THE THIRD-MOMENT RADEMACHER STABILITY CONSTANT IS EXACT ON THE FLAT FAMILY, ON THE SPIKE REGION AND ON THE FOURTH-MOMENT REGION $\sum_i a_i^4 \geq \frac{1}{2}$

KOREA SUPERINTELLIGENCE LABS

Abstract. Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher signs, let $a \in \mathbb{R}^n$ satisfy $\sum_i a_i^2 = 1$, write $\overline{S}_n = n^{-1/2}(\varepsilon_1 + \cdots + \varepsilon_n)$ and $\Delta_n(a) = \sum_i (a_i^2 - \frac{1}{n})^2$. Gao and Qian introduce the optimal dimension-free third-moment stability constant $C_3^{\mathrm{opt}} = \inf\big(\mathbb{E}|\overline{S}_n|^3 - \mathbb{E}|\sum_i a_i\varepsilon_i|^3\big)/\Delta_n(a)$, prove $\frac{3(5\sqrt{2}-7)}{16} \leq C_3^{\mathrm{opt}} \leq 5\sqrt{3} - 6\sqrt{2}$ — a bracket of width a factor just over 13 — and conjecture that the upper end $C_\star := 5\sqrt{3} - 6\sqrt{2} = 0.1749726636\dots$ is the truth. We prove their conjectured inequality on three explicit regions that hold at *every* dimension. First, on the flat coefficient vectors: for all integers $1 \leq k < n$, $\mathbb{E}|\overline{S}_n|^3 - \mathbb{E}|\overline{S}_k|^3 \geq C_\star(\frac{1}{k} - \frac{1}{n})$, with strict inequality except at $(k, n) = (2, 3)$, so $C_\star$ is the unique extremal chord slope of the sequence $\mathbb{E}|\overline{S}_n|^3$. Second, on the vectors in which one coordinate carries at least the ℓ^1 mass of all the others: there the conjectured inequality holds for every $n \geq 2$, and the restricted infimum is *exactly* $C_\star$, attained at Gao and Qian's own dimension-three vector $(a_1^2, a_2^2, a_3^2) = (\frac{1}{2}, \frac{1}{2}, 0)$ — so on that region the factor-13 bracket collapses to a point. In dimension four the sharp constant on that region is the new exact value $6 - 4\sqrt{2} = 0.3431457505\dots$. Third, on the fourth-moment region $\sum_i a_i^4 \geq \frac{1}{2}$, which is provably *not* contained in the second: there too the conjectured inequality holds at every $n \geq 2$ with the same constant, the restricted infimum is again exactly $C_\star$, and in dimension four it is again exactly $6 - 4\sqrt{2}$. That third region contains the conjectured extremiser, so in dimension four the value of the optimum is in doubt only on the complementary set $\sum_i a_i^4 < \frac{1}{2}$. The engine is a closed form for $\mathbb{E}|S_n|^3$ coming from two telescoping binomial identities, after which the whole flat family reduces to one sequence of consecutive slopes; on the spike region the extremiser is exhibited by an exact quadratic factorisation rather than by calculus, and on the fourth-moment region a single Cauchy–Schwarz step against $\mathbb{E}S^4 = 3 - 2\sum_i a_i^4$ reduces everything to the same chord bound at $k = 2$. Nothing here is an exhaustive computation: C_3^{opt} is an infimum over infinitely many dimensions, and all three regions are settled at all of them at once. Every theorem below is machine-checked in Lean 4.

1. Introduction

1.1. The stability quantity. Let $\varepsilon_1, \varepsilon_2, \dots$ be independent Rademacher signs, $\Pr(\varepsilon_i = \pm 1) = \frac{1}{2}$. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ on the unit sphere, $\sum_{i=1}^n a_i^2 = 1$, put

$$S(a) = \sum_{i=1}^n a_i \varepsilon_i, \qquad \overline{S}_n = \frac{\varepsilon_1 + \cdots + \varepsilon_n}{\sqrt{n}} = S\big(n^{-1/2}, \dots, n^{-1/2}\big),$$

and let the *fourth-order mass* and the *flatness defect* be

$$q(a) = \sum_{i=1}^n a_i^4, \qquad \Delta_n(a) = \sum_{i=1}^n \Big(a_i^2 - \frac{1}{n}\Big)^2 = q(a) - \frac{1}{n}. \tag{1}$$

So Δ_n is the squared Euclidean distance from the vector of squared masses to the uniform one, and it vanishes exactly at the flat vector. Among all a on the sphere the third absolute moment $\mathbb{E}|S(a)|^3$ turns out to be largest at the flat vector, and the question is by how much it drops as a moves away: is the drop at least *quadratic* in the displacement, with a constant that does not deteriorate as n grows?

Gao and Qian [1] answer yes, and name the optimal constant. (Throughout, a numbered statement of another paper is cited together with its source, so that a bare number is one of ours.) Theorem 1.4 of

[1], on dimension-free stability at the third moment, states that for every $n \geq 1$ and every unit $a \in \mathbb{R}^n$,

$$\mathbb{E}\Big|\sum_{i=1}^{n} a_i\varepsilon_i\Big|^3 \;\leq\; \mathbb{E}|\overline{S}_n|^3 - \tfrac{1}{100}\,\Delta_n(a), \tag{2}$$

and it defines

$$C_3^{\mathrm{opt}} \;:=\; \inf_{\substack{n\geq 2,\ \sum_i a_i^2=1\\ \Delta_n(a)>0}} \frac{\mathbb{E}|\overline{S}_n|^3 - \mathbb{E}\big|\sum_i a_i\varepsilon_i\big|^3}{\Delta_n(a)}. \tag{3}$$

The same theorem proves

$$\frac{3(5\sqrt{2}-7)}{16} \;\leq\; C_3^{\mathrm{opt}} \;\leq\; 5\sqrt{3}-6\sqrt{2}, \qquad \text{i.e.} \qquad 0.0133252147\ldots \leq C_3^{\mathrm{opt}} \leq 0.1749726636\ldots, \tag{4}$$

settles dimension three completely — for every unit $a \in \mathbb{R}^3$, $\mathbb{E}|S(a)|^3 \leq \mathbb{E}|\overline{S}_3|^3 - (5\sqrt{3}-6\sqrt{2})\Delta_3(a)$ — and records that the upper bound of (4) is attained in dimension three at $(a_1^2, a_2^2, a_3^2) = (\frac{1}{2}, \frac{1}{2}, 0)$, up to signs and permutations. Remark 6.5 of [1], which follows that proof, says, verbatim:

> "The proof establishes dimension-free stability but does not determine the exact value of C_3^{opt}. The dimension-three vector above is a natural sharpness candidate. Any improvement of the constant 3/16 in [their Lemma 6.1] immediately improves the lower bound obtained by this method."

— the constant 3/16 is the small-ball estimate that Lemma 6.1 of [1] quotes from Dzindzalieta and Götze, not one of their own — and Conjecture 6.6 of [1], which closes it, reads, verbatim:

> "We conjecture that, for every $n \geq 1$ and every $a \in \mathbb{R}^n$ with $\sum_i a_i^2 = 1$, $\mathbb{E}\big|\sum_{i=1}^n a_i\varepsilon_i\big|^3 \leq \mathbb{E}|\overline{S}_n|^3 - (5\sqrt{3}-6\sqrt{2})\,\Delta_n(a)$."

Throughout we write

$$C_\star := 5\sqrt{3}-6\sqrt{2} = 0.1749726636\ldots, \qquad F_n := \mathbb{E}|\overline{S}_n|^3,$$

and we call the displayed inequality *the conjecture*. The bracket (4) has width a factor $13.13\ldots$, and the universal constant $\frac{1}{100}$ of (2) is a factor $17.49\ldots$ below the conjectured value.

The stability question itself is Jakimiuk's [2], and it is worth being precise about which half of it he settles. He proves two estimates, both for every $p \geq 3$ and both with a constant depending only on p: *Gaussian* stability, his Theorem 1,

$$\mathbb{E}|S(a)|^p \;\leq\; \mathbb{E}|G|^p - c_p \sum_i a_i^4 \qquad (p \geq 3,\ G \text{ standard Gaussian}),$$

and *diagonal* stability, his Theorem 2, which is the quadratic deficit below the flat value,

$$\mathbb{E}|S(a)|^p \;\leq\; \mathbb{E}|\overline{S}_n|^p - c_p\,\Delta_n(a) \qquad \text{— but only for } p > 3.$$

At the critical exponent $p = 3$ his Theorem 3 gives instead a different deficit which, as he says there, "is not comparable with the quantity $\sum_{i=1}^n \big(a_i^2 - \frac{1}{n}\big)^2$", and his Conjecture 2 asks only for the *existence* of a universal constant $C_3 > 0$ with $\mathbb{E}|S(a)|^3 \leq \mathbb{E}|\overline{S}_n|^3 - C_3\Delta_n(a)$, naming no candidate value. It is that conjecture which [1] settles, with the constant $\frac{1}{100}$ of (2). Numbers of statements of [2] are those of arXiv:2503.07001v2. The constant C_3^{opt}, and with it the candidate $C_\star$, is introduced for the first time in [1], and is open there.

For context, the search for optimal constants in Khintchine-type inequalities goes back to Khintchine [3] and, for the sharp L_p constants, to Szarek [5] and Haagerup [4], with the later distribution-function approach of Nazarov and Podkorytov [6]; the questions here are one order finer than those, being about the *rate* at which the extremal configuration is approached rather than about the extremal value itself.

1.2. **Three regions, and what we prove.** The conjecture ranges over all dimensions at once, so it is not a finite computation (§6). What we do is settle it on three explicit regions of coefficient vectors, each at *every* dimension.

The first region is the *flat family*: for $1 \le k \le n$ let $e^{(n,k)} \in \mathbb{R}^n$ be the vector with k coordinates equal to $k^{-1/2}$ and the remaining $n-k$ equal to 0. Then $\Delta_n(e^{(n,k)}) = \frac{1}{k} - \frac{1}{n}$ and $\mathbb{E}|S(e^{(n,k)})|^3 = F_k$, so on this family the quotient in (3) is the chord slope of the sequence F plotted against $1/n$.

Theorem 1.1 (flat family; Theorems 4.3 and 4.4)**.** *For all integers $1 \le k < n$,*

$$F_n - F_k \;\ge\; C_\star\Big(\frac{1}{k} - \frac{1}{n}\Big),$$

and the inequality is strict except at $(k,n) = (2,3)$, where it is an equality.

