COUNTING THE MARKINGS OF A GRAPH THAT REALIZE RACKS AND QUANDLES

KOREA SUPERINTELLIGENCE LABS

Abstract. A *marking* of a graph Γ with vertex set V is a map $R\colon V \to \operatorname{Aut}\Gamma$; it makes V into a right quasigroup, and by a theorem of Ta that right quasigroup is a rack exactly when $R_{R_v(w)} = R_v R_w R_v^{-1}$ for all $v, w \in V$. Let $\mu_{\mathrm{rack}}(\Gamma)$ and $\mu_{\mathrm{qnd}}(\Gamma)$ be the number of markings realizing a rack, respectively a quandle. Ta tabulated these numbers for complete graphs, star graphs and cycles on few vertices and left five entries of the table undetermined. We determine four of them:

$$\mu(K_5) = (1708, 404), \qquad \mu(K_6) = (36538, 6658),$$
$$\mu(K_{1,5}) = (7628, 1708), \qquad \mu(K_{1,6}) = (223378, 36538),$$

where $\mu = (\mu_{\mathrm{rack}}, \mu_{\mathrm{qnd}})$. The blind sweep over all $|\operatorname{Aut}\Gamma|^{|V|}$ markings is 10^{20} maps in the largest of these cases; we replace it by a search that propagates the forcing $R_{R_v(w)} = R_v R_w R_v^{-1}$ and whose tree has at most $2.7 \cdot 10^5$ nodes. We also record an identity that is visible in Ta's own printed data but is not remarked on there, $\mu_{\mathrm{qnd}}(K_{1,n-1}) = \mu_{\mathrm{rack}}(K_{n-1})$, and explain it: the row of the centre of a star is a *shift*, and racks on a set are in bijection with pairs consisting of a quandle and a compatible shift. The identity relates two of the previously undetermined entries to two others, so the four new values check each other. All statements below are machine-checked in Lean 4.

1. Introduction

A *rack* is a set with a self-distributive, right-invertible binary operation, and a *quandle* is a rack whose operation is idempotent. Both arose from knot theory [5, 1], and both are, at bottom, combinatorial objects: a rack structure on a finite set V is a map R from V to the symmetric group $\mathrm{Sym}(V)$ satisfying one equation.

Ta [9] studies these structures through graphs. If Γ is a (di)graph with vertex set V, a *marking* of Γ is a map $R\colon V \to \operatorname{Aut}\Gamma$, written $R_v := R(v)$; it realizes the right quasigroup $V_R^\Gamma = (V, R)$ with operation $v \triangleleft w = R_w(v)$, and it is a *q-marking* if $R_v(v) = v$ for every v. The central result of [9] that we use, its Theorem 5.1, says that V_R^Γ is a rack (respectively a quandle) if and only if R is a magma homomorphism from V_R^Γ to $\mathrm{Conj}(\operatorname{Aut}\Gamma)$, the automorphism group with its conjugation operation. Unfolded, this is the single equation

$$R_{R_v(w)} \;=\; R_v\, R_w\, R_v^{-1} \qquad \text{for all } v, w \in V. \tag{1}$$

Writing $\mu_{\mathrm{rack}}(\Gamma)$ for the number of markings of Γ that realize racks and $\mu_{\mathrm{qnd}}(\Gamma)$ for the number of q-markings that realize quandles, [9] computes the pair $\mu(\Gamma) = (\mu_{\mathrm{rack}}(\Gamma), \mu_{\mathrm{qnd}}(\Gamma))$ for three families of graphs. The method described there is a direct search, in GAP with the GRAPE package [2, 3], over all $|\operatorname{Aut}\Gamma|^{|V|}$ maps R. It produces Table 1, which is Table 1 of [9]; the entries printed as "?" are exactly those the search did not reach.

Table 1. Table 1 of [9]: $(\mu_{\mathrm{rack}}, \mu_{\mathrm{qnd}})$ for the complete graph K_n, the star $K_{1,n-1}$ and the cycle C_n, indexed by the number n of vertices.

n	0	1	2	3	4	5	6	7
K_n	$(1,1)$	$(1,1)$	$(2,1)$	$(13,5)$	$(114,36)$	?	?	?
$K_{1,n-1}$	—	$(1,1)$	$(2,1)$	$(4,2)$	$(31,13)$	$(390,114)$	?	?
C_n	—	—	—	$(13,5)$	$(32,8)$	$(41,7)$	$(108,13)$	$(113,9)$

The five undetermined entries are $\mu(K_5)$, $\mu(K_6)$, $\mu(K_7)$, $\mu(K_{1,5})$ and $\mu(K_{1,6})$; they are still printed as "?" in the published version of [9]. Filling them in is asked for there. The two invariants are introduced in [9] as an application of its Theorem 5.1, which answers a question of Bardakov recorded as [9, Problem 1.1] ("Under what conditions does a marked graph realize a rack or a quandle?"), and the list of open problems that closes [9] includes [9, Problem 8.2], "Add more entries to Table 1."

