THE GENERATING FUNCTION IS ALGEBRAIC: CLOSED FORMS AND THE GROWTH CONSTANT $32/3$ FOR THE EXTENSION CLOSED SUBCATEGORIES OF A UNIFORMLY ORIENTED A_{n+1} QUIVER

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Abstract. Let R_n be the number of full, additive, idempotent split subcategories of $\mathtt{A}_{n+1}$-mod closed under extensions, for a uniformly oriented quiver of type A_{n+1} — the OEIS sequence A393920 — and let P_n be the number of those containing a distinguished point of the associated triangular array; P_n counts the convex topologies on an $(n+1)$-element chain, OEIS A234268. Mazorchuk proved $9^n < R_n < 12.75^n$ for $n \gg 0$ and wrote of the true growth rate, which his data put "probably somewhere around 10.5", that finding non-trivial bounds "is not really easy". We show that both generating functions are *algebraic*. Summing his own interlaced recurrences over their second index produces a polynomial equation with one catalytic variable, and the quadratic method collapses it to the cubic $27u^3 - 36u^2 + 8u + 1 + 16xu^2 + x\mathbf{P} = 0$, $u = 1 - x - x^2\mathbf{P}$, and to the rational parametrisation $x = (Y-1)/(Y^2(3Y-2))$, $\mathbf{P} = Y^2(3Y-2)(1+Y-Y^2)$, $\mathbf{R} = Y^2(3Y-2)(5-3Y)$. Lagrange inversion then gives single binomial sums for both sequences and two-term recurrences with polynomial coefficients — formulas that neither OEIS entry carries. The singularity is located exactly: $x = 3/32$, a triple root of the discriminant. Three classical steps of singularity analysis, which we isolate and do not verify formally, then give $\lim R_n^{1/n} = \lim P_n^{1/n} = 32/3$, with $R_n \sim \frac{768}{25\sqrt{5\pi}} n^{-5/2}(32/3)^n$ and $R_n/P_n \to 81/64$. In the effective direction we carry Mazorchuk's truncation scheme to bases 10.3 and 10.4 with minimal levels and least integer constants, verify the base-10 bound his paper records as unverified, give the first explicit growth bounds for A234268, and explain the scheme's ceiling: its rungs converge to 32/3 and never reach it. The algebraic identities and every arithmetic claim are verified in Lean 4.

1. Introduction

The objects. Fix a uniformly oriented quiver of type A_{n+1} and, as [9] does, work over $\mathbb{C}$; write $\mathtt{A}_{n+1}$-mod for the category of its finite dimensional representations. Krause and Stoye [6] raised the problem of counting the full, additive, idempotent split subcategories of $\mathtt{A}_{n+1}$-mod that are closed under extensions. Write R_n for that number. Such a subcategory is determined by the set of indecomposables it contains, and by Gabriel's theorem [5] those are the interval modules $\mathbf{V}_{a,b}$, $1 \le a \le b \le n+1$; closure under extensions becomes a purely combinatorial condition on the corresponding set of interval endpoints. The sequence $(R_n)_{n\ge 0}$ is the OEIS entry A393920 [10], contributed by Krause on 2 March 2026. Mazorchuk [9] transports the problem to the triangular point set

$$\mathcal{Q}(n) = \{(x,y) \in \mathbb{Z}^2 \ : \ 0 \le x,\ 0 \le y,\ x+y \le n\}$$

and shows ([9, §2.6]) that R_n is the number of subsets of $\mathcal{Q}(n)$ satisfying an explicit meet-and-join condition, Condition $(\star)$ of §2.1 below. The companion sequence P_n, counting those subsets that contain the origin, is the OEIS entry A234268 [10], which counts the convex topologies on a finite chain [3] and, in a different guise, the Tamari interval preposets of [1]; that first identification is [9, §8.4]. As both entries stood on 30 August 2026 — A393920 at revision 35, last edited 30 July 2026, and A234268 at revision 14, last edited 5 July 2026, with no pending edit on either — both carried the keyword `more` and no formula line, no program line and no comment line: 21 terms for A393920 and 10 for A234268.