So $C_\star$ is the unique extremal chord slope of the third-moment sequence, and the dimension-three vector singled out by [1] as "a natural sharpness candidate" is, inside the flat family, the *only* candidate. A numerical check of the inequality of Theorem 1.1 for $n \le 60$ was available before this work; here it is a theorem for every pair of dimensions.

The second region is the *spike region*: call a a *spike vector* if some coordinate carries at least the ℓ^1 mass of all the others,

$$\sum_{j\ne i_0} |a_j| \;\le\; |a_{i_0}| \qquad \text{for some } i_0. \tag{5}$$

Every vector supported on at most two coordinates is a spike vector, at every ambient dimension; in particular every unit vector of $\mathbb{R}^2$ is one, and so is the extremiser of [1].

Theorem 1.2 (spike region; Theorems 5.5 and 5.8)**.** *For every $n \ge 2$ and every unit spike vector $a \in \mathbb{R}^n$,*

$$\mathbb{E}|S(a)|^3 \;\le\; \mathbb{E}|\overline{S}_n|^3 - C_\star\,\Delta_n(a).$$

Moreover the infimum in (3) *restricted to spike vectors equals exactly $C_\star$ and is attained, at $(a_1^2, a_2^2, a_3^2) = (\frac{1}{2}, \frac{1}{2}, 0)$ in dimension three.*

On the spike region, therefore, the factor-13 bracket (4) collapses to a point. Let us be exact about what is new here. At $n = 3$ Theorem 1.2 is contained in [1]: Theorem 1.4 of [1] settles the whole of $\mathbb{R}^3$ with the constant $C_\star$, and the spike case is the first of the two branches in the proof of its Lemma 6.4. Consequently the equality statement — the value $C_\star$ at $(\frac{1}{2}, \frac{1}{2}, 0)$ — is theirs as well; only its kernel certificate is ours. The new content of Theorem 1.2 is $n \ge 4$ and the uniformity in n: the general-n constant of [1] is the $\frac{1}{100}$ of (2), a factor $17.49\ldots$ smaller. What is not available to their argument, and is supplied here, is the uniform chord bound of Theorem 1.1 at $k = 2$.

Restricting to a fixed dimension sharpens the constant, and in dimension four the sharp value is an exact algebraic number that does not appear in [1].

Theorem 1.3 (Theorem 5.9)**.** *For every unit spike vector $a \in \mathbb{R}^4$, $\mathbb{E}|S(a)|^3 \le \mathbb{E}|\overline{S}_4|^3 - (6-4\sqrt{2})\Delta_4(a)$, and equality holds at $a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$. So $6 - 4\sqrt{2} = 0.3431457505\ldots$ is the sharp constant on the spike region of $\mathbb{R}^4$.*

The third region is a condition on the fourth-order mass $q(a) = \sum_i a_i^4$ rather than on the ℓ^1 geometry of a, and the argument on it is different in kind: one Cauchy–Schwarz step against the fourth moment. We call

$$\Big\{a \in \mathbb{R}^n :\ \textstyle\sum_i a_i^2 = 1,\ q(a) \ge \frac{1}{2}\Big\}$$

the *fourth-moment region*. It contains the extremiser of [1], which has $q = \frac{1}{2}$ exactly, and it is not contained in the spike region.

Theorem 1.4 (fourth-moment region; Theorems 7.3, 7.4 and 7.5)**.** *For every $n \ge 2$ and every unit $a \in \mathbb{R}^n$ with $q(a) \ge \frac{1}{2}$,*

$$\mathbb{E}|S(a)|^3 \;\le\; \mathbb{E}|\overline{S}_n|^3 - C_\star\,\Delta_n(a).$$

Moreover the infimum in (3) *restricted to that region equals exactly $C_\star$, attained at the same vector* $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$*; and in dimension four the sharp constant on the region is again* $6 - 4\sqrt{2}$.

The two regions of Theorems 1.2 and 1.4 really are different: the unit vector $b \in \mathbb{R}^4$ with $b_1^2 = \frac{7}{10}$ and $b_2^2 = b_3^2 = b_4^2 = \frac{1}{10}$ has $q(b) = \frac{13}{25} > \frac{1}{2}$ and fails the spike condition (5) at every coordinate (Proposition 7.6). Whether the containment holds the other way we do not know; numerically it does (Remark 7.9). As with Theorem 1.2, the case $n = 3$ of the statement above is contained in Theorem 1.4 of [1], which settles the whole of $\mathbb{R}^3$; the new content is $n \geq 4$ and the uniformity in n.

To summarise, with $C_\star = 5\sqrt{3} - 6\sqrt{2}$ and $q(a) = \sum_i a_i^4$:

region of the unit sphere of $\mathbb{R}^n$	the conjecture	restricted infimum
$\lvert a_i\rvert = k^{-1/2}$ on a k-set, 0 off it	Thm. 1.1, Cor. 4.8	$C_\star$ (Thm. 4.4)
$\sum_{j \neq i_0} \lvert a_j\rvert \leq \lvert a_{i_0}\rvert$ for some i_0	Thm. 1.2	$C_\star$ (Thm. 5.8)
$q(a) \geq \frac{1}{2}$	Thm. 1.4	$C_\star$ (Thm. 7.4)

In dimension four the last two regions both carry the sharp constant $6 - 4\sqrt{2}$ (Theorems 5.9 and 7.5), so the conjecture holds with that constant on their union.

Behind all of this is a closed form for the third absolute moment of the unnormalised Rademacher sum (Proposition 3.1), which turns every quantity above into central binomial coefficients and square roots of small integers, after which the arguments are comparisons of squares.

1.3. **What is not claimed.** Four limits, stated here rather than left to be inferred.

First, the conjecture itself remains open, and so does the value of C_3^{opt}. Nothing below improves the lower bound $\frac{3(5\sqrt{2}-7)}{16}$ of (4), which rests on the small-ball constant $\frac{3}{16}$ of Lemma 6.1 of [1]; a certified improvement of that constant would improve the bracket immediately, as [1] observes, and we do not touch it. What we prove is that the conjecture cannot fail on any of the three regions: if it fails, it fails at a vector with at least three nonzero coordinates, none of which dominates the others, and with $q < \frac{1}{2}$.

Second, on the reading of Theorem 1.1. It is proved, and stated, as an inequality between the two moment values F_n and F_k. Reading it literally as "the conjecture holds at every flat-on-k vector of $\mathbb{R}^n$" needs in addition that $\mathbb{E}|S(a)|^3$ is unchanged when a is padded with zero coordinates, so that $\mathbb{E}|S(e^{(n,k)})|^3 = F_k$. That invariance, and the companion invariance under permutations and sign changes, are Proposition 2.2; the literal reading is Corollary 4.8. (In the first version of this paper the invariance was stated as a caveat outside the verified development; it is now inside it.)

Third, the fixed-dimension optima are known exactly only on the three regions. Write Φ_n for the infimum of the quotient in (3) over the unit sphere of $\mathbb{R}^n$ alone. The source determines $\Phi_3 = C_\star$. We determine the analogous infimum over the spike region and over the fourth-moment region of $\mathbb{R}^n$ for every n, and in dimension four both are $6 - 4\sqrt{2}$; whether $\Phi_4 = 6 - 4\sqrt{2}$ — that is, whether the part of the sphere of $\mathbb{R}^4$ left over, where $q < \frac{1}{2}$ and no coordinate dominates, contains nothing smaller — is left open, and the exact algebraic values listed in Remark 6.1 for $4 \leq n \leq 12$ are recorded as measurements, not as theorems. The leftover part does not contain the conjectured extremiser — that vector lies in the region of Theorem 1.4 — but it does come arbitrarily close to it, which is exactly what makes a certificate there delicate (Remark 7.9).

Fourth, on provenance. The source [1] was re-fetched from the arXiv API on 3 September 2026 and is still at v1 with an empty journal-reference field; every quotation above was read out of the LaTeX e-print of that version, and the statement numbers used here are the ones that e-print compiles to. Jakimiuk's preprint [2] was read directly, to confirm that the constant C_3^{opt} and the candidate value $C_\star$ are not stated there. A search of a large local corpus of recent preprints for five LaTeX spellings of $5\sqrt{3} - 6\sqrt{2}$ and for the symbol C_3^{opt} returned exactly one paper, [1] itself; arXiv full-text queries for the pairing of Rademacher, Khintchine and stability returned [1] and [2] and nothing else; and the integer sequences of §3 were checked against OEIS. So the results below are reported as *not found* in the literature, which is not the same as reporting them as new: every ingredient — the closed form, the central binomial bound, the sign decomposition on a dominated coordinate — is classical, and the chord computation of §4 could well have been written down somewhere unindexed. What is certainly new is that these are theorems for all dimensions rather than tables for small ones.

Changes from version 1. This is the second version of this paper. No statement of the first version is withdrawn or weakened, and the numbering of every statement it contained is unchanged. Three things are new.

(i) A third region. Theorem 1.4 and §7 settle the conjecture, with its conjectured constant, on the fourth-moment region $q \geq \frac{1}{2}$ at every dimension, and determine the restricted infimum there exactly — $C_\star$ over all dimensions, $6-4\sqrt{2}$ in dimension four. The region is not contained in the spike region, and it contains the conjectured extremiser.

(ii) The padding caveat is discharged. The invariance of $\mathbb{E}|S(a)|^3$ under padding with zero coordinates, which the first version carried as a stated caveat outside the verified development, is now Proposition 2.2, together with the invariances under permutations and sign changes; Corollary 4.8 is the resulting literal reading of Theorem 1.1 as a statement about coefficient vectors of $\mathbb{R}^n$. Remark 4.5 has been rewritten accordingly.

(iii) One correction. The first version asserted, in its closing discussion, that no relaxation with rational coefficients can certify $\Phi_4 = 6-4\sqrt{2}$, because the bound is tight at a corner with irrational coordinates. That is too strong: the polynomial $12AB - A^2 - 4B^2$, with $A = F_4 - \mathbb{E}|S(a)|^3$ and $B = \Delta_4(a)$, has rational coefficients and its nonnegativity implies $A \geq (6-4\sqrt{2})B$ wherever $A/B < 6+4\sqrt{2}$. The price of that route is degree 8 instead of degree 5; both routes are counted in Remark 7.9.

2. The objects

Definition 2.1. For $n \geq 0$ and $a \in \mathbb{R}^n$ put

$$T_3(a) \;=\; \frac{1}{2^n} \sum_{s \subseteq \{1,\dots,n\}} \Big| \sum_{i \notin s} a_i - \sum_{i \in s} a_i \Big|^3,$$

the exact average over all 2^n sign patterns; thus $T_3(a) = \mathbb{E}|S(a)|^3$. Write $F_n = T_3(e^{(n,n)}) = \mathbb{E}|\overline{S}_n|^3$ for the value at the flat vector, and $C_\star = 5\sqrt{3} - 6\sqrt{2}$.