Results. *Four entries* (Section 4). We determine

$$\mu(K_5) = (1708, 404), \qquad \mu(K_6) = (36538, 6658),$$
$$\mu(K_{1,5}) = (7628, 1708), \qquad \mu(K_{1,6}) = (223378, 36538),$$

where $K_{1,5}$ has 6 vertices and $K_{1,6}$ has 7, so that the last two are the two "?" entries of the star row of Table 1. Each value is an exact cardinality, not a sample: it is the number of elements of the set defined by (1). The reason the values were out of reach is arithmetic — the sweep over all maps $V \to \operatorname{Aut}\Gamma$ is $(5!)^5 \approx 2.5 \cdot 10^{10}$ maps at K_5 and $(6!)^7 \approx 1.0 \cdot 10^{20}$ at $K_{1,6}$ — and the reason they are now in reach is that (1) is a forcing rule: once the rows R_v and R_w are known, the row $R_{R_v(w)}$ is determined. Propagating that rule turns the sweep into a search tree with 2 142 nodes at K_5 and 265 631 nodes at $K_{1,6}$ (Section 3).

An identity in the printed data (Section 5). Read the first two rows of Table 1 together:

$$\mu_{\text{qnd}}(K_{1,n-1}) \;=\; \mu_{\text{rack}}(K_{n-1}). \tag{2}$$

On the data [9] prints this says $2 = 2$, $13 = 13$ and $114 = 114$; on the entries it leaves open it says $1708 = 1708$ and $36538 = 36538$. The source does not remark on it. We prove the structure behind it. For $n \geq 3$ every automorphism of a star fixes the centre, and (1) forces the centre's row R_0 to commute with every other row and to satisfy $R_{R_0(u)} = R_u$; a q-marking of a star is therefore a quandle on the leaves together with such a *shift*. Racks, in turn, are in bijection with pairs (quandle, compatible shift), by the map sending a rack R to $(R\pi^{-1}, \pi)$ where $\pi(v) = R_v(v)$ is the kink map. The bijection itself is folklore — the kink map is the standard canonical automorphism of a rack [4] — but it is what (2) is made of, and it makes the two star values above consequences of the two complete-graph values, computed independently.

Controls (Section 3). Every one of the twelve entries [9] prints is re-derived here from its own definitions, and this is not a formality: the opposite reading $R_{R_v(w)} = R_v^{-1} R_w R_v$ of (1) reproduces the printed pair $(13, 5)$ at K_3 and only fails at K_4, where it gives $(112, 34)$ against the printed $(114, 36)$. Agreement on the small entries is therefore no evidence at all about the convention, and the check has to run over the whole printed table before any new value is quoted.

The one entry of Table 1 we do not settle is $\mu(K_7)$; see Remark 4.5.

2. Markings, racks and quandles

Throughout, V is a finite set, $\mathrm{Sym}(V)$ is the group of permutations of V acting on the left, and Γ is a graph with vertex set V, with $\operatorname{Aut}\Gamma \leq \mathrm{Sym}(V)$ its automorphism group. We follow the definitions of [9].

Definition 2.1 ([9, Def. 2.1])**.** A *magma* is a pair (V, R), written V_R, where R maps V to the set of functions $V \to V$. We write $R_v := R(v)$ and call it a *right-multiplication map*, and we write $v \triangleleft w := R_w(v)$. The magma V_R is a *right quasigroup* if $R(V) \subseteq \mathrm{Sym}(V)$.

Definition 2.2 ([9, Def. 2.5])**.** A *magma homomorphism* $V_R \to W_T$ is a map $\varphi\colon V \to W$ with $\varphi R_v = T_{\varphi(v)}\varphi$ for all $v \in V$.

A right quasigroup V_R is a *rack* if every R_v is an endomorphism of V_R, i.e. if $R_v R_w = R_{R_v(w)} R_v$ for all v, w, and a rack is a *quandle* if $R_v(v) = v$ for all v. In the $\triangleleft$ notation the rack condition is the self-distributive law

$$(x \triangleleft w) \triangleleft v \;=\; (x \triangleleft v) \triangleleft (w \triangleleft v),$$

and the quandle condition is idempotency $v \triangleleft v = v$.

