_0, \ldots, c_5 \in \mathbb{Q}$. Then*

$$\sum_{p \in L} \sum_{k=0}^{5} c_k p^k \;=\; |L| \sum_{k=0}^{5} c_k a_k.$$

That is, the average over L of every polynomial of degree at most 5 equals the corresponding moment functional.

Proof. Exchange the two finite sums and apply condition (iii) of Definition 2.2 to each k separately. No search is involved. □

The integer system. Let L be an antipodal degree-five design for the Chebyshev measure with $|L| = 2N + 1$ odd. Since L is a set of distinct points invariant under $x \mapsto -x$ and $|L|$ is odd, the involution has a fixed point in L, so $0 \in L$ and $L = \{0\} \cup \{\pm x_1, \ldots, \pm x_N\}$ with $x_1, \ldots, x_N > 0$ distinct. Halving the surviving even power sums, condition (iii) becomes the pair of equations displayed as [9, §8, eq. (90)]:

$$x_1^2 + \cdots + x_N^2 = \frac{2N+1}{4}, \qquad x_1^4 + \cdots + x_N^4 = \frac{3(2N+1)}{16}. \tag{9}$$

Likewise for $|L| = 2N$ even, $L = \{\pm x_1, \ldots, \pm x_N\}$ and $\sum x_i^2 = N a_2$, $\sum x_i^4 = N a_4$.

Lemma 2.4. *Let $x_1, \dots, x_N$ be positive rationals satisfying* (9) *and let k be any positive integer with $kx_i \in \mathbb{Z}$ for all i. Then k is even; and writing $k = 2h$ and $m_i = kx_i$, the pair $(m_i; h)$ satisfies* (5) *with $c_2 = 2N+1$ and $c_4 = 3(2N+1)$.*

Proof. Multiplying the first equation of (9) by $4k^2$ gives $4\sum m_i^2 = (2N{+}1)k^2$, so 4 divides $(2N{+}1)k^2$; as $2N+1$ is odd, $4 \mid k^2$ and hence k is even. Substituting $k = 2h$ into the two equations of (9) multiplied by k^2 and k^4 gives $\sum m_i^2 = (2N+1)h^2$ and $\sum m_i^4 = 3(2N+1)h^4$. No search is involved. This lemma is elementary and, unlike everything labelled a theorem below, it is not part of the formal development; see Section 8. □

For an even configuration the same substitution, applied to a common denominator $k = 2h$ that happens to be even, gives (5) with $(c_2, c_4) = (2N, 6N)$ for the Chebyshev measure and $(2N, 12N)$ for the Hermite measure. Here the parity of k is not automatic for every N, so we do not assert it; all the even configurations appearing below have an even denominator by inspection.

Note that Lemma 2.4 imposes no primitivity: k may be *any* common denominator, not only the least one. The congruence results of the next section are likewise stated for arbitrary integer solutions of (5), which is what makes them apply to every common denominator at once.

3. Congruence obstructions

The whole of this section rests on one observation. On $\mathbb{Z}/3\mathbb{Z}$ every square lies in $\{0,1\}$ and squaring fixes $\{0,1\}$, so

$$x^4 = x^2 \quad \text{identically on } \mathbb{Z}/3\mathbb{Z}; \tag{10}$$

the same identity holds on $\mathbb{Z}/4\mathbb{Z}$, for the same reason. Each is verified by evaluating both sides at each of the 3, respectively 4, residues; these are the smallest of the three exhaustive checks in this section, and none of them ranges over a set of more than nine elements. Under (10) the two equations of (5) collapse onto each other:

$$c_4 h^4 \equiv \sum_i m_i^4 \equiv \sum_i m_i^2 \equiv c_2 h^2 \pmod 3,$$

and likewise modulo 4. If in addition h is invertible modulo the relevant modulus, its powers disappear and a condition on $c_4 - c_2$ alone is left.

We record the 2-adic case first. It is not new: it is the mechanism of [9, Lemmas 6.6–6.8], repackaged as a divisibility of h rather than as a count of odd numerators. We state it because the 37-point conclusion uses both primes.

Proposition 3.1. *Let $m_1, \dots, m_N, h, c_2, c_4$ be integers satisfying* (5)*. If $4 \nmid c_4 - c_2$, then h is even.*

Proof sketch. Suppose h is odd. Then $h \bmod 4$ is a unit of $\mathbb{Z}/4\mathbb{Z}$, and both units satisfy $u^2 = u^4 = 1$ there, which is again a check on a 4-element set. Reducing (5) modulo 4 and using $x^4 = x^2$ on $\mathbb{Z}/4\mathbb{Z}$ gives $c_4 \equiv c_4 h^4 \equiv c_2 h^2 \equiv c_2$, so $4 \mid c_4 - c_2$. □

Theorem 3.2. *Let $m_1, \dots, m_N, h, c_2, c_4$ be integers satisfying* (5)*. If $3 \nmid c_4 - c_2$, then $3 \mid h$.*

Proof sketch. Suppose $3 \nmid h$. Then $h \bmod 3$ is a unit of $\mathbb{Z}/3\mathbb{Z}$ and both units satisfy $u^2 = u^4 = 1$ there, a check on a 3-element set. Summing (10) over the list gives $\sum m_i^4 \equiv \sum m_i^2 \pmod 3$; substituting (5) and cancelling the powers of h leaves $c_4 \equiv c_2 \pmod 3$, contradicting the hypothesis. □

The hypothesis is not removable, and neither is the conclusion improvable; see Proposition 5.4. Specialising:

Corollary 3.3. *Let h and $m_1, \dots, m_N$ be integers.*

(1) (Chebyshev, $2N+1$ points) *If $\sum m_i^2 = (2N+1)h^2$ and $\sum m_i^4 = 3(2N+1)h^4$, then h is even; and if moreover $3 \nmid 2N+1$, then $3 \mid h$, so $6 \mid h$.*

(2) (Chebyshev, $2N$ points) *If* $\sum m_i^2 = 2Nh^2$ *and* $\sum m_i^4 = 6Nh^4$ *and* $3 \nmid N$*, then* $3 \mid h$.
(3) (Hermite, $2N$ points) *If* $\sum m_i^2 = 2Nh^2$ *and* $\sum m_i^4 = 12Nh^4$ *and* $3 \nmid N$*, then* $3 \mid h$.

Proof. In case (i), $c_4 - c_2 = 2(2N+1)$, which is never divisible by 4, and is divisible by 3 exactly when $3 \mid 2N+1$. In cases (ii) and (iii), $c_4 - c_2 = 4N$ and $10N$ respectively, each divisible by 3 exactly when $3 \mid N$. Apply Proposition 3.1 and Theorem 3.2. □

Case (i) explains the pattern noted in the introduction: in the odd Chebyshev system the half-denominator h is *always* even, and it is divisible by 3 except possibly when $3 \mid 2N+1$.