Taking the expectation as a finite average is not a restriction: the underlying probability space is $\{\pm 1\}^n$ with the uniform measure, and it is the form in which every statement below is verified. On the unit sphere the second expression for Δ_n in (1) follows by expanding the square, which is the identity $\Delta_n(a) = q(a) - \frac{1}{n}$ used throughout.

Two elementary facts about T_3 are used repeatedly and are part of the formal development: $T_3(-a) = T_3(a)$ and $\Delta_n(-a) = \Delta_n(a)$, which let us assume a distinguished coordinate nonnegative; and the second-moment identity $\sum_s \big(\sum_{i \notin s} a_i - \sum_{i \in s} a_i\big)^2 = 2^n \sum_i a_i^2$, which is the usual orthogonality of the signs written as a sum over sign patterns.

Three further invariances of T_3 are used below, and it is worth isolating them, because one of them is what turns an inequality between two numbers F_n and F_k into a statement about coefficient vectors. For $k \leq n$ write $\iota a \in \mathbb{R}^n$ for the vector $a \in \mathbb{R}^k$ padded with $n-k$ zero coordinates.

Proposition 2.2 (invariances of T_3). *Let $0 \leq k \leq n$ and $a \in \mathbb{R}^k$. Then $T_3(\iota a) = T_3(a)$, and $\sum_{i=1}^n G(\iota a_i) = \sum_{j=1}^k G(a_j)$ for every $G : \mathbb{R} \to \mathbb{R}$ with $G(0) = 0$; in particular padding changes neither $\sum_i a_i^2$ nor $q(a)$. Moreover, for $a \in \mathbb{R}^n$, T_3 is unchanged when the coordinates are permuted, and when the signs of an arbitrary subset of the coordinates are reversed; in particular $T_3(|a_1|, \dots, |a_n|) = T_3(a)$. The same two invariances hold for Δ_n, which depends on the coordinates only through their squares.*

Proof sketch. For the padding it is enough to add one zero coordinate. Sending a sign pattern $s \subseteq \{1, \dots, n+1\}$ to the pair consisting of its trace on $\{1, \dots, n\}$ and the sign it gives the new coordinate is a bijection onto pairs, under which the signed sum gains the term $\pm a_{n+1}$; at $a_{n+1} = 0$ that term vanishes, so the two members of each pair contribute equally and the 2^{n+1}-term average of Definition 2.1 equals the 2^n-term one. Iterating $n-k$ times gives the general statement, and the same induction gives the statement about $\sum_i G(\iota a_i)$, the new coordinate contributing $G(0) = 0$. For a permutation σ of the coordinates the sign patterns are permuted by σ itself and the signed sums match term by term; for a sign reversal on a set U they are permuted by the symmetric difference with U, an involution. In each case the average of Definition 2.1 is over the same multiset of values, and Δ_n, being a symmetric function of the squares, is invariant for the same reason. $\square$

Since $\Delta_n(a) = q(a) - \frac{1}{n}$ on the unit sphere, the padded vector ιa of a unit $a \in \mathbb{R}^k$ has $\Delta_n(\iota a) = q(a) - \frac{1}{n}$: padding leaves T_3 alone and changes Δ in a controlled way. Two negative controls delimit the proposition, and both are verified in the same way as the theorems. Padding with a *nonzero* coordinate does change T_3: $T_3(1) = 1$ while $T_3(1,1) = 4$, by the homogeneity $T_3(ca) = |c|^3 T_3(a)$. And Δ itself is *not* padding-invariant — the flat-on-two vector has $\Delta_3 = \frac{1}{6}$ and $\Delta_4 = \frac{1}{4}$ — so the invariance above is a property of T_3 alone.

Finally, for the integer arithmetic of §3 put

$$W_n = \sum_{j=0}^{n} \binom{n}{j} |n-2j|^3 = 2^n\, \mathbb{E}|S_n|^3, \qquad U_n = \sum_{j=0}^{n} \binom{n}{j} |n-2j| = 2^n\, \mathbb{E}|S_n|, \qquad \beta_k = \frac{1}{4^k}\binom{2k}{k},$$

where $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ is the unnormalised sum, so that $F_n = W_n/(2^n n^{3/2})$ and $\beta_k = \Pr(S_{2k} = 0)$.

3. A closed form for the third absolute moment

Everything below rests on the following identities. They are elementary, and we record them as a proposition only because the two telescoping steps are the mechanism of all three main theorems.

Proposition 3.1. *For all $n, m \geq 0$, in $\mathbb{Z}$,*

$$\sum_{j \leq m} \binom{n}{j}(n-2j) = (m+1)\binom{n}{m+1}, \qquad \sum_{j \leq m} \binom{n}{j}(n-2j)^3 = (m+1)\binom{n}{m+1}\pi_n(m+1), \tag{6}$$

where $\pi_n(x) = n^2 + 4n - (4n+4)x + 4x^2$. Consequently, for every $k \geq 0$,

$$W_{2k} = 8k^2\binom{2k}{k}, \qquad W_{2k+1} = (k+1)(4k+1)\binom{2k+2}{k+1},$$

and therefore

$$F_{2k} = 2\sqrt{2k}\,\beta_k, \qquad F_{2k+1} = \frac{(4k+1)\,\beta_k}{\sqrt{2k+1}}. \tag{7}$$

Moreover $16^k \leq 4k\binom{2k}{k}^2$ for every $k \geq 1$, equivalently $4k\,\beta_k^2 \geq 1$.

Proof sketch. Both identities in (6) are telescopings driven by the absorption law $\binom{n}{j+1}(j+1) = \binom{n}{j}(n-j)$. For the first, $\binom{n}{j}(n-2j) = h(j+1) - h(j)$ with $h(j) = j\binom{n}{j}$. For the second, $\binom{n}{j}(n-2j)^3 = p(j+1) - p(j)$ with $p(j) = j\binom{n}{j}\pi_n(j)$; the quadratic π_n is forced, not guessed — expanding $(n-j)\pi_n(j+1) - j\pi_n(j) = (n-2j)^3$ matches four coefficients and determines it uniquely. At $m = n$ both right-hand sides vanish because $\binom{n}{n+1} = 0$, so the part of the sum above the middle index is the negative of the part below it; taking absolute values doubles the lower part, giving $W_n = 2\sum_{j \leq n/2}\binom{n}{j}(n-2j)^3$. Evaluating π_n at $m+1$ with $m = \lfloor n/2 \rfloor$ collapses the bracket to $4k$ when $n = 2k$ and to $2n-1$ when n is odd, and $\binom{2k+2}{k+1} = 2\binom{2k+1}{k+1}$ finishes the odd case. Dividing by $2^n n^{3/2}$ gives (7). The last claim is an induction on k: the ratio of consecutive sides is governed by $(2m+1)^2 > 4m(m+1)$. No case analysis and no search is involved; all of it is identity manipulation in $\mathbb{Z}[n,j]$. $\square$

The first values, from (7):

$$F_1 = 1, \quad F_2 = \sqrt{2}, \quad F_3 = \frac{5}{2\sqrt{3}}, \quad F_4 = \frac{3}{2}, \quad F_5 = \frac{27\sqrt{5}}{40}, \quad F_6 = \frac{5\sqrt{6}}{8}, \quad F_7 = \frac{65\sqrt{7}}{112}. \tag{8}$$

The value $F_3 = \frac{5}{2\sqrt{3}}$ is the one computed in [1], and it agrees.

Remark 3.2. Outside the formal development, and recorded only because it is the cleanest way to say what (6) gives: the same telescoping applied to the first identity yields $U_{2k} = 2k\binom{2k}{k}$ and $U_{2k+1} = (k+1)\binom{2k+2}{k+1}$, so

$$\mathbb{E}|S_n|^3 = \big(2n - (n \bmod 2)\big)\,\mathbb{E}|S_n|,$$

twice the dimension times the first absolute moment for even n, one less for odd n. Only the W_n half of this is inside the formal development: the two U_n values and the displayed identity were checked outside it, by direct summation of the defining sums for $n \le 14$, and they agree exactly. The two half-sums are recorded in OEIS: A107233 is $\sum_{j\le n/2}\binom{n}{j}(n-2j)^3 = \frac{1}{2}W_n$ and A100071 is $\sum_{j\le n/2}\binom{n}{j}(n-2j) = \frac{1}{2}U_n$. The sequence $W_n = 2, 16, 60, 192, 540, 1440, 3640, 8960, \ldots$ itself has no OEIS match, and neither of the two entries lists the relation above (all three searches run 3 September 2026).

4. The flat family, at every pair of dimensions

For $1 \le k < n$ the flat-on-k vector $e^{(n,k)}$ has $\Delta_n(e^{(n,k)}) = \frac{1}{k} - \frac{1}{n}$ and $\mathbb{E}|S(e^{(n,k)})|^3 = F_k$, so the quotient of (3) evaluated there is the chord slope

$$R(n,k) = \frac{F_n - F_k}{\frac{1}{k} - \frac{1}{n}}, \qquad 1 \le k < n,$$

and we write $r_j = R(j+1, j) = j(j+1)(F_{j+1} - F_j)$ for the consecutive slopes. Since

$$\frac{1}{k} - \frac{1}{n} = \sum_{j=k}^{n-1}\Big(\frac{1}{j} - \frac{1}{j+1}\Big) \quad \text{with all summands positive,}$$

$R(n,k)$ is a positively weighted average of $r_k, \ldots, r_{n-1}$: the two-parameter family of chords collapses to the single sequence r_j, and $\inf_{k<n} R(n,k) = \inf_j r_j$.

Proposition 3.1 rationalises r_j completely. With $s = \sqrt{2m}$, $t = \sqrt{2m+1}$ (so $t^2 - s^2 = 1$ and $s^2 + t^2 = 4m+1$), and with $u = \sqrt{2m+1}$, $v = \sqrt{2m+2}$,

$$r_{2m} = \frac{\beta_m\, s^2 t}{(s+t)^2}, \qquad r_{2m+1} = \frac{\beta_m\, uv(2u+v)}{(u+v)^2}. \tag{9}$$

The first follows from $F_{2m+1} - F_{2m} = \beta_m (t-s)^2/t = \beta_m/\big(t(s+t)^2\big)$; the second from the polynomial identity $(2u^3 - 2u^2 v + v)(u+v)^2 = 2u + v$, valid modulo $v^2 = u^2 + 1$, together with $\beta_{m+1} = \beta_m u^2/v^2$. At $m = 1$ the first formula reads

$$r_2 = \frac{\frac{1}{2}\sqrt{3}}{(\sqrt{2}+\sqrt{3})^2} = \frac{\sqrt{3}}{5 + 2\sqrt{6}} = 5\sqrt{3} - 6\sqrt{2} = C_\star,$$

which is the following statement, and explains where the conjectured constant comes from: it is the second consecutive slope of the third-moment sequence.