Definition 2.3 ([9, Def. 3.3])**.** A *marking* of Γ is a map $R\colon V \to \operatorname{Aut}\Gamma$; the right quasigroup $V_R^\Gamma := (V, R)$ is said to be *realized* by the marked graph (Γ, R). A marking is a *q-marking* if $R_v(v) = v$ for all $v \in V$.

Since $\operatorname{Aut}\Gamma \subseteq \operatorname{Sym}(V)$, a marking always realizes a right quasigroup. The condition for it to realize a rack is the content of [9, Thm. 5.1]: V_R^Γ is a rack (resp. a quandle, for a q-marking) if and only if R is a magma homomorphism $V_R^\Gamma \to \operatorname{Conj}(\operatorname{Aut}\Gamma)$, where $\operatorname{Conj}(G)$ is the group G with $C_g(h) = ghg^{-1}$. Unfolding the definition of a magma homomorphism with $\varphi = R$ and $T = C$ gives exactly (1).

Definition 2.4. For a finite graph Γ,

$$\mu_{\text{rack}}(\Gamma) = \#\{\, R\colon V \to \operatorname{Aut}\Gamma \text{ satisfying (1)} \,\},$$
$$\mu_{\text{qnd}}(\Gamma) = \#\{\, R \text{ as above with } R_v(v) = v \text{ for all } v \,\},$$

and $\mu(\Gamma) = (\mu_{\text{rack}}(\Gamma), \mu_{\text{qnd}}(\Gamma))$. Both sets are subsets of the finite set of maps $V \to \operatorname{Aut}\Gamma$, so both numbers are finite.

Two elementary facts are used constantly, and both are proved rather than assumed.

Proposition 2.5. *Let X be a set and $R\colon X \to \operatorname{Sym}(X)$. Then*

$$\bigl(\forall v, w\colon R_{R_v(w)} = R_v R_w R_v^{-1}\bigr) \iff \bigl(\forall v, w, x\colon R_v(R_w(x)) = R_{R_v(w)}(R_v(x))\bigr).$$

Proof. Left to right, apply both sides of $R_{R_v(w)} = R_v R_w R_v^{-1}$ to $R_v(x)$. Right to left, fix v, w and $y \in X$ and put $x = R_v^{-1}(y)$; the pointwise identity then reads $R_v(R_w(R_v^{-1}(y))) = R_{R_v(w)}(y)$, which is the asserted equality of permutations evaluated at y. $\square$

The point of Proposition 2.5 is that the inverse-free form is the one a machine can test in $O(|V|^3)$ table lookups, while the conjugation form is the one that forces rows; both readings of [9, Thm. 5.1] are used below and they must be the same condition.

Lemma 2.6. *Let $n \geq 3$ and let $K_{1,n-1}$ be the star on n vertices with centre 0. Then every $\sigma \in \operatorname{Aut}(K_{1,n-1})$ satisfies $\sigma(0) = 0$; consequently $\operatorname{Aut}(K_{1,n-1})$ is the symmetric group on the $n-1$ leaves.*

Proof. A leaf has degree 1 and the centre has degree $n - 1 \geq 2$, so an automorphism cannot send the centre to a leaf. Concretely, if $\sigma(0) \neq 0$ then σ maps the two distinct leaves 1 and 2 both to the centre — each of them is adjacent to 0 only, and $\sigma(0)$ is a leaf — contradicting injectivity. Conversely every permutation of the leaves preserves adjacency. $\square$

Remark 2.7. For the complete graph $\operatorname{Aut} K_n = \operatorname{Sym}(V)$, so a marking of K_n is an arbitrary map $V \to \operatorname{Sym}(V)$ and (1) is nothing but the rack axiom. Hence $\mu_{\text{rack}}(K_n)$ is the number of rack structures on a fixed n-element set and $\mu_{\text{qnd}}(K_n)$ the number of quandle structures on it. The number of *isomorphism classes* of racks and of quandles of order at most 7 is known [8, A181770, A181769], so the labelled counts are in principle recoverable from that classification together with the automorphism orders, by $\sum_{\text{classes}} n!/|\operatorname{Aut}|$. We are not aware of any place where the labelled counts themselves are printed; the sequences $1, 2, 13, 114, 1708, 36538$ and $1, 1, 5, 36, 404, 6658$ of Section 4 returned no match in the OEIS when queried on 2026-08-23.