Once $3 \mid h$ the collapse can be pushed one step further. The identity (10) fails on $\mathbb{Z}/9\mathbb{Z}$, so the next statement is not a collapse of one equation onto the other but a count.

Theorem 3.4. *Let* $m_1, \dots, m_N, h, c_2, c_4$ *be integers satisfying* (5)*. If* $3 \mid h$*, then*

$$9 \;\Big|\; \#\{i : 3 \nmid m_i\}.$$

Proof sketch. If $3 \mid h$ then $9 \mid h^2$ and $81 \mid h^4$, so both right-hand sides of (5) vanish modulo 9 and $\sum_i (m_i^2 + m_i^4) \equiv 0 \pmod 9$. The pointwise input is that

$$m^2 + m^4 \equiv \begin{cases} 0 \pmod 9, & 3 \mid m, \\ 2 \pmod 9, & 3 \nmid m, \end{cases}$$

which is uniform over all six residues prime to 3 and is verified by evaluating $m^2 + m^4$ at each of the 9 residues modulo 9 — the only exhaustive computation in the proof, over a set of size 9. Summing, $2 \cdot \#\{i : 3 \nmid m_i\} \equiv 0 \pmod 9$; since $5 \cdot 2 = 1$ in $\mathbb{Z}/9\mathbb{Z}$, the factor 2 cancels. □

Combining the three statements at the parameters of Problem 1.1 gives the constraint that made the search of Section 4 feasible. For $2N + 1 = 37$ we have $N = 18$, $c_2 = 37$ and $c_4 = 111$, so $c_4 - c_2 = 74$, which is divisible by neither 4 nor 3.

Theorem 3.5. *Let* $m_1, \dots, m_{18}$ *and* h *be integers with*

$$\sum_{i=1}^{18} m_i^2 = 37h^2, \qquad \sum_{i=1}^{18} m_i^4 = 111h^4.$$

Then $6 \mid h$ *and* $9 \mid \#\{i : 3 \nmid m_i\}$.

Proof. $c_4 - c_2 = 74$ and $4 \nmid 74$, so $2 \mid h$ by Proposition 3.1; $3 \nmid 74$, so $3 \mid h$ by Theorem 3.2; hence $6 \mid h$, and Theorem 3.4 applies. □

Corollary 3.6. *Let* L *be an antipodal* 5*-design with* 37 *rational points for the Chebyshev measure, let* $x_1, \dots, x_{18} > 0$ *be its positive points, and let* k *be any positive integer with* $kx_i \in \mathbb{Z}$ *for all* i*. Then* $12 \mid k$*, and the numerators* $m_i = kx_i$ *satisfy* $\#\{i : 3 \nmid m_i\} \in \{0, 9, 18\}$*. If* k *is the least such integer, then* $\#\{i : 3 \nmid m_i\} \in \{9, 18\}$.

Proof. Lemma 2.4 puts $(m_i; h)$ with $k = 2h$ into the system of Theorem 3.5, whence $6 \mid h$ and $12 \mid k$; the count is divisible by 9 and bounded by 18. For the last clause, if every m_i were divisible by 3 then, since $3 \mid h$ divides k, the integer $k/3$ would again be a common denominator, contradicting minimality. The passage through Lemma 2.4 and the minimality argument are elementary and are not part of the formal development; the statement of Theorem 3.5 is. □

4. A 37-point antipodal rational design

Theorem 4.1. *Let* M *and* X_{37} *be as in* (3) *and* (4)*. Then* X_{37} *consists of* 37 *distinct rational numbers, all lying in the open interval* $(-1, 1)$*; it is closed under* $x \mapsto -x$*; and*

$$\sum_{p \in X_{37}} p^k = 37\, a_k^{\mathrm{C}} \qquad \text{for } k = 0, 1, 2, 3, 4, 5.$$

Consequently, by Theorem 2.3, X_{37} is an antipodal 5-design with 37 rational points for the Chebyshev measure $(1-t^2)^{-1/2}dt/\pi$ on $(-1,1)$; this answers Problem 1.1.

Proof sketch. Distinctness, membership in $(-1,1)$ and antipodality are structural: for any list of *distinct* integers m with $0 < m < D$, the list $\{0\} \cup \{\pm m/D\}$ has pairwise distinct entries lying in $(-1,1)$ and is closed under negation. Here $D = 144$, the 18 elements of M are distinct, and $0 < m \le 143 < 144$ for each. The odd power sums vanish by antipodality. For $k \in \{2,4\}$ the identity to be checked is $2\sum_{m\in M}(m/144)^k = 37\, a_k^{\mathrm{C}}$, that is

$$\sum_{m\in M} m^2 = 37\cdot 72^2 = 191808, \qquad \sum_{m\in M} m^4 = 111\cdot 72^4 = 2982998016,$$

a pair of exact integer identities on 18 terms. The whole verification is a finite exact computation in $\mathbb{Q}$; it involves no search, and in particular the configuration's correctness does not depend on the program that found it. □

Remark 4.2 (how the configuration was found). The set M was produced by a randomized meet-in-the-middle search over the integer system of Theorem 3.5, run at the half-denominator $h = 72$; it returned in about forty minutes of wall time on one machine, on 2026-08-29. The search space is the set of 18 distinct integers m_i with $0 < m_i < 2h$ and $\sum m_i^2 = 37h^2$, $\sum m_i^4 = 111h^4$, restricted by three cuts. The first, $6 \mid h$, is Theorem 3.5. The second is that exactly 16 of the m_i are odd: reducing the fourth-power equation modulo 16 and using $m^4 \equiv 1$ or $0 \pmod{16}$ according as m is odd or even — this is [9, Lemma 6.6(ii)] — gives $T \equiv 0 \pmod{16}$ for the number T of odd m_i, since h is even; and $1 \le T \le 18$ forces $T = 16$, the lower bound $T \ge 1$ being the primitivity of the representation. The third is that exactly 9 of the m_i are prime to 3, one of the two values allowed by Theorem 3.4. Together the last two cuts reduce the sampling volume by a factor of about $2.7\cdot 10^4$. Both search programs were validated on planted targets before use, and every candidate they report is re-verified from scratch inside the program; the exact rational arithmetic was then redone independently for Theorem 4.1. Neither the programs nor their output form part of the formal development, and no statement of this paper depends on them.