Proposition 4.1. *$F_3 - F_2 = C_\star\big(\frac{1}{2} - \frac{1}{3}\big)$; equivalently $r_2 = C_\star$ exactly, with no slack.*

Proof. $F_2 = \sqrt{2}$ and $F_3 = \frac{5\sqrt{3}}{6}$ by (8), and $\frac{5\sqrt{3}}{6} - \sqrt{2} = \frac{5\sqrt{3} - 6\sqrt{2}}{6}$. □

The content of the section is that every other consecutive slope is strictly larger, by a margin visible to rational arithmetic.

Proposition 4.2. *For every $j \ge 1$, $r_j \ge C_\star$; and for every $j \ge 1$ with $j \ne 2$, $r_j \ge \frac{351}{2000} > C_\star$, so the inequality is strict there.*

Proof sketch. The single inequality input is $4m\beta_m^2 \ge 1$ (Proposition 3.1); after that everything is a comparison of squares, so no square root is ever estimated except through the two rational brackets $1.4142135 < \sqrt{2} < 1.4142136$ and $1.7320508 < \sqrt{3} < 1.7320509$, which give $C_\star < \frac{351}{2000}$.

Even $j = 2m$ with $m \ge 2$. From $(s+t)^2 \le 2(s^2+t^2)$ and (9), $r_{2m} \ge \beta_m s^2 t/\big(2(s^2+t^2)\big)$. Squaring, $(\beta_m s^2 t)^2 = 4m\beta_m^2 \cdot m(2m+1) \ge m(2m+1)$, while $\big(2 \cdot \frac{351}{2000}(s^2+t^2)\big)^2 = \frac{123201}{10^6}(4m+1)^2$, so it suffices that

$$123201\,(4m+1)^2 \;\le\; 10^6\, m(2m+1) \qquad (m \ge 2),$$

a quadratic inequality in m with 20719 to spare at $m = 2$.

Odd $j = 2m+1$ with $m \geq 1$. Write $N = 2u^3 - 2u^2v + v > 0$, so that $r_{2m+1} = \beta_m Nuv$ by the identity above. From $(u+v)^2 \leq 2(u^2+v^2) = 2(2u^2+1)$ one gets $3u \leq 2N(2u^2+1)$, and squaring $2(2u^2+1) \leq 15\beta_m u^3$ reduces — again by $4m\beta_m^2 \geq 1$ — to the polynomial inequality

$$16m(4m+3)^2 \;\leq\; 225\,(2m+1)^3 \qquad (m \geq 1).$$

Combining, $r_{2m+1} \geq \beta_m N u^2 \geq \frac{351}{2000}$.

$j = 1$. Directly, $r_1 = 2(F_2 - F_1) = 2\sqrt{2} - 2 > \frac{351}{2000}$.

Finally $r_2 = C_\star$ by Proposition 4.1, which gives the non-strict claim at $j = 2$. ☐

Theorem 4.3. *For all integers $1 \leq k < n$,*

$$\mathbb{E}|\overline{S}_n|^3 - \mathbb{E}|\overline{S}_k|^3 \;=\; F_n - F_k \;\geq\; C_\star\Big(\frac{1}{k} - \frac{1}{n}\Big).$$

Proof. Induction on $n-k$, adding the consecutive bounds $r_j \geq C_\star$ of Proposition 4.2 for $j = k, \dots, n-1$ and telescoping both sides. ☐

Theorem 4.4. *For all integers $1 \leq k < n$ with $(k,n) \neq (2,3)$ the inequality of Theorem 4.3 is strict; at $(k,n) = (2,3)$ it is an equality. Consequently $C_\star$ is the unique extremal chord slope of the sequence $(F_n)_{n\geq 1}$: the infimum of $R(n,k)$ over all $1 \leq k < n$ equals $C_\star$ and is attained only at $(k,n) = (2,3)$.*

Proof. If $(k,n) \neq (2,3)$ then the range $k \leq j < n$ contains some $j \neq 2$: if $k \neq 2$ take $j = k$; if $k = 2$ then $n \geq 4$ and $j = 3$ works. Split the telescoping sum at that j, use the strict bound of Proposition 4.2 on that step and the non-strict bound on the others. Equality at $(2,3)$ is Proposition 4.1. ☐

Remark 4.5 (reading Theorem 4.3 as the conjecture). Since $\Delta_n(e^{(n,k)}) = \frac{1}{k} - \frac{1}{n}$ and $\mathbb{E}|S(e^{(n,k)})|^3 = F_k$, Theorem 4.3 says exactly that the conjecture holds at every flat-on-k vector of $\mathbb{R}^n$, for all $1 \leq k < n$ at once, with equality only at $e^{(3,2)}$. The identification $\mathbb{E}|S(e^{(n,k)})|^3 = F_k$ is the invariance of the moment under padding with zero coordinates, which is Proposition 2.2; the theorem as stated and verified above is the inequality between the two moment values F_n and F_k, and Corollary 4.8 below is the reading of it as a statement about coefficient vectors, for an arbitrary support and arbitrary signs. Nothing between here and there depends on the identification: Theorem 1.2 is proved in the ambient dimension directly.

Corollary 4.6. *For every $n \geq 2$, $F_n \geq \sqrt{2} + C_\star\big(\frac{1}{2} - \frac{1}{n}\big)$, equivalently $\sqrt{2} + \frac{C_\star}{2} \leq F_n + \frac{C_\star}{n}$.*

Proof. Theorem 4.3 at $k = 2$, with $F_2 = \sqrt{2}$; the case $n = 2$ is trivial. ☐

Remark 4.7 (the exact slopes, and why $\frac{351}{2000}$). By (7) each r_j is an explicit element of a real quadratic field. The first eight values are

j	r_j	$\approx$
1	$2\sqrt{2} - 2$	0.828427
2	$5\sqrt{3} - 6\sqrt{2}$	**0.174973**
3	$18 - 10\sqrt{3}$	0.679492
4	$\frac{27\sqrt{5}}{2} - 30$	0.186918
5	$\frac{75\sqrt{6}}{4} - \frac{81\sqrt{5}}{4}$	0.647556
6	$\frac{195\sqrt{7}}{8} - \frac{105\sqrt{6}}{4}$	0.191082
7	$\frac{245\sqrt{8}}{8} - \frac{65\sqrt{7}}{2}$	0.633663
8	$\frac{1785}{16} - \frac{315\sqrt{8}}{8}$	0.193182

The decimal column is exact arithmetic on the surd expression beside it. The even-indexed slopes increase and the odd-indexed ones decrease — checked here for $j \leq 40$, not proved — towards $\frac{1}{4}\sqrt{2/\pi} = 0.199471\ldots$ and $\frac{3}{4}\sqrt{2/\pi} = 0.598413\ldots$ respectively; these two limits are what (9) gives under $\beta_m \sim 1/\sqrt{\pi m}$, and they are stated for orientation only, outside the formal development. That r_2 is the unique minimum, on the other hand, is Proposition 4.2, and $r_2 = C_\star$.

The rational constant $\frac{351}{2000} = 0.1755$ in Proposition 4.2 is not cosmetic: the chain of lossy steps used there cannot deliver more than $\frac{1}{4\sqrt{2}} = 0.176777\ldots$. Indeed the even branch, after $(s+t)^2 \le 2(s^2+t^2)$ and $\beta_m \ge 1/(2\sqrt{m})$, yields only $\sqrt{m(2m+1)}/\big(2(4m+1)\big)$, which increases in m with supremum $\frac{\sqrt{2}}{8} = \frac{1}{4\sqrt{2}}$ and never attains it. So any rational strict constant provable that way lies in $(C_\star, 0.176777)$, and $\frac{351}{2000}$ is the round number inside it. A constant such as $\frac{9}{50} = 0.18$, which the true values would permit — among the r_j with $j \ne 2$ the smallest over the range checked is $r_4 = 0.186918\ldots$ — is *not* reachable through those bounds at any m.

4.1. **The flat cone.** Proposition 2.2 turns Theorem 4.3 into a statement about coefficient vectors — which is the form the conjecture is stated in — with no restriction on where the support sits or on the signs.

Corollary 4.8 (the conjecture on the flat cone)**.** *Let $1 \le k \le n$, let $S \subseteq \{1,\ldots,n\}$ have exactly k elements, and let $a \in \mathbb{R}^n$ satisfy $|a_i| = k^{-1/2}$ for $i \in S$ and $a_i = 0$ for $i \notin S$. Then $\sum_i a_i^2 = 1$, $\Delta_n(a) = \frac{1}{k} - \frac{1}{n}$, and*

$$\mathbb{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^3 \;\le\; \mathbb{E}|\overline{S}_n|^3 - (5\sqrt{3} - 6\sqrt{2})\,\Delta_n(a).$$

Proof. By Proposition 2.2 we may replace a by $(|a_1|,\ldots,|a_n|)$ and then permute S onto $\{1,\ldots,k\}$, neither step changing T_3 or Δ_n. What is left is $e^{(n,k)}$, the flat vector of $\mathbb{R}^k$ padded with $n-k$ zeros; padding leaves T_3 unchanged, so its moment is F_k, and leaves q unchanged, so $\Delta_n = q - \frac{1}{n} = \frac{1}{k} - \frac{1}{n}$. Apply Theorem 4.3 if $k < n$; if $k = n$ both sides equal F_n. □

By Theorem 4.4 the inequality is strict for every $1 \le k < n$ except $(k,n) = (2,3)$, where it is an equality — and there the vector of the corollary is, up to the permutations and sign changes just used, the vector w of [1] (Proposition 5.7). So on the flat cone the constant $C_\star$ is attained, and cannot be raised.

5. The spike region, at every dimension

Definition 5.1. A vector $a \in \mathbb{R}^n$ is a *spike vector at i_0* if $\sum_{j \ne i_0} |a_j| \le |a_{i_0}|$. We call a a spike vector if it is one at some coordinate.

The name is descriptive: one coordinate is at least as large as everything else put together in ℓ^1. Every vector with at most two nonzero coordinates is a spike vector, so in particular every unit vector of $\mathbb{R}^2$ is, and so is the vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ of [1].

5.1. **The moment on the region is a cubic polynomial.** The structural reason the region is tractable is that the sign of $S(a)$ is frozen.