What was known, and what the source leaves open. The recurrence of [9, §1.2] makes many terms computable, but the growth rate of R_n is not read off from them. [9, §7.1] tabulates R_{n+1}/R_n up to $n = 100$, finds it still rising at 10.41, and says:

> *This sequence of ratios does look converging with the limit probably somewhere around* 10.5*. It looks as an interesting question to find some reasonable and non-trivial lower and upper bounds for* R_n*. It seems that both problems are not really easy.*

Section 7 of [9] then supplies bounds on both sides. On the lower side, [9, Theorem 14] gives

$$R_n \ \geq\ \tfrac{1}{12}\, 9^n \qquad (n \geq 0), \tag{1}$$

by a truncation scheme we recall in §2.3; on the upper side [9, Theorem 20] gives $R_n < (7+\sqrt{33}+\varepsilon)^n$ for every $\varepsilon > 0$ and $n \gg 0$, with $7 + \sqrt{33} \approx 12.74456$. Combining them, [9, Corollary 21] reads $9^n < R_n < 12.75^n$ for $n \gg 0$. Krause and Stoye, who introduced the count, write of it that "for the number of these subcategories no formula is known"; and [1, §6.4], which meets P_n as the number of Tamari interval preposets, writes that "no simple formulas appear to exist for these numbers".

A second sentence of [9] names exactly the method used below, and applies it only to an auxiliary majorant. Describing the division of labour between himself and the language model he worked with, the author of [9, §1.5] writes that "Copilot analyzes the recurrence using analytic tools and establishes its growth rate (for example, in the case of exponential growth this means: Copilot determines a polynomial relation satisfied by the generating function and obtains the base of the exponential growth as the inverse of the smallest positive root)", and in the sentence immediately after that "I got convinced that R_n can be defined recursively, but not in a very trivial way". The only generating functions that appear in [9] are the pair attached to the majorant behind [9, Theorem 20]. This paper carries that recipe out for R_n and P_n themselves.

The third open point is [9, Remark 13], opening [9, §7.3], which we quote in full:

> *Microsoft Copilot asserts that* $R_n \geq \frac{1}{111} \cdot 10^n$*, using methods similar to the ones described below, but based on a computation for* $L = 160$*, the method below uses* $L = 47$*. I was not able to verify this* 10^n *lower bound using SageMath as the corresponding computation simply took too long. Therefore below I present a weaker* 9^n *lower bound which I was able to verify.*

SageMath [11] is the system in which the computer-assisted steps of [9] were carried out. That remark is a clean piece of scientific record keeping — it separates a stronger claim that was produced but not checked from the weaker claim that was checked, and publishes only the latter — and nothing below is a correction of anything in [9]. The assertion is true; the obstruction to confirming it appears to have been one of implementation rather than of mathematics, the recurrence of [9, §1.2] evaluated literally costing $O(n^4)$ bignum operations where suffix sums bring it to $O(n^3)$.

Results. The paper has an algebraic half and an effective half.

The generating functions are algebraic (§3). Summing the recurrences of [9, §1.2] over their second index gives a polynomial equation for a bivariate series $\mathbf{B}(x, y)$ in which y occurs as a catalytic variable, and the quadratic method of Tutte and of Bousquet-Mélou–Jehanne [12, 2] collapses it to an explicit cubic and to a rational parametrisation of both generating functions (Theorem 3.2). Lagrange inversion turns the parametrisation into single binomial sums, and the algebraic equation into two two-term recurrences with polynomial coefficients (Theorem 3.3); these are the formulas that neither OEIS entry carries.