Remark 4.3 (a bracket on the least denominator, outside the formal development). We do *not* claim that 144 is the least denominator of an antipodal rational 5-design with 37 points, and we record here exactly what was computed. By Corollary 3.6 the admissible denominators are the multiples of 12; the six of them that are at most 72, namely $12, 24, 36, 48, 60, 72$, were all settled. Two are impossible outright: at denominator $2h$ the odd numerators available are $1, 3, \ldots, 2h-1$, only h in number, and $h \in \{6, 12\}$ leaves fewer than the 16 required by the odd-count cut of Remark 4.2. The remaining four, $h \in \{18, 24, 30, 36\}$, were enumerated completely by meet-in-the-middle over the 2 even and 16 odd numerators, in 806 seconds of wall time with a peak of 0.95 GB, with no solution found. Hence the least denominator of an antipodal rational 5-design with 37 points, if the odd-count cut is granted, lies in $\{84, 96, 108, 120, 132, 144\}$. The values 84, 108 and 132 were not searched at all, and 120 saw only 113 randomized trials. This paragraph is an engineering measurement of what was run on one machine on 2026-08-29 — about 2.3 CPU-hours in total, peak resident set 6.4 GB — and not a theorem: the search programs are not machine-checked, and the negative slices rest on the odd-count cut, hence on the primitivity of the representation, which the theorems of Section 3 do not require.

The new configuration satisfies the obstructions of Section 3, which is the non-vacuity check for them.

Proposition 4.4. *The numerators* (3) *satisfy $\sum_{m\in M} m^2 = 37\cdot 72^2$ and $\sum_{m\in M} m^4 = 111\cdot 72^4$ with $h = 72$; and $6 \mid 72$, and exactly 9 of the 18 elements of M are prime to 3 — the values predicted by Proposition 3.1 and Theorems 3.2 and 3.4.*

Proof. Direct evaluation on the 18 integers of M. The nine elements divisible by 3 are 3, 18, 39, 81, 99, 111, 117, 123 and 141. □

The one remaining prediction, the odd count of Remark 4.2, holds too: the only even elements of M are 18 and 122, so exactly 16 of the 18 are odd. That count is an inspection of (3) and, unlike Proposition 4.4, is not part of the formal development.

Proposition 4.5. *X_{37} is not a 7-design: $\sum_{p\in X_{37}} p^6 \neq 37\, a_6^{\mathrm{C}} = \frac{185}{16}$. In fact the sixth power sum equals* $688300668749/61917364224$.

Proof. An exact rational computation of one power sum over the 37 points; the two sides differ. This is the negative control against the too-large reading of Theorem 4.1: the configuration is a 5-design and no more. □

5. The published configurations, and controls

Four antipodal rational 5-designs are printed in [9]. We re-certify all four, in exact rational arithmetic, as a check on the framework in which Theorem 4.1 is stated: the same definition, the same structural lemmas and the same power-sum computation are applied to configurations whose correctness is independent of us. These are not new results; the configurations are the source's, and the 38-point one is recorded there as the output of the authors' own computer search [9, Remark 6.4].

Theorem 5.1. *The* 35 *points* $\{0\} \cup \{\pm m/1092\}$, *with m running over*

$$9,\ 65,\ 91,\ 195,\ 531,\ 669,\ 689,\ 729,\ 837,\ 871,\ 923,\ 933,\ 1001,\ 1027,\ 1053,\ 1066,\ 1079$$

(*the configuration displayed as* [9, §8, eq. (91)]), *are distinct, lie in* $(-1, 1)$, *are closed under negation, and form an antipodal* 5*-design for the Chebyshev measure.*

Theorem 5.2. *The* 22 *points* $\{\pm m/90\}$ *with* $m \in \{2, 8, 16, 34, 72, 73, 76, 77, 80, 84, 86\}$ *and the* 38 *points* $\{\pm m/126\}$ *with*

$$m \in \{2, 4, 20, 32, 40, 44, 56, 83, 88, 100, 104, 106, 109, 110, 116, 118, 120, 122, 124\}$$

(*the two configurations displayed as* [9, §6.1, eqs. (59) and (60)]) *are, in each case, distinct, contained in* $(-1, 1)$, *closed under negation, and antipodal* 5*-designs for the Chebyshev measure.*

Theorem 5.3. *The* 28 *points* $\{\pm m/30\}$ *with* $m \in \{1, 2, 3, 4, 5, 8, 9, 13, 14, 15, 25, 39, 40, 42\}$ (*the configuration displayed as* [9, §8, eq. (92)] *for the exceptional case $N = 14$ of its Theorem 8.6*) *are distinct, closed under negation, and form an antipodal* 5*-design for the Hermite measure* $e^{-t^2}dt/\sqrt{\pi}$. (*The Hermite measure is supported on all of* $\mathbb{R}$, *so no interval condition is imposed.*)

Proof sketch for Theorems 5.1–5.3. Identical in each case to the proof of Theorem 4.1: distinctness, range and antipodality from the structural lemmas, odd power sums zero, and two exact integer identities for $k = 2$ and $k = 4$. In the notation of (5) the four cases are $(c_2, c_4; h) = (35, 105; 546)$, $(22, 66; 45)$, $(38, 114; 63)$ and $(28, 168; 15)$. Each configuration was first reproduced in exact rational arithmetic from the displays of [9] and then certified; no search is involved. □

All four satisfy the constraints of Section 3, which is the compute-first-values control for those results: 3 divides 546, 45, 63 and 15; the numbers of numerators prime to 3 are 9 of 17, 9 of 11, 18 of 19 and 9 of 14, all multiples of 9; and among the five configurations certified in this paper, the two with $4 \nmid c_4 - c_2$ — the odd Chebyshev ones, at 35 and 37 points — are exactly the two with even h. The following records the first of these formally, together with the two refutations that keep the congruence statements from being read too strongly.

Proposition 5.4. (1) (non-vacuity) *The* 17 *numerators of Theorem 5.1 satisfy* $\sum m_i^2 = 35 \cdot 546^2$ *and* $\sum m_i^4 = 105 \cdot 546^4$, *and exactly* 9 *of them are prime to* 3.

(2) (the conclusion is not improvable) *The conclusion* $6 \mid h$ *of Theorem 3.5 cannot be strengthened to* $4 \mid h$ *or to* $9 \mid h$: *the witness of* (i) *has* $h = 546$, *which is divisible by neither. Nor can the count in Theorem 3.4 be strengthened to "all m_i": that witness has* 9 *numerators prime to* 3, *not* 17.

(3) (the hypothesis is not removable) *The hypothesis* $3 \nmid c_4 - c_2$ *of Theorem 3.2 cannot be dropped: with* $N = 1$, $m_1 = 1$, $c_2 = c_4 = 1$ *and* $h = 1$, *both equations of* (5) *hold,* $3 \nmid h$, *and indeed* $3 \mid c_4 - c_2$.