Lemma 5.2. *Let $a \in \mathbb{R}^n$ be a unit spike vector at i_0 with $a_{i_0} \ge 0$, and put $c = a_{i_0}$. Then*

$$\mathbb{E}|S(a)|^3 \;=\; 3c - 2c^3.$$

Proof sketch. Write $R = \sum_{j \ne i_0} a_j\varepsilon_j$ and $\varepsilon_0 = \varepsilon_{i_0}$. The hypothesis gives $|R| \le c$ pointwise, so $S(a) = \varepsilon_0 c + R$ has the sign of ε_0 and $|S(a)| = \varepsilon_0(\varepsilon_0 c + R) = c + \varepsilon_0 R$. Expanding the cube,

$$|S(a)|^3 = c^3 + 3c^2(\varepsilon_0 R) + 3cR^2 + \varepsilon_0 R^3.$$

Summing over sign patterns, the involution $s \mapsto s \triangle \{i_0\}$ flips ε_0 and fixes R, so the two terms odd in ε_0 cancel in pairs; and $\mathbb{E}R^2 = \sum_{j \ne i_0} a_j^2 = 1 - c^2$ by the second-moment identity of §2. Hence $\mathbb{E}|S(a)|^3 = c^3 + 3c(1-c^2) = 3c - 2c^3$. □

The hypothesis cannot be dropped: at the flat vector of $\mathbb{R}^3$ the left side is $\frac{5\sqrt{3}}{6}$ and the right side is $\frac{7\sqrt{3}}{9}$ (Proposition 5.10).

Combining Lemma 5.2 with the elementary bound $q(a) \le c^4 + (1-c^2)^2$ — valid because $\sum_{j \ne i_0} a_j^4 \le (\sum_{j \ne i_0} a_j^2)^2$ — reduces the conjectured inequality on the region to one variable. That one variable is handled by an exact factorisation rather than by differentiation, which is what makes the extremal point visible.

Lemma 5.3. *For all real C and c,*

$$\sqrt{2}+\frac{C}{2}-\big(3c-2c^3+C(c^4+(1-c^2)^2)\big) \;=\; \Big(c-\frac{1}{\sqrt{2}}\Big)^2\big(-2Cc^2+(2-2\sqrt{2}\,C)c+2\sqrt{2}-C\big).$$

In particular, if $0\le C\le \frac{7}{10}$ then the second factor is at least $2\sqrt{2}-3C>0$ on $[0,1]$, and hence

$$3c-2c^3+C\big(c^4+(1-c^2)^2\big) \;\le\; \sqrt{2}+\frac{C}{2} \qquad (0\le c\le 1),$$

with equality at $c=\frac{1}{\sqrt{2}}$.

Proof. The identity is a polynomial identity in c over $\mathbb{R}$ once $\sqrt{2}^2=2$ is used; the first factor is $c^2-\sqrt{2}c+\frac{1}{2}$. On $[0,1]$ the second factor is at least $-2C+0+2\sqrt{2}-C$ because $c^2\le 1$ and $2-2\sqrt{2}C\ge 0$ for $C\le\frac{7}{10}$. □

Note where the double root sits: at $c=\frac{1}{\sqrt{2}}$, which is exactly the spike height of the extremiser of [1]. The extremal configuration is therefore exhibited algebraically, not located by a derivative.

5.2. **The conjecture on the region.**

Proposition 5.4. *Let $n\ge 2$ and let C satisfy*

$$0\le C\le\tfrac{7}{10} \qquad \textit{and} \qquad \sqrt{2}+\frac{C}{2}\;\le\;F_n+\frac{C}{n}.$$

Then for every unit spike vector $a\in\mathbb{R}^n$,

$$\mathbb{E}|S(a)|^3\;\le\;\mathbb{E}|\overline{S}_n|^3-C\,\Delta_n(a).$$

Proof. Replacing a by $-a$ changes neither side, so assume $c=a_{i_0}\ge 0$; then $c\le 1$. By Lemma 5.2, $\Delta_n(a)=q(a)-\frac{1}{n}$ and $q(a)\le c^4+(1-c^2)^2$,

$$\mathbb{E}|S(a)|^3+C\Delta_n(a)\le 3c-2c^3+C\big(c^4+(1-c^2)^2\big)-\frac{C}{n}\le\sqrt{2}+\frac{C}{2}-\frac{C}{n}\le F_n$$

by Lemma 5.3 and the hypothesis on C. □

Both hypotheses on C are exactly what the argument needs, and both are sharp inputs: the second is an equality precisely when $C=R(n,2)$, the chord slope of §4 at $k=2$. Feeding it Corollary 4.6 — which is where the flat-family analysis enters, and which is what is not available to [1] — gives the conjecture on the region at every dimension.

Theorem 5.5. *For every $n\ge 2$ and every unit spike vector $a\in\mathbb{R}^n$,*

$$\mathbb{E}\Big|\sum_{i=1}^{n}a_i\varepsilon_i\Big|^3\;\le\;\mathbb{E}|\overline{S}_n|^3-(5\sqrt{3}-6\sqrt{2})\,\Delta_n(a).$$

Proof. Apply Proposition 5.4 with $C=C_\star$: it is positive and $C_\star<\frac{351}{2000}<\frac{7}{10}$ by the rational brackets, and the chord condition is Corollary 4.6. □

Corollary 5.6. *The conjecture holds for every unit $a\in\mathbb{R}^n$ supported on at most two coordinates, at every $n\ge 2$; in particular it holds on all of $\mathbb{R}^2$.*

Proof. If a has at most two nonzero coordinates a_{i_0},a_{i_1} with $|a_{i_0}|\ge|a_{i_1}|$, then $\sum_{j\ne i_0}|a_j|=|a_{i_1}|\le|a_{i_0}|$. □

5.3. **The restricted infimum is exactly $C_\star$.** The vector singled out by [1] lies in the region, and there the inequality of Theorem 5.5 is an equality.

Proposition 5.7. *Let $w = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \in \mathbb{R}^3$, so $(w_1^2, w_2^2, w_3^2) = (\frac{1}{2}, \frac{1}{2}, 0)$. Then $\mathbb{E}|S(w)|^3 = \sqrt{2}$, $\Delta_3(w) = \frac{1}{6}$, and*

$$\mathbb{E}|\overline{S}_3|^3 - \mathbb{E}|S(w)|^3 = C_\star \cdot \Delta_3(w), \qquad \textit{i.e.} \qquad \frac{\mathbb{E}|\overline{S}_3|^3 - \mathbb{E}|S(w)|^3}{\Delta_3(w)} = 5\sqrt{3} - 6\sqrt{2}.$$

In particular $C_3^{\mathrm{opt}} \le 5\sqrt{3} - 6\sqrt{2}$.

Proof. w is a spike vector at the first coordinate with $c = \frac{1}{\sqrt{2}}$, so $\mathbb{E}|S(w)|^3 = 3c - 2c^3 = \sqrt{2}$ by Lemma 5.2; $q(w) = \frac{1}{2}$ gives $\Delta_3(w) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$; and $F_3 - \sqrt{2} = \frac{5\sqrt{3}-6\sqrt{2}}{6}$. $\square$

This reproduces, from the definitions of [1] alone, the upper bound of (4); the statement is theirs, and only the certificate is ours. Together with Theorem 5.5 it pins the restricted infimum.

Theorem 5.8. *Let*

$$C_3^{\mathrm{spike}} := \inf\Big\{ \frac{\mathbb{E}|\overline{S}_n|^3 - \mathbb{E}|S(a)|^3}{\Delta_n(a)} \;:\; n \ge 2,\ \textstyle\sum_i a_i^2 = 1,\ \Delta_n(a) > 0,\ a \textit{ a spike vector} \Big\}.$$

Then $C_3^{\mathrm{spike}} = 5\sqrt{3} - 6\sqrt{2}$, and the infimum is attained, at the vector w of Proposition 5.7.

Proof. Every competitor has quotient $\ge C_\star$ by Theorem 5.5, since $\Delta_n(a) > 0$; and w is an admissible competitor with quotient exactly $C_\star$ by Proposition 5.7. $\square$

So on the spike region the bracket (4) — a factor 13 wide — collapses to the single point $C_\star$, and the extremiser of [1] is the extremiser of the restricted problem. Since $C_3^{\mathrm{opt}} \le C_3^{\mathrm{spike}}$, this also says that if the conjecture is false then it is false at a vector outside the region: one with at least three nonzero coordinates, no one of which carries the ℓ^1 mass of the rest.

5.4. **The sharp constant in dimension four.** Fixing the dimension improves the constant, because the chord condition of Proposition 5.4 becomes an equality at $C = R(n, 2)$. At $n = 4$, $F_4 = \frac{3}{2}$ and

$$R(4,2) = \frac{F_4 - F_2}{\frac{1}{2} - \frac{1}{4}} = 4\Big(\frac{3}{2} - \sqrt{2}\Big) = 6 - 4\sqrt{2},$$

which is admissible since $6 - 4\sqrt{2} = 0.343145\ldots \le \frac{7}{10}$.

Theorem 5.9. *For every unit spike vector $a \in \mathbb{R}^4$,*

$$\mathbb{E}|S(a)|^3 \;\le\; \mathbb{E}|\overline{S}_4|^3 - (6 - 4\sqrt{2})\,\Delta_4(a),$$

and equality holds at $w_4 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$, where $\mathbb{E}|S(w_4)|^3 = \sqrt{2}$ and $\Delta_4(w_4) = \frac{1}{4}$. Hence the sharp constant on the spike region of $\mathbb{R}^4$ is exactly

$$6 - 4\sqrt{2} = 0.3431457505\ldots,$$

almost twice the conjectured global constant $C_\star$.

Proof. The inequality is Proposition 5.4 with $n = 4$ and $C = 6 - 4\sqrt{2}$, whose two hypotheses hold as computed above (the chord condition being an equality). For equality, w_4 is a spike vector with $c = \frac{1}{\sqrt{2}}$, so $\mathbb{E}|S(w_4)|^3 = \sqrt{2}$ by Lemma 5.2 and $\Delta_4(w_4) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$; and $\frac{3}{2} - \sqrt{2} = (6 - 4\sqrt{2}) \cdot \frac{1}{4}$. $\square$

5.5. **Controls.** The following are not results; they are the checks that the statements above are not satisfied vacuously or by a misplaced quantifier, and each is verified in the same way as the theorems.