3. Counting markings

3.1. **The forcing rule and the search.** The equation (1) is deterministic in the following sense: it does not merely constrain a marking, it *computes* rows from other rows. If the rows R_v and R_w of a marking are known, then the row indexed by the vertex $R_v(w)$ is known too, namely $R_v R_w R_v^{-1}$. This is the whole of the algorithm.

Call a *partial marking* an assignment of elements of $\operatorname{Aut}\Gamma$ to some of the vertices, the others being unassigned. The search maintains a partial marking and does three things.

- *Propagation.* For every ordered pair (v, w) of assigned vertices such that the vertex $R_v(w)$ is unassigned, assign it the permutation $R_v R_w R_v^{-1}$. Repeat until nothing changes; $|V|$ sweeps suffice, since each sweep that changes anything fills at least one of the $|V|$ rows. Propagation only ever fills an empty row, never overwrites an assigned one.
- *Pruning.* For every ordered pair (v, w) of assigned vertices such that $R_v(w)$ is *also* assigned, test $R_{R_v(w)} = R_v R_w R_v^{-1}$ and discard the partial marking if the test fails. This is a necessary condition on every total marking extending the partial one, so no solution is lost.
- *Branching.* Choose the least unassigned vertex and branch over the $|\operatorname{Aut}\Gamma|$ possible values of its row.

At the leaves, all rows are assigned; there the pruning test is precisely (1), so a leaf that survives is a marking realizing a rack. Distinct branches assign different permutations to the vertex they branch on, so no marking is produced twice. These three statements — completeness, soundness and irredundancy — are what turn the output of a program into a statement about a cardinality.

Proposition 3.1. *Let Γ be a graph on n vertices. The search above outputs each marking of Γ realizing a rack exactly once, and no other table. Consequently $\mu_{\mathrm{rack}}(\Gamma)$ is the length of its output and $\mu_{\mathrm{qnd}}(\Gamma)$ the number of outputs satisfying $R_v(v) = v$ for all v.*

Proof sketch. Completeness is an induction on the number of unassigned vertices. Let R be a marking realizing a rack and let P be a partial marking whose assigned rows agree with R. Every pruning test performed on P is an instance of (1) for R, hence passes; every propagation step fills an empty row of P with the value $R_v R_w R_v^{-1}$, which by (1) is the row of R at that vertex, so the enlarged partial marking again agrees with R; and the branch that assigns to the branching vertex its row in R — one of the $|\operatorname{Aut}\Gamma|$ candidates, since R is a marking — again agrees with R. So R survives to a leaf. Soundness is the observation above that the full pruning test on a total table is the self-distributive law, together with the check that each row is an automorphism. Irredundancy: two distinct branches from the same node differ at the branching vertex, and neither propagation nor later branching changes an assigned row. A list containing every solution, only solutions, and no repetitions has length equal to the cardinality of the solution set. $\square$

3.2. **The blind sweep, and the convention.** The search of Proposition 3.1 shares its definitions, but nothing else, with the method described in [9]: enumerate all $|\operatorname{Aut}\Gamma|^n$ maps $V \to \operatorname{Aut}\Gamma$ and filter. For $\Gamma = K_n$ the blind sweep is $(n!)^n$ maps, which is $331\,776$ at $n = 4$ and $2.5 \cdot 10^{10}$ at $n = 5$; it is a genuinely independent route to the small entries.

Proposition 3.2. *The enumeration of all $(n!)^n$ tables of n permutations of an n-element set is complete and repetition-free, so filtering it by (1) computes μ_{rack} and μ_{qnd} as well. Carried out over the $331\,776$ tables at $n = 4$ and the $1\,296$ at $n = 3$, it returns $\mu_{\mathrm{rack}}(K_3) = 13$, $\mu_{\mathrm{qnd}}(K_3) = 5$, $\mu_{\mathrm{rack}}(K_4) = 114$, $\mu_{\mathrm{qnd}}(K_4) = 36$, in agreement with the search.*

The agreement is a check on the completeness proof of Proposition 3.1: the two enumerations have no pruning, no propagation and no branching in common. The blind sweep stops at $n = 4$, which is exactly where Table 1 turns into question marks.

The same sweep carries the family's main negative control. Definition 2.5 of [9] can be read with the conjugating permutation on either side, and the wrong reading is not detectable on small graphs.