Proof. Three exact evaluations; in (ii) note $546 = 2 \cdot 3 \cdot 7 \cdot 13$. □

Finally, three controls on the design framework itself.

Proposition 5.5. (1) *The* 35*-point configuration of Theorem 5.1 is not a* 7*-design: its sixth power sum is* $9632093236737695/905795139255552$, *against the required* $35\, a_6^{\mathrm{C}} = 175/16$.

(2) *Replacing the single numerator* 9 *by* 11 *in Theorem 5.1 destroys the design property: the fourth power sum of the perturbed configuration is* $291615021905/22218287364$, *against the required* $35\, a_4^{\mathrm{C}} = 105/8$.

(3) *The positivity hypothesis in the distinctness lemma for* $\{0\} \cup \{\pm m/D\}$ *is not vacuous: for* $D = 4$ *and numerators* $0, 1$ *the resulting list repeats the entry* 0 *and is not a set of distinct points.*

Proof. Three exact rational evaluations. Item (i) is a too-large refutation, (ii) a too-small one — the design equations are not satisfied on a neighbourhood of the witness in the numerator lattice — and (iii) shows that the structural lemma used in every proof above carries a real hypothesis. □

6. Two-orbit weighted 9-designs on $\mathbb{S}^3$

This section is independent of Sections 2–5. The measure is now the surface measure ρ on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, the configurations are orbits of the hyperoctahedral group, and the weights are no longer required to be equal. Only the shape of the question is the same: whether a cubature formula of a prescribed combinatorial shape exists. Here the answer is negative, and the object of the section is the case $n = 4$ of a question raised in [14].

Setup. Let B_n be the group of signed permutations of the coordinates of $\mathbb{R}^n$ and, for $x \in \mathbb{R}^n$, let x^{B_n} be its orbit. For $a > 0$ and $0 \le s \le n-1$, the generalized corner vector $v_{a,s} \in \mathbb{S}^{n-1}$ of (6) has a in the first coordinate, 1 in the next s coordinates and 0 elsewhere, normalised; the orbit $v_{a,s}^{B_n}$ is proper when $a \ne 1$. A pair of orbits together with weights $W_1, W_2 > 0$ is a *weighted t-design of two-orbit type* when (7) holds. This is the object of [14, §5], displayed there at the head of that section; *exactly two* orbits are involved, and that is the reading used here throughout. In particular nothing below says anything about configurations built from two proper orbits *together with* further corner-vector orbits.

Three facts, all quoted from [14], reduce (7) to finitely many polynomial equations. First, Sobolev's theorem, stated as [14, Theorem 2.13] without an original reference there and cited here only through that paper: for a union of B_n-orbits carrying a weight constant on each orbit, (7) for all $f \in \mathcal{P}_t$ is equivalent to (7) for all B_n-*invariant* $f \in \mathcal{P}_t$. Second, the harmonic Molien series [14, Proposition 2.14],

$$\sum_{i \ge 0} \bigl(\dim \mathrm{Harm}_i(\mathbb{S}^{n-1})^{B_n}\bigr)\lambda^i = \frac{1}{(1-\lambda^4)(1-\lambda^6)\cdots(1-\lambda^{2n})} = 1 + \lambda^4 + \lambda^6 + 2\lambda^8 + \lambda^{10} + \cdots \quad (n = 4),$$

so that for $n = 4$ the invariant harmonics of degree between 1 and 9 form a space of dimension 4, spanned by the polynomials f_4, f_6, $f_{8,1}$, $f_{8,2}$ of [14, Lemma 2.16], which at $n = 4$ read

$$\begin{aligned} f_4 &= m_{(4)} - 2\, m_{(2,2)}, & f_{8,1} &= m_{(8)} - \tfrac{28}{3} m_{(6,2)} + \tfrac{70}{3} m_{(4,4)}, \\ f_6 &= m_{(6)} - 5\, m_{(4,2)} + 30\, m_{(2,2,2)}, & f_{8,2} &= m_{(4,4)} - 3\, m_{(4,2,2)} + 54\, m_{(2,2,2,2)}, \end{aligned} \tag{11}$$

m_λ denoting the monomial symmetric polynomial in $x_1^2, \ldots, x_4^2$ indexed by λ — for instance $m_{(4,2)} = \sum_{i \ne j} x_i^4 x_j^2$ has twelve terms and $m_{(2,2,2,2)} = x_1^2 x_2^2 x_3^2 x_4^2$. Third, a harmonic polynomial of positive degree has sphere average 0, and an invariant f is constant on each orbit, so (7) at $t = 9$, $n = 4$ becomes the pair of conditions

$$\widetilde{W}_1 + \widetilde{W}_2 = 1, \qquad \widetilde{W}_1 f_j(v_{a_1,s_1}) + \widetilde{W}_2 f_j(v_{a_2,s_2}) = 0 \quad \big(j = 4,\ 6,\ (8,1),\ (8,2)\big), \tag{12}$$

where $\widetilde{W}_i = W_i\,|v^{B_4}_{a_i,s_i}|$. The four equations on the right are homogeneous in $(\widetilde{W}_1, \widetilde{W}_2)$, so the normalisation on the left can always be arranged by scaling and plays no further role: a two-orbit weighted 9-design on $\mathbb{S}^3$ exists if and only if the four homogeneous equations admit a solution with $\widetilde{W}_1, \widetilde{W}_2 > 0$.

Each f_j is homogeneous of degree d_j, so that

$$f_j(v_{a,s}) = \frac{f_j(u_s)}{(a^2+s)^{d_j/2}}, \qquad u_s = (a, \underbrace{1, \dots, 1}_{s}, 0, \dots, 0) \in \mathbb{R}^4,$$

u_s being the unnormalised corner vector. Multiplying the four equations by the positive number $3(a_i^2 + s_i)^4$ — one factor per orbit, the same factor for all four equations, so that neither solvability nor any sign is affected — clears every denominator and every third. This produces the integer polynomials of the next definition.