Proposition 5.10. (i) (non-vacuity) *w is a unit spike vector with $\Delta_3(w) = \frac{1}{6} > 0$.*

(ii) (no larger constant) $\mathbb{E}|S(w)|^3 \le \mathbb{E}|\overline{S}_3|^3 - \frac{18}{100}\Delta_3(w)$ *is* false*: at w the deficit is exactly $C_\star \Delta_3(w)$, so no constant above $C_\star$ can work, and $\frac{18}{100}$ already fails.*

(iii) (the hypothesis is not removable) *For the flat vector of $\mathbb{R}^3$, $\mathbb{E}|S(a)|^3 = \frac{5\sqrt{3}}{6} \ne \frac{7\sqrt{3}}{9} = 3a_1 - 2a_1^3$, so the identity of Lemma 5.2 fails without the dominance hypothesis.*

(iv) (strictness is not vacuous) *At $(k, n) = (4, 5)$ the chord bound of Theorem 4.3 is strict.*

6. What is finite, and what is not

It is worth being explicit about which parts of this problem are compact optimisations and which are not, because the two are easy to conflate.

Each fixed dimension is finite. Let Φ_n be the infimum of the quotient in (3) over the unit sphere of $\mathbb{R}^n$ alone. The function $a \mapsto \mathbb{E}|S(a)|^3$ is piecewise polynomial: it is a polynomial on each cell of the arrangement of the 2^{n-1} hyperplanes $\{\varepsilon \cdot a = 0\}$, intersected with the sorted cone $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ (to which one may reduce by signs and permutations). So Φ_n is the infimum of finitely many ratios of polynomials over compact semialgebraic sets, the flat vector itself — where Δ_n vanishes — being excluded: a finite problem in principle.

Which of those hyperplanes actually meets the interior of the sorted cone is easy to say; normalise $\varepsilon_1 = +1$, one representative per global flip. The extreme rays of the cone are the vectors $v^{(m)} = (1, \ldots, 1, 0, \ldots, 0)$ with m ones, so writing $a = \sum_{m=1}^{n} c_m v^{(m)}$ with $c_m \ge 0$ gives $\varepsilon \cdot a = \sum_m c_m(\varepsilon_1 + \cdots + \varepsilon_m)$: the sign of $\varepsilon \cdot a$ is constant on the cone exactly when all the partial sums $\varepsilon_1 + \cdots + \varepsilon_m$ have one sign, and $\{\varepsilon \cdot a = 0\}$ cuts it otherwise. At $n = 3$ only $\varepsilon = (+, -, -)$ fails the test, so there is one cutting hyperplane and hence two cells, $a_1 \ge a_2 + a_3$ and $a_1 \le a_2 + a_3$, which is the case split of [1]. At $n = 4$ exactly $(+, -, -, -)$ and $(+, -, -, +)$ fail, giving the two hyperplanes $a_1 = a_2 + a_3 + a_4$ and $a_1 + a_4 = a_2 + a_3$; the first region is contained in the second, so they cut the cone into exactly three cells: $a_1 \ge a_2 + a_3 + a_4$; then $a_1 \le a_2 + a_3 + a_4$ together with $a_1 + a_4 \ge a_2 + a_3$; and finally $a_1 + a_4 \le a_2 + a_3$. The first cell in each list is the spike region, and it is the one settled here in closed form at every n.

The conjecture is not finite. $C_3^{\mathrm{opt}} = \inf_n \Phi_n$ is an infimum of infinitely many such problems, and no argument here bounds Φ_n away from $C_\star$ uniformly in n — such a bound would essentially be the conjecture. Nothing about the problem truncates the family: the best available universal lower bound is still the $\frac{3(5\sqrt{2}-7)}{16}$ of (4), a factor 13 below the conjectured value. So the conjecture is not a computation that can be run; it is a statement about all dimensions at once.

What is done is done at all dimensions at once. Theorem 4.3 settles the flat family for every pair (k, n), Theorem 5.5 settles the spike region for every n, and Theorem 7.3 of §7 settles the fourth-moment region for every n. None of them rests on an exhaustive search of any kind: all are identities and inductions, and the only numerical inputs anywhere are the two rational brackets for $\sqrt{2}$ and $\sqrt{3}$ used in Proposition 4.2.

Remark 6.1 (measured values of Φ_n). The following is a computation reported outside the formal development, and outside the theorems above; it is included because it says where the remaining difficulty is. Minimising $R(n, k)$ over k — which by §4 is the minimum over the whole flat family of $\mathbb{R}^n$

— gives $k = 2$ for even n and $k = n - 1$ for odd n, with exact values

n	flat-family minimum in $\mathbb{R}^n$	$\approx$
4	$6 - 4\sqrt{2}$	0.343146
5	$\frac{27\sqrt{5}}{2} - 30$	0.186918
6	$\frac{15\sqrt{6}}{8} - 3\sqrt{2}$	0.350153
7	$\frac{195\sqrt{7}}{8} - \frac{105\sqrt{6}}{4}$	0.191082
8	$\frac{\sqrt{2}}{4}$	0.353553
9	$\frac{1785}{16} - \frac{315\sqrt{2}}{4}$	0.193182
10	$\frac{315\sqrt{10}}{256} - \frac{5\sqrt{2}}{2}$	0.355550
11	$\frac{6615\sqrt{11}}{128} - \frac{3465\sqrt{10}}{64}$	0.194444
12	$\frac{693\sqrt{12}}{640} - \frac{12\sqrt{2}}{5}$	0.356860

in exact rational-plus-surd arithmetic — every entry is a few lines of arithmetic on (7), and the argument of the minimum was checked to be $k = 2$ for even n and $k = n - 1$ for odd n throughout $n \le 80$ — all of them above $\Phi_3 = C_\star = 0.174973\ldots$. A direct minimisation over the unit sphere — 200 random restarts of 600 pairwise-transfer steps, with the moment evaluated as an exact 2^n sum — finds nothing below these values for $3 \le n \le 8$, and returns $\Phi_3 = C_\star$ at $n = 3$, the value [1] determines. These are *measurements*, and the sphere search is a heuristic with no completeness guarantee: the identification of Φ_n with the flat-family minimum is proved here only on the spike region (Proposition 5.4) and on the fourth-moment region (Proposition 7.2), so for $n \ge 4$ the table is a conjecture. The entry $\Phi_4 = 6 - 4\sqrt{2}$ is a theorem on each of those two regions (Theorems 5.9 and 7.5), the second of which contains the vector where the value is attained; what is open is the complement, and Remark 7.9 says what a certificate there would cost.

Two questions this leaves. First, the complementary region at a fixed small dimension. Proving $\Phi_4 = 6 - 4\sqrt{2}$ means handling the part of the sphere of $\mathbb{R}^4$ that §7 does not reach, where the deficit vanishes both at the corner w_4 and at the flat vector; that is a polynomial optimisation whose price is counted in Remark 7.9. Second, the general three-support case at every n, which by Theorem 4.3 at $k = 3$ would follow from Lemma 6.4 of [1] together with Proposition 2.2; that lemma's second branch is a genuine two-variable optimisation and is not reproved here. Both are natural next steps, and neither is attempted here. Separately, and independently of all of this, any certified improvement of the small-ball constant $\frac{3}{16}$ of Lemma 6.1 of [1] raises the lower end of (4) at once.

7. The fourth-moment region, at every dimension

The spike region is an ℓ^1 condition on the coefficients. The region of this section is an ℓ^4 condition — $q(a) = \sum_i a_i^4 \ge \frac{1}{2}$ — and the route to it is different in kind: a single Cauchy–Schwarz step against the fourth moment, which is exactly computable. What the two arguments share is their one nontrivial input, the chord bound at $k = 2$ (Corollary 4.6).

7.1. **One Cauchy–Schwarz step.** The fourth moment of a Rademacher sum is a polynomial in the coefficients, and on the unit sphere it is an affine function of q alone — this is equation (1) of [1], the identity through which the parameter q enters that paper in the first place.

Lemma 7.1. *For every $n \ge 0$ and every $a \in \mathbb{R}^n$,*

$$\frac{1}{2^n} \sum_{s \subseteq \{1,\ldots,n\}} \Big(\sum_{i \notin s} a_i - \sum_{i \in s} a_i \Big)^4 \;=\; 3\Big(\sum_i a_i^2\Big)^2 - 2\sum_i a_i^4, \tag{10}$$

that is, $\mathbb{E}S(a)^4 = 3 - 2q(a)$ on the unit sphere. Consequently, for every unit $a \in \mathbb{R}^n$,

$$T_3(a)^2 \;\le\; 3 - 2q(a).$$

Proof. The identity (10) is classical, and for unit a it is equation (1) of [1]; we prove it by induction on n through the padding bijection of Proposition 2.2. Splitting the sum over the 2^{n+1} sign patterns of

$\mathbb{R}^{n+1}$ at the last coordinate pairs each pattern of $\mathbb{R}^n$ with the two values $Y \pm x$, where Y is its signed sum and $x = a_{n+1}$, and

$$(Y-x)^4 + (Y+x)^4 = 2Y^4 + 12x^2Y^2 + 2x^4;$$

the middle term is evaluated by the second-moment identity of §2, and the induction closes. For the second claim, write $X = S(a)$ and apply the Cauchy–Schwarz inequality to the 2^n-term average in the form $\big(\mathbb{E}[|X| \cdot X^2]\big)^2 \le \mathbb{E}[X^2]\,\mathbb{E}[X^4]$; on the unit sphere $\mathbb{E}X^2 = 1$ and $\mathbb{E}X^4 = 3 - 2q(a)$ by (10), while $\mathbb{E}[|X| \cdot X^2] = \mathbb{E}|X|^3 = T_3(a)$. ☐

7.2. **The conjecture on the region.** The bound of Lemma 7.1 decreases in q, and the linear term $C\Delta_n$ increases in q only slowly; so the whole region is settled by its worst point $q = \frac{1}{2}$, and there the requirement is precisely the chord bound at $k = 2$. We do not argue by differentiating: the comparison is made on squares, through an exact factorisation, so no square root is ever estimated.