Proposition 3.3. *Let $\mu^{\mathrm{op}}_{\mathrm{rack}}$ and $\mu^{\mathrm{op}}_{\mathrm{qnd}}$ be defined as above but with (1) replaced by $R_{R_v(w)} = R_v^{-1} R_w R_v$. Then*

$$\left(\mu^{\mathrm{op}}_{\mathrm{rack}}(K_3), \mu^{\mathrm{op}}_{\mathrm{qnd}}(K_3)\right) = (13, 5) = \left(\mu_{\mathrm{rack}}(K_3), \mu_{\mathrm{qnd}}(K_3)\right), \qquad \left(\mu^{\mathrm{op}}_{\mathrm{rack}}(K_4), \mu^{\mathrm{op}}_{\mathrm{qnd}}(K_4)\right) = (112, 34),$$

while $\left(\mu_{\mathrm{rack}}(K_4), \mu_{\mathrm{qnd}}(K_4)\right) = (114, 36)$. In particular the two readings agree on the $n = 3$ entry printed in [9] and disagree on the $n = 4$ entry.

So the $n = 4$ entry of Table 1 is the first that pins the convention, and a computation that agrees with [9] only up to $n = 3$ has demonstrated nothing. This is why the verification below runs over every printed entry rather than a sample.

3.3. **Non-degeneracy.** A counting problem stated with quantifiers can be satisfied vacuously; the following witnesses show that no clause of the definitions is empty or redundant.

Proposition 3.4. (1) *The table on K_3 with $R_0 = (0\,1\,2)$ and $R_1 = R_2 = \mathrm{id}$ is a marking — every permutation is an automorphism of K_3 — but does not satisfy* (1)*: from $R_0(2) = 0$ one gets $R_0 = R_0R_2R_0^{-1} = \mathrm{id}$.*

(2) *The table on K_2 with both rows the transposition satisfies* (1) *but is not a q-marking; it is the single marking separating $\mu_{\mathrm{rack}}(K_2) = 2$ from $\mu_{\mathrm{qnd}}(K_2) = 1$.*

(3) *The permutation exchanging the centre of $K_{1,3}$ with a leaf gives a table that is a marking of K_4 and not a marking of the star, so the requirement $R_v \in \operatorname{Aut}\Gamma$ is not vacuous.*

(4) *$\mu_{\mathrm{rack}}(K_5) \notin \{1707, 1709\}$, $\mu_{\mathrm{rack}}(K_6) \notin \{36537, 36539\}$, $\mu_{\mathrm{rack}}(K_{1,5}) \notin \{7627, 7629\}$ and $\mu_{\mathrm{rack}}(K_{1,6}) \notin \{223377, 223379\}$; each new value of Section 4 is refuted on both sides.*

3.4. **The published entries.**

Proposition 3.5. *All twelve entries printed in Table 1 are correct:*

$$\begin{aligned}
\mu(K_n) &= (1,1),\ (1,1),\ (2,1),\ (13,5),\ (114,36) && (n = 0,\dots,4),\\
\mu(K_{1,n-1}) &= (4,2),\ (31,13),\ (390,114) && (n = 3,4,5),\\
\mu(C_n) &= (13,5),\ (32,8),\ (41,7),\ (108,13),\ (113,9) && (n = 3,\dots,7).
\end{aligned}$$

Each of these is computed from the definitions of Section 2 through the search of Proposition 3.1, and the five complete-graph and cycle entries with $n \le 4$ come additionally out of the blind sweep of Proposition 3.2. (The coincidence $\mu(C_3) = \mu(K_3)$ in Table 1 is not one: $C_3 = K_3$.)

4. Four new values

Theorem 4.1. $\mu_{\mathrm{rack}}(K_5) = 1708$ *and* $\mu_{\mathrm{qnd}}(K_5) = 404$.

Theorem 4.2. $\mu_{\mathrm{rack}}(K_6) = 36538$ *and* $\mu_{\mathrm{qnd}}(K_6) = 6658$.

Theorem 4.3. $\mu_{\mathrm{rack}}(K_{1,5}) = 7628$ *and* $\mu_{\mathrm{qnd}}(K_{1,5}) = 1708$*, where $K_{1,5}$ is the star on 6 vertices.*

Theorem 4.4. $\mu_{\mathrm{rack}}(K_{1,6}) = 223378$ *and* $\mu_{\mathrm{qnd}}(K_{1,6}) = 36538$*, where $K_{1,6}$ is the star on 7 vertices.*

Proof of Theorems 4.1–4.4. Each value is the cardinality of the set of maps $V \to \operatorname{Aut}\Gamma$ satisfying (1), respectively of its subset of q-markings. By Proposition 3.1 that cardinality is the number of leaves the search of Section 3 reaches, so the proof is the exhaustive execution of a finite search, and what has to be reported is its size. The four searches are those of Table 2: in the largest, $K_{1,6}$, every branching is over the $6! = 720$ automorphisms of the star, and the tree has 265 631 nodes and 30 422 160 candidate tests in all. Both coordinates come from one traversal, the quandle count being the number of surviving tables with $R_v(v) = v$ for all v. The executions are carried out and checked inside a proof assistant; see Section 6. □

Filling in Theorems 4.1–4.4, the first two rows of Table 1 become Table 3.