Definition 6.1. For $s \in \{0, 1, 2, 3\}$ and $A \in \mathbb{R}$ set

s	$F_4(s,A)$	$F_6(s,A)$
0	$3A^4$	$3A^4$
1	$3A^4 - 6A^2 + 3$	$3A^4 - 12A^3 - 30A^2 - 12A + 3$
2	$3A^4 - 36A^2 - 48A$	$3A^4 - 24A^3 + 96A - 48$
3	$3A^4 - 90A^2 - 216A - 81$	$3A^4 - 36A^3 + 90A^2 + 684A + 27$

s	$F_{8,1}(s,A)$	$F_{8,2}(s,A)$
0	$3A^4$	0
1	$3A^4 - 28A^3 + 70A^2 - 28A + 3$	$3A^2$
2	$3A^4 - 56A^3 + 140A^2 - 56A + 20$	$-3A^2 - 18A + 3$
3	$3A^4 - 84A^3 + 210A^2 - 84A + 51$	$-18A^2 + 108A - 18$

Theorem 6.2. *Let $a \in \mathbb{R}$ and write $u_0 = (a,0,0,0)$, $u_1 = (a,1,0,0)$, $u_2 = (a,1,1,0)$, $u_3 = (a,1,1,1)$. Then for each $s \in \{0,1,2,3\}$, with $f_4, f_6, f_{8,1}, f_{8,2}$ as in* (11),

$$F_4(s,a^2) = 3(a^2+s)^2 f_4(u_s), \quad F_6(s,a^2) = 3(a^2+s)\, f_6(u_s), \quad F_{8,i}(s,a^2) = 3 f_{8,i}(u_s)\ (i = 1,2).$$

Equivalently, $F_j(s,a^2) = 3(a^2+s)^4 f_j(v_{a,s})$ for all four j whenever $a^2 + s > 0$.

Proof. Each of the sixteen identities is an identity between polynomials in the single variable a, obtained by expanding both sides; no search and no case distinction is involved. The quantities $3(a^2+s)^{4-d_j/2} f_j(u_s)$ are, up to the factor 3, the quantities $\tilde{f}_j(v_{a,s})$ of [14, Lemma 2.17], and for $j = 4, 6, (8,1)$ the closed forms above agree with the ones printed there, specialised to $n = 4$. For $j = (8,2)$ they agree with the printed one only after the correction recorded in Proposition 6.14(i). □

The reduction leading to (12) replaces an orbit sum by $|\text{orbit}| \cdot f(v)$, which needs the f_j to be genuinely B_4-invariant. They are written in (11) as symmetric functions of the squares of the coordinates, so this is visible; we record two representative instances of it.

Proposition 6.3. *For all $a, b, c, d \in \mathbb{R}$, the polynomials f_4 and $f_{8,2}$ of* (11) *are unchanged by the transposition of the first two coordinates and by a sign change in the first coordinate:*

$$f_j(a,b,c,d) = f_j(b,a,c,d) \quad \text{and} \quad f_j(-a,b,c,d) = f_j(a,b,c,d), \qquad j = 4,\ (8,2).$$

Proof. Expansion of both sides in each of the four cases. □

The nonexistence theorem.

Theorem 6.4. *Let $s_1, s_2 \in \{0,1,2,3\}$ and let A_1, A_2, w_1, w_2 be real numbers with $A_1, A_2 > 0$ and $w_1, w_2 > 0$. Then the four equations*

$$w_1 F_j(s_1, A_1) + w_2 F_j(s_2, A_2) = 0, \qquad j = 4,\ 6,\ (8,1),\ (8,2),$$

cannot all hold.

Corollary 6.5. *There is no weighted 9-design of two-orbit type (7) on $\mathbb{S}^3$: for no $a_1, a_2 > 0$, no $s_1, s_2 \in \{0,1,2,3\}$ and no weights $W_1, W_2 > 0$ does (7) hold for every $f \in \mathcal{P}_9(\mathbb{S}^3)$. In particular there is none with two proper orbits, and, an 11-design being in particular a 9-design, there is none of degree 11 either. This answers the sentence quoted in Section 1 from [14, §6], negatively.*

Proof. Substitute $A_i = a_i^2 > 0$ and $w_i = \widetilde{W}_i/(A_i + s_i)^4 > 0$ in (12) and apply Theorem 6.2 and then Theorem 6.4. The passage from (7) to (12), which uses Sobolev's theorem, the Molien series and the statement that (11) spans the invariant harmonics of degree at most 9 for $n = 4$, is quoted from [14] and is not part of the formal development; see Section 8. The statement of Theorem 6.4 is. $\square$

Proof sketch of Theorem 6.4. There are ten unordered pairs (s_1, s_2), and the system is symmetric in the two orbits, so ten cases suffice; the whole argument is a finite chain of polynomial identities and sign estimates, with no search anywhere.

Eight cases by one linear functional each. Let $\Lambda = F_4 - F_6 - 8F_{8,2}$. From Definition 6.1,

$$\Lambda(0, A) = 0, \qquad \Lambda(1, A) = 12A(A^2 + 1),$$
$$\Lambda(2, A) = 12(2A^3 - A^2 + 2), \qquad \Lambda(3, A) = 36(A^3 - A^2 - 49A + 1).$$

The middle two are strictly positive for $A > 0$: this is clear for $\Lambda(1, \cdot)$, and for the other one from the identity $16(2A^3 - A^2 + 2) = 8A(2A - 1)^2 + (4A - 1)^2 + 31$. Applying Λ to the four equations kills any $s = 0$ orbit and leaves a strictly positive contribution from any $s \in \{1, 2\}$ orbit, so the pairs $(0, 2)$, $(1, 2)$ and $(2, 2)$ are impossible. Next, $F_6(0, A) = 3A^4 > 0$ and $F_6(3, A) = 3(A + 3)\big(A\left(A^2 - 15A + 75\right) + 3\big) > 0$ for $A > 0$, since $A^2 - 15A + 75 = (A - \frac{15}{2})^2 + \frac{75}{4} > 0$; hence the f_6 equation alone kills $(0, 0)$, $(0, 3)$ and $(3, 3)$. Finally $F_{8,2}(0, A) = 0$ and $F_{8,2}(1, A) = 3A^2 > 0$, so the $f_{8,2}$ equation kills $(0, 1)$ and $(1, 1)$.

The case $(1, 3)$. Write $x = A_1$, $y = A_2$, $p = w_1$, $q = w_2$. The four polynomials at $s = 1$ are palindromic of degree at most 4 and hence satisfy the linear relation $4F_4 - 7F_6 + 3F_{8,1} - 132F_{8,2} \equiv 0$. Applying that functional to the four equations annihilates the $s = 1$ orbit and leaves $2016\, q\, (y^2 - 10y + 1) = 0$, so $y^2 - 10y + 1 = 0$. Modulo that relation the $f_{8,2}$ equation reads $px^2 = 24qy$; substituting it into the f_6 equation gives $p(3x^4 - 12x^3 + 10x^2 - 12x + 3) = 0$, hence $3x^4 - 12x^3 + 10x^2 - 12x + 3 = 0$; and the f_4 equation gives $y(3x^4 + 70x^2 + 3) = 12x^2$, which, reduced by the previous quartic, becomes $y(x^2 + 5x + 1) = x$. Feeding this into $y^2 - 10y + 1 = 0$ yields $x^4 - 22x^2 + 1 = 0$, and three times that minus the earlier quartic gives $4x(3x^2 - 19x + 3) = 0$, so $3x^2 - 19x + 3 = 0$. Combining once more, $27(x^4 - 22x^2 + 1) - (9x^2 + 57x + 154)(3x^2 - 19x + 3)$ is $2755x - 435$, so $x = 3/19$; but then $3x^2 - 19x + 3 = 27/361 \neq 0$, a contradiction.