Proposition 7.2. *Let $n \ge 1$ and let C satisfy*

$$0 \le C \le \tfrac{7}{10} \qquad \text{and} \qquad \sqrt{2} + \frac{C}{2} \;\le\; F_n + \frac{C}{n}.$$

Then for every unit $a \in \mathbb{R}^n$ with $q(a) \ge \frac{1}{2}$,

$$\mathbb{E}|S(a)|^3 \;\le\; \mathbb{E}|\overline{S}_n|^3 - C\,\Delta_n(a).$$

Proof. Put $e = q(a) - \frac{1}{2}$ and $L = \sqrt{2} - Ce = \sqrt{2} + \frac{C}{2} - Cq(a)$. The hypothesis on a gives $e \ge 0$, and $\sum_i a_i^4 \le (\sum_i a_i^2)^2 = 1$ gives $e \le \frac{1}{2}$; hence $L \ge \sqrt{2} - \frac{7}{20} > 0$. Expanding both sides gives the identity

$$L^2 - \big(3 - 2q(a)\big) \;=\; e\,\big(2 - 2\sqrt{2}\,C + C^2 e\big),$$

whose right-hand side is nonnegative because $e \ge 0$ and $2\sqrt{2}\,C \le \frac{7\sqrt{2}}{5} = 1.9798\ldots < 2$. With Lemma 7.1, $T_3(a)^2 \le 3 - 2q(a) \le L^2$, and both $T_3(a)$ and L are nonnegative, so $T_3(a) \le L$. Finally $\Delta_n(a) = q(a) - \frac{1}{n}$, so

$$L = \sqrt{2} + \frac{C}{2} - Cq(a) \;\le\; F_n + \frac{C}{n} - Cq(a) = F_n - C\,\Delta_n(a)$$

by the second hypothesis. ☐

The two hypotheses on C are word for word those of Proposition 5.4: the ℓ^1 region and the ℓ^4 region are two different arguments run from one input. Feeding it Corollary 4.6 again gives the conjecture, this time on the fourth-moment region.

Theorem 7.3. *For every $n \ge 2$ and every unit $a \in \mathbb{R}^n$ with $\sum_i a_i^4 \ge \frac{1}{2}$,*

$$\mathbb{E}\Big|\sum_{i=1}^{n} a_i\varepsilon_i\Big|^3 \;\le\; \mathbb{E}|\overline{S}_n|^3 - (5\sqrt{3} - 6\sqrt{2})\,\Delta_n(a).$$

Proof. Proposition 7.2 with $C = C_\star$, which is positive and at most $\frac{351}{2000} < \frac{7}{10}$ by the rational brackets of Proposition 4.2; the second hypothesis is Corollary 4.6. ☐

As with Theorem 5.5, the case $n = 3$ is contained in Theorem 1.4 of [1], which settles all of $\mathbb{R}^3$ with $C_\star$; what is new is $n \ge 4$ and the uniformity in n. The threshold $\frac{1}{2}$ is exactly what that uniformity costs, and it is attained: the extremiser of [1] has $q = \frac{1}{2}$, and there every inequality in the proof above is an equality — $|S(w)|$ takes only the two values 0 and $\sqrt{2}$, which is the equality case of the Cauchy–Schwarz step. Hence the restricted infimum, exactly as in §5.

Theorem 7.4. *Let*

$$C_3^{\ell^4} := \inf\Big\{\frac{\mathbb{E}|\overline{S}_n|^3 - \mathbb{E}|S(a)|^3}{\Delta_n(a)} \;:\; n \ge 2,\ \textstyle\sum_i a_i^2 = 1,\ \Delta_n(a) > 0,\ q(a) \ge \frac{1}{2}\Big\}.$$

Then $C_3^{\ell^4} = 5\sqrt{3} - 6\sqrt{2}$, and the infimum is attained, at the vector w of Proposition 5.7.

Proof. Every competitor has quotient $\geq C_\star$ by Theorem 7.3, since $\Delta_n(a) > 0$; and w is admissible, because $q(w) = \frac{1}{2}$, with quotient exactly $C_\star$ by Proposition 5.7. $\square$

Theorem 7.5. *For every unit $a \in \mathbb{R}^4$ with $q(a) \geq \frac{1}{2}$,*

$$\mathbb{E}|S(a)|^3 \ \leq \ \mathbb{E}|\overline{S}_4|^3 - (6 - 4\sqrt{2})\,\Delta_4(a),$$

and equality holds at $w_4 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$, which satisfies $q(w_4) = \frac{1}{2}$. So the sharp constant on the fourth-moment region of $\mathbb{R}^4$ is $6 - 4\sqrt{2}$, the same value as on the spike region of $\mathbb{R}^4$.

Proof. Proposition 7.2 with $n = 4$ and $C = 6 - 4\sqrt{2}$, whose two hypotheses were checked before Theorem 5.9 (the second is the equality $C = R(4, 2)$). For equality, $q(w_4) = \frac{1}{2}$, $\mathbb{E}|S(w_4)|^3 = \sqrt{2}$ and $\Delta_4(w_4) = \frac{1}{4}$, and $\frac{3}{2} - \sqrt{2} = (6 - 4\sqrt{2}) \cdot \frac{1}{4}$. $\square$

Since the two regions of $\mathbb{R}^4$ are different and carry the same sharp constant, Theorems 5.9 and 7.5 together give the conjectured inequality with the constant $6 - 4\sqrt{2}$ on their union.

7.3. **Controls.** As in §5, these are not results but checks, verified in the same way as the theorems.

Proposition 7.6. (i) (the region is not contained in the spike region) *The vector*

$$b = \left(\sqrt{\tfrac{7}{10}},\ \sqrt{\tfrac{1}{10}},\ \sqrt{\tfrac{1}{10}},\ \sqrt{\tfrac{1}{10}}\right) \in \mathbb{R}^4$$

is a unit vector with $q(b) = \frac{13}{25} > \frac{1}{2}$, so Theorem 7.5 applies to it, and it fails the spike condition (5) *at* every *coordinate, since $3\sqrt{1/10} > \sqrt{7/10}$.*

(ii) (no larger constant) *w lies in the region, $q(w) = \frac{1}{2}$, and $\mathbb{E}|S(w)|^3 \leq \mathbb{E}|\overline{S}_3|^3 - \frac{18}{100}\Delta_3(w)$ is false, so no constant above $C_\star$ works on the region either.*

(iii) (the hypothesis is not vacuous in the other direction) *The flat vector of $\mathbb{R}^4$ has $q = \frac{1}{4} < \frac{1}{2}$ and lies outside the region. It has to: there $\sqrt{3 - 2q} = \sqrt{5/2} = 1.5811\ldots$ exceeds $F_4 = \frac{3}{2}$, so near the flat vector Lemma 7.1 alone cannot give the conjecture, whatever the constant.*

Remark 7.7 (measured: what the same proof gives at a fixed dimension). Outside the formal development. For fixed n the chain of the proof of Theorem 7.3 closes for every $q \geq q_n^*$, where q_n^* is the root of $\sqrt{3 - 2q} + C_\star(q - \frac{1}{n}) = F_n$; by bisection in exact arithmetic on the closed form (7),

$$q_2^* = q_3^* = \tfrac{1}{2}, \quad q_4^* = 0.4188900\ldots, \quad q_5^* = 0.4177071\ldots, \quad q_6^* = 0.3861917\ldots, \quad q_8^* = 0.3687793\ldots$$

So the uniform threshold is $\frac{1}{2}$, forced by $n = 3$, and at every fixed $n \geq 4$ the same argument covers strictly more than Theorem 7.3 claims. The verified statements carry the uniform threshold.

Remark 7.8 (the fixed-q envelope of [1] and the third moment). It is worth saying why the region above is not an instance of the machinery of [1], whose subject is exactly the fourth-order geometry of these sums. Proposition 4.1 of [1] computes $\max\{\mathbb{E}\Phi(S(a)) : a \in \mathbb{R}^N,\ \sum_i a_i^2 = 1,\ q(a) = q\}$ and locates the maximiser at the one-spike-plus-flat-tail vector $Y^+_{N,q} = c\varepsilon_1 + d(\varepsilon_2 + \cdots + \varepsilon_N)$, and Theorem 1.2 of [1] passes from it to the limit, in which the tail becomes Gaussian — but both for $\Phi \in C^4(\mathbb{R})$ even with convex fourth derivative, and $\Phi(x) = |x|^3$ is not in that class. [1] says so itself, opening its §6: "At $p = 3$, the fourth-order convexity used above is no longer available." Outside the formal development we checked that for $\Phi(x) = |x|^3$ the conclusion of that proposition, and not only its proof, fails: on the level set $q = \frac{1}{2}$ the vector $Y^+_{N,1/2}$ has $\mathbb{E}|Y^+|^3 = 1.351213932\ldots$ for $N = 4$, $1.348580288\ldots$ for $N = 5$ and $1.347243942\ldots$ for $N = 6$, while the two-block vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$ on the same level set has $\mathbb{E}|S|^3 = \sqrt{2} = 1.414213562\ldots$; the corner beats the claimed maximiser. The Cauchy–Schwarz step of Lemma 7.1 runs the other way — it bounds $\mathbb{E}|X|^3$ by $(\mathbb{E}X^4)^{1/2}$, which is *not* attained at Y^+ — and its equality case is the corner, which is the conjectured extremiser. Accordingly [1] proves its general-n third-moment theorem by a different route (repeated averaging of the extreme squared coefficients, with the small-ball estimate of its Lemma 6.1), reaching the constant $\frac{3(5\sqrt{2}-7)}{16}$ of (4). One incidental point: the proof of Theorem 1.2 in [1] begins "Fix $N \geq n$, append zeros to the coefficient vector of S, and apply [its Proposition 4.1]", so the padding invariance of Proposition 2.2 is a step that paper takes too, and also leaves unstated.