Two of the four new values check the other two. The searches for K_5 and for $K_{1,5}$ run over different graphs, different automorphism groups and different numbers of vertices, and they are executed separately; nevertheless $\mu_{\mathrm{qnd}}(K_{1,5}) = 1708 = \mu_{\mathrm{rack}}(K_5)$. The same happens one step further along: $\mu_{\mathrm{qnd}}(K_{1,6}) = 36538 = \mu_{\mathrm{rack}}(K_6)$. This is (2), and Section 5 explains why it must happen.

Remark 4.5. The remaining undetermined entry of Table 1 is $\mu(K_7)$. Two independent implementations of the search in C agree on $\mu(K_7) = (1164056, 152900)$. We record this as a computation and not as a theorem: the tree has 1 330 371 nodes and 838 227 600 candidate tests, roughly 27 times the largest execution we verified, and we have not run it under the verification regime of Section 6.

Table 2. The size of each search, against the blind sweep over all $|\operatorname{Aut}\Gamma|^n$ maps that [9] describes. "Tests" counts candidate rows tried, that is, branches attempted. The sweep column is $|\operatorname{Aut}\Gamma|^n$ rounded to two significant figures; here $\operatorname{Aut} K_5$ and $\operatorname{Aut} K_{1,5}$ have order $5! = 120$ and $\operatorname{Aut} K_6$ and $\operatorname{Aut} K_{1,6}$ have order $6! = 720$, so the four entries are $120^5 = 2.488 \cdot 10^{10}$, $120^6 = 2.986 \cdot 10^{12}$, $720^6 = 1.393 \cdot 10^{17}$ and $720^7 = 1.003 \cdot 10^{20}$.

Γ	n	blind sweep	search nodes	tests
K_5	5	$(5!)^5 \approx 2.5 \cdot 10^{10}$	2 142	52 080
$K_{1,5}$	6	$(5!)^6 \approx 3.0 \cdot 10^{12}$	9 688	247 200
K_6	6	$(6!)^6 \approx 1.4 \cdot 10^{17}$	43 655	5 124 240
$K_{1,6}$	7	$(6!)^7 \approx 1.0 \cdot 10^{20}$	265 631	30 422 160

Table 3. The first two rows of Table 1 completed; the new values are in bold.

n	0	1	2	3	4	5	6	7
K_n	$(1,1)$	$(1,1)$	$(2,1)$	$(13,5)$	$(114,36)$	$(\mathbf{1708},\mathbf{404})$	$(\mathbf{36538},\mathbf{6658})$	?
$K_{1,n-1}$	—	$(1,1)$	$(2,1)$	$(4,2)$	$(31,13)$	$(390,114)$	$(\mathbf{7628},\mathbf{1708})$	$(\mathbf{223378},\mathbf{36538})$

5. Racks are quandles with a shift

This section explains the identity (2). Throughout, X is any set — no finiteness is used — and a *rack on* X means a map $R\colon X \to \operatorname{Sym}(X)$ satisfying (1), a *quandle on* X a rack with $R_v(v) = v$ for all v.

Definition 5.1. Let Q be a quandle on X. A *shift* for Q is a permutation $\sigma \in \operatorname{Sym}(X)$ such that $\sigma Q_v = Q_v\sigma$ for every $v \in X$ and $Q_{\sigma(v)} = Q_v$ for every $v \in X$.

Theorem 5.2. *Let R be a rack on X and put $\pi(v) := R_v(v)$. Then π is a permutation of X, with inverse $v \mapsto R_v^{-1}(v)$; it commutes with every R_v; and the assignment*

$$R \longmapsto \big(v \mapsto R_v\pi^{-1},\ \pi\big)$$

is a bijection from the racks on X to the pairs (Q,σ) consisting of a quandle Q on X and a shift σ for Q; its inverse sends (Q,σ) to $v \mapsto Q_v\sigma$. Consequently the racks on X and the pairs (quandle on X, shift) are equinumerous.

Proof sketch. Everything follows from one observation: *if $R_v(w) = v$ then $R_w = R_v$.* Indeed (1) at (v,w) gives $R_v = R_{R_v(w)} = R_vR_wR_v^{-1}$, whence $R_w = R_v$.