The case $(2, 3)$. Two members of the pencil $4F_4 - 7F_6 + 3F_{8,1} + dF_{8,2}$ are quadratic on *both* curves: at $d = -20$ it equals $336(x - 1)^2$ at $s = 2$ and $-8064y$ at $s = 3$, and at $d = 92$ it equals $-672(4x - 1)$ and $-2016(y - 1)^2$. So the four equations imply

$$p(x - 1)^2 = 24qy \qquad \text{and} \qquad p(4x - 1) + 3q(y - 1)^2 = 0.$$

The first forces $(x - 1)^2 > 0$, since $q, y > 0$. Eliminating p and q between the two gives $8y(4x - 1) + (y - 1)^2(x - 1)^2 = 0$; combining this with the image of Λ and using $y^3 - y^2 - 49y + 1 = (y + 1)(y - 1)^2 - 48y$ gives $y(4x - 1) = (x - 1)^2(2x - 3)$, and the same manipulation on the f_4 equation, using $x^4 - 12x^2 - 16x = (x^2 + 2x - 9)(x - 1)^2 - 9(4x - 1)$ and $y^4 - 30y^2 - 72y - 27 = (y^2 + 2y - 27)(y - 1)^2 - 128y$, gives $y(2x - 3) = 3x^2 + 2x - 37$. The last two together eliminate y:

$$(x - 1)^2(2x - 3)^2 = (3x^2 + 2x - 37)(4x - 1), \qquad \text{i.e.} \qquad x^4 - 8x^3 + 8x^2 + 30x - 7 = 0.$$

Separately, eliminating p between the two displayed relations and the previous ones gives $(y-1)^2 = 24-16x$, while $(y-1)(2x-3) = 3x^2-34$; squaring the latter and substituting the former gives $9x^4+64x^3-492x^2+432x+940 = 0$. Subtracting nine times the quartic above leaves

$$136x^3 - 564x^2 + 162x + 1003 = 0,$$

which has no root with $x \geq 0$, because $25(136x^3-564x^2+162x+1003) = 136x(5x-13)^2+(3580x^2-18934x+25075)$ and $3580(3580x^2-18934x+25075) = (3580x-9467)^2+144411$. Contradiction. □

Remark 6.6 (what is new here). Proposition 5.5 of [14] shows that a weighted 9-design of two-orbit type exists if and only if one of five cases (i)–(v) holds, distinguished by how many of the values $f_4(v_{a_i,s_i})$, $f_6(v_{a_i,s_i})$, $f_{8,1}(v_{a_i,s_i})$ vanish, and its Theorem 5.7 proves that cases (iii), (iv) and (v) do not occur, for every $n \geq 3$. Those three cases are therefore already published, and the proof above reproves them at $n=4$ only incidentally, by a different route. The content of Theorem 6.4 that is not in [14] is cases (i) and (ii) at $n=4$, which that paper leaves open. What it records instead is an expectation, printed after its Table 3: "*What is remarkable here is that all these examples belong to Case (ii), which implies that Case (i) may not occur even for $n \geq 4$.*" Remark 6.7 concerns that sentence.

Remark 6.7 (on Table 3 of [14]; outside the formal development). The two $\mathbb{S}^2$ examples of [14, Table 3] belong to case (i), not to case (ii). Case (ii) requires $f_4(v_{a_1,s_1}) = f_4(v_{a_2,s_2}) = 0$; at the parameters printed in that table the two values of $\tilde{f}_4$ are $+0.5525$ and -2.2784 for the first example and $+137.38$ and -1.8852 for the second — nonzero and of opposite sign, which is exactly case (i). Consistently, the value $A_2 = a_2^2 = 0.221229$ read off the table is a root of the quartic $5A^4-44A^3+78A^2-92A+17$ printed in [14] for that example, and is not a root of A^2-6A-1, which is $\tilde{f}_4(v_{a,2})$ at $n=3$. This does not affect [14, Theorem 5.7], which is about cases (iii)–(v); it removes the ground from the inference quoted in Remark 6.6, and it makes the $n=4$ question sharper rather than weaker. The evidence is numerical: the two printed quartics of [14] for its $n=3$ table were reproduced exactly by two independent exact routes, a lexicographic Gröbner basis over $\mathbb{Q}$ and a resultant computation, and the table's weights were recovered from its printed parameters to all six printed digits. None of this is part of the formal development, and no statement of this paper depends on it; it is recorded because it bears on what Theorem 6.4 adds.

Controls. The first control is the one that keeps Theorem 6.4 from being vacuous: the same four equations *do* have positive solutions once four orbits are allowed.

Theorem 6.8. *With $(s,A) = (2,4)$, $(0,1)$, $(1,1)$, $(3,1)$ and weights $w = \frac{1}{1920}$, $\frac{1}{15}$, $\frac{1}{128}$, $\frac{1}{1920}$ respectively, all four sums*

$$\tfrac{1}{1920}F_j(2,4) + \tfrac{1}{15}F_j(0,1) + \tfrac{1}{128}F_j(1,1) + \tfrac{1}{1920}F_j(3,1), \qquad j = 4,\ 6,\ (8,1),\ (8,2),$$

vanish.

Proof. Four exact rational evaluations of Definition 6.1. □

Remark 6.9. These four orbits and weights are Schur's classical weighted 11-design on $\mathbb{S}^3$, printed as [14, Example 2.5] and, in generalized corner-vector form, as its Example 2.22: the orbits are $v_{2,2}^{B_4}$, $v_{1,0}^{B_4}$, $v_{1,1}^{B_4}$, $v_{1,3}^{B_4}$, of sizes 96, 8, 24, 16, with the weights of Proposition 6.14(ii) halved; the numbers above are $w_i = W_i|\mathrm{orbit}_i|/(A_i+s_i)^4$. So Theorem 6.4 is a statement about a system that is solvable in general, and its "two orbits" cannot be raised to "four". We reproduced the configuration in exact rational arithmetic before formalising it, checking all 1365 monomials of degree at most 11 against the exact sphere moments: the defect is 0 at every degree up to 11, and $2.7 \cdot 10^{-4}$ at degree 12, so it is an 11-design and not a 12-design. That monomial check is a computation outside the formal development; Theorem 6.8 is the machine-checked part, and it certifies exactly the four equations of Theorem 6.4.