Remark 7.9 (measured: what remains in dimension four, and what it would cost). Outside the formal development; this is a report on the part of the sphere of $\mathbb{R}^4$ that none of the three regions covers, namely $q < \frac{1}{2}$ with no dominating coordinate. Reduce by signs and permutations (Proposition 2.2) to $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$ and put $u = a_1 - a_2 - a_3 - a_4$ and $v = a_1 - a_2 - a_3 + a_4$. On the three cells of §6,

$$T_3(a) = 3a_1 - 2a_1^3 - \tfrac{1}{4}(u_-)^3 - \tfrac{1}{4}(v_-)^3, \qquad x_- := \min(x, 0),$$

the three cases being $u \geq 0$, then $u \leq 0 \leq v$, then $v \leq 0$. Hence, with $C = 6 - 4\sqrt{2}$, the deficit $D(a) = F_4 - T_3(a) - C\Delta_4(a)$ is

$$D(a) = \Big(a_1 - \tfrac{1}{\sqrt{2}}\Big)^2 Q_C(a_1) + 2C\big(a_2^2a_3^2 + a_2^2a_4^2 + a_3^2a_4^2\big) + \tfrac{1}{4}(u_-)^3 + \tfrac{1}{4}(v_-)^3,$$

where Q_C is the second factor of Lemma 5.3 — that lemma applies here because $C = R(4,2)$ makes $\sqrt{2} + \frac{C}{2} = F_4 + \frac{C}{4}$ — so the deficit is the spike factorisation, plus one positive quartic term and two negative cubic corrections. (Both displays were checked against the defining 2^4-term average at 20 000 random sorted unit vectors, worst discrepancy below $2 \cdot 10^{-15}$; a 4000-restart minimisation of the quotient over the sphere of $\mathbb{R}^4$ returns $6 - 4\sqrt{2}$ to $7 \cdot 10^{-15}$.) The obstruction to a certificate is visible in that display. What the minimisation says is that $D \geq 0$ on the sorted sphere — that assertion *is* $\Phi_4 = 6 - 4\sqrt{2}$, and is what a certificate would have to establish — with equality at *two* points: the corner w_4, where $\Delta_4 = \frac{1}{4}$, and the flat vector, where $\Delta_4 = 0$ and the quotient tends to $\frac{3}{4}$. A certificate on the leftover region is therefore tight at both, i.e. lives on a proper face of the cone of sums of squares. Counted, without running a semidefinite program: carrying $\sqrt{2}$ as a variable s with $s^2 = 2$, the target is a form of degree 5 in the five variables $a_1, \ldots, a_4, s$, so a degree-6 sum-of-squares certificate needs a 35×35 Gram matrix per sign cell (the degree-3 monomials in five variables), with the face defined over $\mathbb{Q}(\sqrt{2})$; rationalising instead through

$$12AB - A^2 - 4B^2 = (A - CB)\big((6 + 4\sqrt{2})B - A\big), \qquad A = F_4 - T_3(a), \quad B = \Delta_4(a),$$

whose left-hand side has rational coefficients and whose nonnegativity gives $A \geq CB$ wherever $A/B < 6+4\sqrt{2}$ (measured: $A/B \leq \frac{3}{4}$ on the sphere, the value $\frac{3}{4}$ being approached only at the flat vector), raises the degree to 8 and the Gram to 70×70. Finally, one measurement that would simplify the picture if it were a theorem: the minimum of q over the spike region appears to be $\frac{1}{2}$ — a 400-restart constrained minimisation returns 0.500000000 for $n = 3, \ldots, 7$. The value is attained along a curve rather than at isolated points: for any $x \in [\frac{1}{2}, \frac{2}{3}]$ let a have one coordinate $\sqrt{x}$ and two further coordinates $\alpha \geq \beta \geq 0$ determined by $\alpha + \beta = \sqrt{x}$ and $\alpha^2 + \beta^2 = 1 - x$; then a is a unit spike vector, its defining inequality (5) holding with equality, and

$$q(a) = x^2 + (1 - x)^2 - \tfrac{1}{2}(2x - 1)^2 = \tfrac{1}{2}$$

identically in x. Its endpoints, at $x = \frac{1}{2}$ and at $x = \frac{2}{3}$, are

$$\Big(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0, \ldots\Big) \qquad \text{and} \qquad \Big(\sqrt{\tfrac{2}{3}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, 0, \ldots\Big).$$

That nothing goes *below* $\frac{1}{2}$ is the measurement, and it was re-checked here by a scan organised around the Lagrange conditions: at a constrained minimiser the tail coordinates satisfy $4b^3 = 2\lambda b + \mu$, a cubic with no quadratic term, so its roots sum to 0 and at most two of them are positive; scanning that two-block family on a fine grid in x for $n \leq 7$ finds nothing below $\frac{1}{2}$. Were $q \geq \frac{1}{2}$ to hold on the spike region at every n, that region would sit inside the fourth-moment region and the leftover would have a single tight point. We record it as a measurement; it is not proved here.

8. Verification

Every statement asserted above as a proposition, lemma, corollary or theorem — Propositions 2.2, 3.1, 4.1, 4.2, 5.4, 5.7, 5.10, 7.2 and 7.6, Lemmas 5.2, 5.3 and 7.1, Corollary 4.8, and Theorems 4.3, 4.4, 5.5, 5.8, 5.9, 7.3, 7.4 and 7.5 — is checked, in the form stated, by the kernel of the Lean 4 proof assistant [9] over the Mathlib library [10]. The development is seven modules and about 2164 lines carrying 177 declarations: definitions of T_3, q, Δ_n, the flat vector and $C_\star$; the two telescoping identities and the closed forms; the slope bounds and the chord theorems; the sign involution, the moment identity, the factorisation and the spike theorems; the padding bijection with the permutation and sign invariances and the flat-cone statement; the fourth-moment identity, the Cauchy–Schwarz step and the region theorems of §7; and a short module that records, without re-proving anything, which declaration corresponds to which statement above. It contains no `sorry` and declares no axiom of its own. A print of the axiom set of each of the 21 headline declarations of the first five modules — of the eleven further ones carrying their auxiliary statements, among them the two telescopings of (6) and the factorisation of Lemma 5.3, and of 31 declarations of the two modules added in this version, namely every theorem and every control they state together with the lemmas those rest on — returns `propext`, `Classical.choice` and `Quot.sound` and nothing else — in particular no `sorryAx`, and no compiled-evaluation axiom: there is no use of `native_decide`, no external solver, no certificate file, and no floating-point or interval arithmetic anywhere. This is not an accident of style but of content, since none of the theorems is a finite check: the expectation T_3 is the exact 2^n-term average of Definition 2.1, so what the kernel verifies is the definition and not a numerical surrogate, and the two rational brackets $1.4142135 < \sqrt{2} < 1.4142136$ and $1.7320508 < \sqrt{3} < 1.7320509$ are themselves theorems about $\sqrt{2}$ and $\sqrt{3}$. Total elaboration was about 0.4 core-hours for the first five modules; the two added in this version elaborated in 13 and 26.5 seconds on an unloaded machine, those being the times recorded when they landed. The axiom prints just described were re-run against the compiled development while this version was written.

Two points of form. The five sharpness statements — the last sentence of Theorem 4.4, Theorems 5.8 and 7.4, and the last sentences of Theorems 5.9 and 7.5 — are verified in the two-part form in which an infimum is proved exact: a lower bound valid on the whole region, together with an explicit vector attaining it. And Corollaries 4.6 and 5.6 are not separately named declarations: the first is Theorem 4.3 at $k = 2$ together with $F_2 = \sqrt{2}$, and it occurs in exactly that form inside the verified proofs of Theorems 5.5 and 7.3; the second is the observation that a vector with at most two nonzero coordinates satisfies Definition 5.1, which is one line and is not formalised. Corollary 4.8, by contrast, is a named declaration, stated there for an arbitrary k-element support and arbitrary signs.

The following are deliberately outside the formal development and are labelled where they appear: the reformulation $\mathbb{E}|S_n|^3 = (2n-(n \bmod 2))\mathbb{E}|S_n|$ of Remark 3.2, for which only the third-moment half is formalised; the asymptotics and the decimal column of Remark 4.7; the whole of Remark 6.1, both the exact table and the sphere minimisation behind it; and the whole of Remarks 7.7, 7.8 and 7.9 — the measured thresholds, the comparison with the fixed-q envelope of [1], and the sign-cell formulas, deficit identity and counted certificate sizes for the leftover region of $\mathbb{R}^4$. (The zero-padding invariance, which the first version of this paper listed here, is now inside the development, as Proposition 2.2.) As an independent check outside the kernel, the values of [1] that this paper touches — $\mathbb{E}|\overline{S}_3|^3 = \frac{5}{2\sqrt{3}}$ and both endpoints of (4), together with the quotient at the dimension-three vector, which [1] states to be the upper endpoint — were recomputed from the definitions of [1] by exact 2^n summation before any claim above was made, and they agree; the quotient came out as $0.174972663605\ldots = C_\star$. ([1] prints no decimal expansions; every decimal in this paper is ours.)

The source of the development is currently held in a private repository: repository reference to be added at submission.

On the reference list. Entries [1] and [2] were read here in full from their arXiv LaTeX e-prints, and [7, 8] were consulted live. The four classical records [3, 4, 5, 6] are cited for attribution and context only and were not read here; their bibliographic data, and the journal data of [2], were verified at zbMATH and Crossref on 3 September 2026.

References

[1] P. Gao and J. Qian, *Fourth-Moment Geometry of Rademacher Sums*, arXiv:2608.17802v1 [cs.LG], 18 August 2026. Read here in full from the arXiv LaTeX e-print of v1; all quotations above are from that source. Numbers of its statements are those of that e-print as it compiles, and are quoted here version-pinned to v1: its Theorem 1.4 (dimension-free stability at the third moment) is (2) together with (3), (4) and the dimension-three sentence following them; Lemma 6.1 is the small-ball estimate it attributes to Dzindzalieta and Götze; Lemma 6.4 is the dimension-three lemma; Remark 6.5 and Conjecture 6.6 are the two passages quoted verbatim in §1; its equation (1) is the fourth-moment identity $\mathbb{E}S^4 = 3 - 2q$ of Lemma 7.1; and its Theorem 1.2 (fixed-fourth-moment extremal principle) with its finite-dimensional form Proposition 4.1, together with the opening sentence of its §6, are what Remark 7.8 discusses. Re-fetched from the arXiv API on 3 September 2026: still v1, no journal reference, no DOI.

[2] J. Jakimiuk, *Stability of Khintchine inequalities with optimal constants between the second and the p-th moment for $p \geq 3$*, Bernoulli **32** (2026), no. 3, 2524–2542, doi:10.3150/25-BEJ1965; preprint arXiv:2503.07001v2 [math.PR], 2025. The arXiv version is the one read here, and its statement numbers are the ones used above; the journal data are from Crossref and zbMATH (Bernoulli has no page numbers in Crossref; the page range is zbMATH's).

[3] A. Khintchine, *Über dyadische Brüche*, Math. Z. **18** (1923), 109–116; Zbl 49.0132.01. Cited for attribution only.

[4] U. Haagerup, *The best constants in the Khintchine inequality*, Studia Math. **70** (1981), no. 3, 231–283, doi:10.4064/sm-70-3-231-283. Cited for attribution only.

[5] S. J. Szarek, *On the best constants in the Khinchin inequality*, Studia Math. **58** (1976), no. 2, 197–208, doi:10.4064/sm-58-2-197-208. Cited for attribution only.

[6] F. L. Nazarov and A. N. Podkorytov, *Ball, Haagerup, and distribution functions*, in: V. P. Havin and N. K. Nikolski (eds.), Complex Analysis, Operators, and Related Topics: The S. A. Vinogradov Memorial Volume, Operator Theory: Advances and Applications **113**, Birkhäuser, Basel, 2000, 247–267, doi:10.1007/978-3-0348-8378-8_21. Cited for attribution only.

[7] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, entry A100071 (`oeis.org/A100071`), consulted 3 September 2026.

[8] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, entry A107233 (`oeis.org/A107233`), consulted 3 September 2026.

[9] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[10] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.