Taking $w = v$ in (1) gives $R_{\pi(v)} = R_{R_v(v)} = R_vR_vR_v^{-1} = R_v$, so $R^{-1}_{\pi(v)}(\pi(v)) = R_v^{-1}(R_v(v)) = v$: the map $v \mapsto R_v^{-1}(v)$ is a left inverse of π. For the other side put $w = R_v^{-1}(v)$, so that $R_v(w) = v$; by the observation $R_w = R_v$ and therefore $\pi(w) = R_w(w) = R_v(w) = v$. Hence π is a bijection. It commutes with each R_v because, for $x \in X$,

$$\pi(R_v(x)) = R_{R_v(x)}(R_v(x)) = \big(R_vR_xR_v^{-1}\big)(R_v(x)) = R_v(R_x(x)) = R_v(\pi(x)).$$

Set $Q_v := R_v\pi^{-1}$. Then $Q_v(v) = R_v(R_v^{-1}(v)) = v$, so Q is idempotent. That Q satisfies (1) is a computation in which π^{-1}, being central for the image of R, is moved to the right past the other factors; the row index is identified by $R_{\pi^{-1}(w)} = R_w$, which is $R_{\pi(v)} = R_v$ read backwards, so that $Q_{Q_v(w)} = R_{R_v(\pi^{-1}(w))}\pi^{-1} = R_vR_wR_v^{-1}\pi^{-1} = Q_vQ_wQ_v^{-1}$. The same centrality gives $\pi Q_v = Q_v\pi$, and $Q_{\pi(v)} = R_{\pi(v)}\pi^{-1} = R_v\pi^{-1} = Q_v$, so π is a shift for Q.

Conversely let σ be a shift for a quandle Q and put $R_v := Q_v\sigma$. Then $R_v(w) = Q_v(\sigma(w))$, so by the quandle axiom and $Q_{\sigma(w)} = Q_w$,

$$R_{R_v(w)} = Q_{Q_v(\sigma(w))}\sigma = Q_vQ_{\sigma(w)}Q_v^{-1}\sigma = Q_vQ_wQ_v^{-1}\sigma,$$

while $R_v R_w R_v^{-1} = Q_v \sigma Q_w \sigma \sigma^{-1} Q_v^{-1} = Q_v \sigma Q_w Q_v^{-1} = Q_v Q_w Q_v^{-1} \sigma$ by centrality of σ; so R is a rack. Its kink map is $R_v(v) = Q_v(\sigma(v)) = \sigma(v)$, using $Q_{\sigma(v)} = Q_v$ and idempotency of Q at $\sigma(v)$. The two constructions are therefore mutually inverse: $Q_v \sigma \pi^{-1} = Q_v$ because $\pi = \sigma$, and $R_v \pi^{-1} \pi = R_v$. ☐

Theorem 5.2 is not new. The permutation π is the canonical automorphism, or kink map, of a rack; that it is an automorphism and that it and its inverse are central for every rack homomorphism is stated by Jackson [4], and the reading "a rack is a quandle precisely when $\pi = \mathrm{id}$" is standard. The packaging above is folklore made explicit. What is new is the use we make of it, which is the following reduction.

Lemma 5.3. *Let $n \geq 3$ and let R be a marking of the star $K_{1,n-1}$, with centre 0, satisfying* (1). *Then for all vertices u, v:*

$$R_0 R_v = R_v R_0, \qquad R_{R_0(u)} = R_u.$$

Proof. By Lemma 2.6 every row of R fixes the centre, so $R_v(0) = 0$ for all v. Taking $w = 0$ in (1) therefore gives $R_0 = R_{R_v(0)} = R_v R_0 R_v^{-1}$, which is the first identity. For the second, let x be any vertex and write $x = R_0(y)$, possible since R_0 is a permutation. Equation (1) at $(0, u)$, applied to y in the form of Proposition 2.5, reads $R_0(R_u(y)) = R_{R_0(u)}(R_0(y)) = R_{R_0(u)}(x)$, while the first identity gives $R_0(R_u(y)) = R_u(R_0(y)) = R_u(x)$. Hence $R_{R_0(u)}(x) = R_u(x)$ for every x. ☐

Lemma 5.3 says precisely that in a marking of a star the centre's row is a shift for the restriction of the marking to the leaves. Combining it with Lemma 2.6 and Theorem 5.2:

- a q-marking of $K_{1,n-1}$ realizing a quandle consists of a quandle Q on the leaf set, namely the restriction of the rows indexed by leaves, together with the centre's row $\sigma = R_0$, which is a shift for Q by Lemma 5.3 (all rows fix the centre, so all of them are permutations of the leaf set, and the q-marking condition at a leaf v says $Q_v(v) = v$, while at the centre it is automatic);
- pairs (quandle on the leaf set, shift) are in bijection with racks on the leaf set by Theorem 5.2;
- racks on an $(n-1)$-element set are exactly the markings of K_{n-1} realizing racks, by Remark 2.7.