Proposition 6.10. *There exist $A_2 > 0$ and $w_1, w_2 > 0$ with*

$$w_1 F_4(1,1) + w_2 F_4(3, A_2) = 0 \qquad \text{and} \qquad w_1 F_6(1,1) + w_2 F_6(3, A_2) = 0;$$

explicitly $A_2 = 3 + 2\sqrt{3}$, $w_1 = 1728 + 1152\sqrt{3}$ and $w_2 = 48$.

Proof. At $A_2 = 3 + 2\sqrt{3}$ both fourth-degree values vanish, $F_4(1,1) = 0$ and $F_4(3, A_2) = 0$, so the first equation holds for any weights; and $F_6(1,1) = -48$ while $F_6(3, A_2) = 1728 + 1152\sqrt{3} > 0$, so the two sixth-degree values have opposite signs and the displayed positive weights solve the second equation. The verification is an exact computation in $\mathbb{Q}(\sqrt{3})$. □

Remark 6.11. This is the negative control against a too-small reading of Theorem 6.4: the degree 9 there is sharp. For $n = 4$ the invariant harmonics of degree at most 7 are f_4 and f_6 alone, so dropping the two degree-8 equations is exactly asking for a 7-design; and by the same reduction as in Corollary 6.5, Proposition 6.10 exhibits a two-orbit weighted 7-design on $\mathbb{S}^3$, supported on $v_{1,1}^{B_4}$ and $v_{a,3}^{B_4}$ with $a^2 = 3 + 2\sqrt{3}$. This is not in conflict with Bajnok's bound $t \leq 7$ [1, Proposition 15]: that bound is for unions of the orbits $v_k^{B_n}$, all with $a = 1$, whereas here one of the two orbits is proper.

Proposition 6.12. *For every $s \in \{0, 1, 2, 3\}$ and every $A > 0$, the two values $F_4(s, A)$ and $F_{8,2}(s, A)$ do not both vanish.*

Proof. For $s = 0$, $F_4 = 3A^4 > 0$; for $s = 1$, $F_{8,2} = 3A^2 > 0$. For $s = 2$, $F_{8,2} = 0$ says $A^2 + 6A - 1 = 0$, and reducing F_4 modulo that relation leaves $75 - 516A = 0$, i.e. $A = 25/172$, which does not satisfy $A^2 + 6A - 1 = 0$. For $s = 3$, $F_{8,2} = 0$ says $A^2 - 6A + 1 = 0$, while $F_4(3, A) = 3(A + 3)^2(A^2 - 6A - 3)$ and $(A + 3)^2 > 0$ force $A^2 - 6A - 3 = 0$; subtracting, $4 = 0$. □

Remark 6.13. This is the non-vacuity control for the hypotheses of Theorem 6.4: the two-orbit system does not degenerate into a one-orbit one, because no single orbit satisfies even two of the four conditions. The corresponding design statement — that no single orbit of a generalized corner vector is a weighted 9-design — is [14, Proposition 4.6], proved there for all $n \geq 3$ by a Gröbner basis computation; the two-quadratic elimination above is only the $n = 4$ instance, by a different route.

Two errata in the source. Both were found while reproducing the printed numbers of [14] before formalising anything, and neither affects any statement above. We record them because a reader working from the printed displays would otherwise be misled.

Proposition 6.14. (1) *The expression printed in* [14, Lemma 2.17] *for $\tilde{f}_{8,2}(v_{a,s})$, namely*

$$a^4 + \frac{s-1}{2} - 3\big(a^4 + 2a^2 + s - 2\big)\frac{s-1}{n-2} + \frac{9}{2}\big(4a^2 + s - 3\big)\frac{(s-1)(s-2)}{(n-2)(n-3)} \qquad (n \geq 4),$$

evaluates at $n = 4$, $s = 0$, $a = 1$ to $13/2$. But $v_{a,0} = e_1$ and every monomial of $f_{8,2}$ involves at least two coordinates, so $f_{8,2}(e_1) = 0$ and the true value is 0.

(2) *The weights printed for Schur's formula in* [14, Example 2.5] *have total mass 2:*

$$96 \cdot \tfrac{9}{640} + 8 \cdot \tfrac{1}{60} + 24 \cdot \tfrac{1}{96} + 16 \cdot \tfrac{1}{60} = 2,$$

against the orbit sizes $96, 8, 24, 16$, whereas the normalisation $(1/|\mathbb{S}^3|)\int 1\, d\rho = 1$ requires mass 1.

Proof. Two exact rational evaluations, together with the expansion showing $f_{8,2}(1,0,0,0) = 0$. □

Remark 6.15. In (i) the printed expression is off by an overall factor s: multiplying it by s reproduces direct evaluation of $\tilde{f}_{8,2}(v_{a,s})$ for $n = 4$, $s = 0, 1, 2, 3$ and for $n = 5$, $s = 0, \ldots, 4$. (That check across two dimensions is a computation outside the formal development; Proposition 6.14(i) is the machine-checked part, and Theorem 6.2 carries the corrected values at $n = 4$.) We do *not* claim that this affects §5.3 of [14]: every use of $\tilde{f}_{8,2}$ that we found there is through a difference at a common $s > 0$, in which an overall factor s cancels, and we did not re-derive its g_3 display. In (ii) only the ratios of the weights enter the harmonic conditions; halving all four gives an exact 11-design, and that halved configuration is the one used in Theorem 6.8.

7. Open questions

(1) **The least denominator of a 37-point design.** Subject to the odd-count cut, Remark 4.3 leaves it in $\{84, 96, 108, 120, 132, 144\}$. Closing the gap is an exhaustive computation whose measured cost grows like $\binom{h}{8}\binom{h-1}{2}$ table lookups in the half-denominator h; on the machine used here $h = 42$ would still be affordable and $h \geq 54$ would not be, absent a better split of the meet-in-the-middle table. Neither was run.

(2) **Enlarging the odd spectrum.** The union principle [9, Proposition 2.6] together with the one-parameter family of 6-point designs, whose members are displayed in [9, Lemma 6.3] and shown to be designs in the proof of [9, Proposition 6.1], should combine with Theorem 4.1 to give antipodal rational 5-designs with $37+6k$ points for every $k \geq 0$, which with [9, Theorem 8.2] would enlarge the known odd spectrum. We neither carry this out nor claim it: the distinctness half of [9, Lemma 6.3] is asymptotic in its parameter, and a usable version needs an explicit range for that parameter.

(3) **A sharper 3-adic obstruction.** Theorem 3.4 uses only $\sum(m_i^2 + m_i^4) \bmod 9$. Since (10) fails on $\mathbb{Z}/9\mathbb{Z}$, a modulus 27 or 81 statement would need a finer bookkeeping by residue class rather than the one-line collapse used here. With the 37-point cell settled, the payoff has moved to the cells listed next.

(4) **Degree seven.** [9, Problem 8.5] asks whether antipodal rational 7-designs with $2N$ points exist beyond the range $1 \leq N \leq 16$ that its Theorem 8.4 excludes. The system adds $\sum x_i^6 = 5N/16$. Since $x^6 = x^4 = x^2$ on $\mathbb{Z}/3\mathbb{Z}$, the collapse of Section 3 plausibly applies to that system as well; we have not carried this out, and state it only as the natural first thing to try.

(5) **The Hermite cell $N = 16$.** [9, Theorem 8.6] rules out antipodal rational 5-designs with $2N$ points for the Hermite measure for $1 \leq N \leq 15$ except $N = 14$, for which that paper supplies the configuration re-certified in Theorem 5.3 above; $N = 16$ is untouched, and by Corollary 3.3(iii) any solution there has $3 \mid h$. A single search probe at half-denominator 30 ran for 21 minutes and returned nothing; that is a bounded negative at one denominator, not evidence about the cell. [9, Problem 8.7] asks for an infinite family in this measure.

(6) **Two-orbit 9-designs on $\mathbb{S}^2$.** For $n = 3$ they exist: [14, Table 3] lists two of them. That source calls the list a classification ("*Such 9-designs can be classified by two examples listed in Table 3*"), but attaches no numbered theorem to the sentence and gives the parameters numerically to six significant digits, so we take from it the existence only and leave the completeness of the list where the source puts it. Existence is all that the contrast with Corollary 6.5 needs — such designs exist in dimension three, and by that corollary not in dimension four — one half of it the source's and the other half ours. We did not certify the two $n = 3$ configurations here, because their parameters are algebraic of degree 4 and a formal witness needs arithmetic in the corresponding quartic field. Doing so would put the two dimensions into a single formal statement.

(7) **Two orbits in dimension $n \geq 5$.** The machinery of Section 6 carries over verbatim, with $s \in \{0, \ldots, n-1\}$ and hence $\binom{n+1}{2}$ unordered pairs, once the analogue of the linear relation $4F_4 - 7F_6 + 3F_{8,1} - 132F_{8,2} \equiv 0$ is worked out at each n. Beyond [14, Theorem 5.7], which handles cases (iii)–(v) for all n, nothing seems to be known. We have not run it.

(8) **Three proper orbits at degree 11.** [14, §6] exhibits a weighted 11-design on $\mathbb{S}^3$ with three proper orbits, given to six digits. An exact algebraic witness for it would be a three-unknown analogue of Theorem 6.8; we did not attempt one. The only numbered open problem of that section, [14, Problem 6.1], asks whether a weighted 15-design on $\mathbb{S}^3$ with more than five

proper orbits exists; that is a system in twelve unknowns whose degree-$\leq$ 15 conditions bring in the invariant harmonics of degrees 10, 12 and 14, and we say nothing about it.

8. Verification

Every statement labelled a theorem or a proposition above has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [7, 8]; repository reference to be added at submission. The development is three modules, one carrying the design predicate of Definition 2.2 with the structural lemmas for the configuration $\{0\} \cup \{\pm m/D\}$, the linearity statement Theorem 2.3 and the five certified configurations, the second carrying Section 3, and the third, independent of the other two, carrying Section 6. `#print axioms` was run on every declaration to which a statement of this paper is pinned; each returns an axiom set contained in {`propext`, `Classical.choice`, `Quot.sound`} — no `sorry`, and in particular no compiled evaluation: there is no use of `native_decide`, no external solver and no stored certificate anywhere in the development. All arithmetic is exact arithmetic in $\mathbb{Q}$ and $\mathbb{Z}$ inside the kernel, or in $\mathbb{R}$ with $\sqrt{3}$ handled symbolically in Proposition 6.10; the power sums of Theorems 4.1 and 5.1–5.3 are computed there rather than recalled from the search or from [9]. In Propositions 4.5 and 5.5(i)–(ii) the formal statements are the inequalities alone; the rational numbers displayed alongside them are the values to which the kernel reduces those power sums while proving the inequalities, and are quoted here for the reader.

The two corollaries of Section 3 are not themselves declarations: Corollary 3.3 substitutes explicit values of (c_2, c_4) into two verified theorems and checks the divisibility of one integer, and only its case $(c_2, c_4) = (37, 111)$ is itself a declaration, namely Theorem 3.5; Corollary 3.6 adds the two elementary steps named in its proof. Corollary 6.5 is in the same position with respect to Theorem 6.4: what is machine-checked is the inconsistency of the four polynomial equations, and the reduction of (7) to them — Sobolev's theorem, the Molien series and the choice of a basis of invariant harmonics, all quoted from [14] — is not formalised. This is why Theorem 6.4 is stated in the algebraic form. The same applies to Remark 6.11, which reads Proposition 6.10 back as a design.

Six things lie outside the formal guarantee, and we name them. First, the moment sequences (8) are *defined* in the development by their closed forms; that these are the moments of the Chebyshev and Hermite measures is classical, is used only to interpret the results, and is not formalised. Second, Lemma 2.4 and the minimality argument in Corollary 3.6 — the passage from a rational design to the integer system (5) — are elementary and are not part of the development, which states Theorems 3.2–3.5 directly about integer solutions of (5); this is why those theorems are stated in that form. Third, the search programs of Remark 4.2 and the measurement of Remark 4.3 are ordinary C programs, not machine-checked; no theorem depends on them, and Remark 4.3 is labelled as a measurement rather than a result. Fourth, the reduction quoted from [14] that turns (7) into (12), as just described. Fifth, the numerical and exact-arithmetic reproductions reported in Remarks 6.7, 6.9 and 6.15 — the two quartics and the weights of [14, Table 3], the 1365-monomial check of Schur's configuration, and the two-dimensional check of the corrected $\tilde{f}_{8,2}$ — are ordinary computer algebra, run before formalising and reported as evidence, not as theorems. Sixth, the literature evidence described in Section 1 is a record of searches run on 2026-08-29 and 2026-08-30, not a proof of absence.

Of the exhaustive computations that do enter the proofs, all are tiny and all are named where they occur: the identity $x^4 = x^2$ on $\mathbb{Z}/3\mathbb{Z}$ and on $\mathbb{Z}/4\mathbb{Z}$ (checks over 3 and 4 residues), the unit computation $u^2 = u^4 = 1$ in each of those rings, the pointwise identity $m^2 + m^4 \equiv 0 \text{ or } 2 \pmod 9$ (a check over the 9 residues modulo 9), and the case distinction over the ten unordered pairs (s_1, s_2) in Theorem 6.4 and the four values of s in Propositions 6.3 and 6.12. Everything else is a finite exact evaluation on an explicit list of at most 37 rationals, or a polynomial identity in at most four real variables.

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