That is the identity (2). We state what we have verified.

Proposition 5.4. $\mu_{\mathrm{qnd}}(K_{1,n-1}) = \mu_{\mathrm{rack}}(K_{n-1})$ *for $n = 3, 4, 5, 6, 7$, that is*

$$2 = 2, \qquad 13 = 13, \qquad 114 = 114, \qquad 1708 = 1708, \qquad 36538 = 36538.$$

The first three are entries printed in [9]*; in the fourth the right-hand side is one of the entries printed as "?", and in the fifth both sides are.*

Remark 5.5. The chain of bijections displayed above proves (2) for every $n \geq 3$ once the leaf set $\{1, \ldots, n-1\}$ of the star is identified with the vertex set $\{0, \ldots, n-2\}$ of K_{n-1}. In our formal development the two halves — the star reduction of Lemma 5.3 and the bijection of Theorem 5.2 — are proved in the stated generality, but the tables the counting layer manipulates are indexed by $\{0, \ldots, n-1\}$ and the re-indexing between the two conventions is not carried out there. This is why Proposition 5.4 is recorded as five verified instances rather than as a single theorem; the missing step is bookkeeping, not mathematics.

6. Verification

Every statement in this paper — the definitions of Section 2, the correctness of the search (Propositions 3.1 and 3.2), the twelve published entries (Proposition 3.5), the four new values (Theorems 4.1–4.4), the convention and non-degeneracy controls (Propositions 3.3 and 3.4), the bijection of Theorem 5.2, the star reduction of Lemma 5.3 and the instances of Proposition 5.4) — has been formalized and machine-checked in Lean 4 on top of its mathematical library [6, 7]. The counts μ_{rack} and μ_{qnd} are *defined* in that development as cardinalities of the sets described in Section 2 and never as the output of a program; the identification of those cardinalities with the length of the search's output list is a proved theorem, so the arguments in Sections 3 and 5 contain no appeal to evaluation. Evaluation enters only at the last step of each numerical statement, where a closed finite computation — for instance the traversal of the 265 631-node tree for $K_{1,6}$, about 13 minutes — is discharged by compiled evaluation inside the proof assistant. The formal sources, the C implementations used for the cross-checks of Remark 4.5, and the machine-checked statements corresponding to each numbered result above are available in the authors' repository; repository reference to be added at submission.

7. Questions

Question 7.1. What is $\mu(K_7)$? The value $(1164056, 152900)$ of Remark 4.5 is supported by two independent implementations but is not verified to the standard of Section 6; the obstruction is the cost of $8.4 \cdot 10^8$ candidate tests under a verified evaluator, not the search itself.

Question 7.2. The cycle row of Table 1 is printed in full up to $n = 7$, and the search reaches C_7 in a fraction of a second, so $\mu(C_n)$ for $n \geq 8$ is a column of the table that is cheap and that nobody has printed. Does it have a closed form? The printed values $13, 32, 41, 108, 113$ of $\mu_{\mathrm{rack}}(C_n)$ are far from monotone in any obvious pattern.

Question 7.3. Lemma 5.3 uses only that one vertex of the star is fixed by every automorphism and is adjacent to all others. For which graphs Γ does a distinguished vertex force the same shift structure, and hence an identity of the shape (2) relating $\mu_{\mathrm{qnd}}(\Gamma)$ to μ_{rack} of a smaller graph?

References

[1] R. Fenn and C. Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992), no. 4, 343–406.
[2] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.14.0, 2024. (The release cited in [9].)
[3] L. H. Soicher, *GRAPE, GRaph algorithms using PErmutation groups*, Version 4.9.2, 2024. GAP package, `https://gap-packages.github.io/grape`. (The release cited in [9].)
[4] N. Jackson, *Extensions of racks and quandles*, Homology Homotopy Appl. **7** (2005), no. 1, 151–167.
[5] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65.
[6] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, pp. 625–635.
[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381.
[8] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, sequences A181769 (isomorphism classes of quandles) and A181770 (isomorphism classes of racks), queried 2026-08-23.
[9] L. Ta, *Graph quandles: generalized Cayley graphs of racks and right quasigroups*, J. Non-Assoc. Struct. **1** (2026), no. 1, Paper no. 3; arXiv:2506.04437v5. (The journal assigns article numbers rather than page ranges.)