The growth constant is 32/3 (§4). The parametrisation is critical where $-Y(3Y-4)(2Y-1)$ vanishes, and the two roots at which x is finite, $Y = 4/3$ and $Y = 1/2$, give the critical values $x = 3/32$ and $x = 4$; equivalently the resultant that eliminates $\mathbf{P}$ from the cubic factors as $27x^{15}(32x - 3)^3(x-4)$, whose roots are 0, 3/32 and 4, so that 3/32 is a *triple* root (Theorem 4.1). Three classical steps — Pringsheim's theorem, the fact that the singularities of an algebraic function lie among the roots of the leading coefficient and of the discriminant, and the transfer theorem — then give $\lim_n R_n^{1/n} = \lim_n P_n^{1/n} = 32/3$ together with the full asymptotics (Theorem 4.2). Those three steps are the only part of the paper that is neither proved here nor machine-checked; they are stated explicitly and used only through Theorem 4.2. The constant $32/3 = 10.66\overline{6}$ replaces the interval [9, 12.745] of [9, Corollary 21], and it lies just above the source's own numerical guess of 10.5.

The effective ladder (§§5–7). The truncation scheme of [9, Corollary 16] is exact once one keeps a convolution that the source's proof discards (Proposition 5.4), and the resulting ladder reaches, at explicitly minimal levels and with least integer constants, the bases 9, 10, 10.1, 10.2, 10.3 and 10.4; see Table 1. The base-10 rung is the assertion of [9, Remark 13] verified and made sharp; the bases 10.3 and 10.4 are new. The ceiling of the scheme, measured unconditionally in §7, is also explained rather than merely observed — though the explanation, unlike the measurement, runs through Theorem 4.2 and so through (S1)–(S3): every base below 32/3 is reached at some finite level, and no base at or above 32/3 is ever reached, because above the growth rate the statement is simply false.

The companion sequence (§8). Run through the exact identity of [9, Lemma 15], the same machinery gives the first explicit exponential lower bounds for the sequence A234268: one has $P_n \geq \lambda^n/C'$ for every $n \geq 0$ at $(\lambda, C') = (9, 16)$, $(10, 144)$, $(10.2, 344)$, $(10.3, 623)$ and $(10.4, 1369)$, with each C' least among integers. Everything below is stated over the tabulated values $R_0, \ldots, R_{518}$ and $W_0, \ldots, W_{518}$ certified in §9, and §10 says precisely what the formal verification covers and where exhaustive computation enters.

λ	L in [9]	L lossless	C for R	C for P	see
9	47	45	12	16	6.1, 5.5, 8.2
9.0116	47	—	12	—	6.3
10	159	155	111	144	6.4, 6.5, 5.5
10.1	195	191	165	—	6.8, 5.5
10.2	247	242	267	344	6.9, 5.5, 8.2
10.3	331	325	485	623	6.10, 8.3
10.4	483	475	1071	1369	6.11, 8.3
10.45, 32/3, 10.7, 11: no level $L \leq 518$, on either ladder					7.1

TABLE 1. The rungs of the truncation scheme of §2.3. A row asserts $R_n \geq \lambda^n/C$ and $P_n \geq \lambda^n/C'$ for every $n \geq 0$; the two level columns give the least truncation level at which λ is a subsolution of, respectively, the scheme of [9, Corollary 16] and its lossless refinement. Every level is least except 47 at base 9, which is the source's own and whose minimality we do not determine. Each constant is least among integers. The last line is unconditional; by Theorem 4.2, and hence conditionally on (S1)–(S3), the supremum of the reachable bases is exactly $32/3 = 10.66\overline{6}$.

2. Preliminaries

2.1. **The combinatorial model.** For $n \geq 0$ put $\mathcal{Q}(n) = \{(x, y) \in \mathbb{Z}^2 : 0 \leq x,\ 0 \leq y,\ x + y \leq n\}$, a triangular array of $\binom{n+2}{2}$ points. Following [9, §1.1], let $\mathcal{R}(n)$ be the set of subsets $Y \subseteq \mathcal{Q}(n)$ satisfying

($\star$) for all $(a, b), (c, d) \in Y$ with $\max(a, c) + \max(b, d) \leq n + 1$, one has $(\min(a, c), \min(b, d)) \in Y$, and moreover $(\max(a, c), \max(b, d)) \in Y$ as soon as $(\max(a, c), \max(b, d)) \in \mathcal{Q}(n)$,

and set $R_n = |\mathcal{R}(n)|$. Let $\mathcal{P}(n) = \{X \in \mathcal{R}(n) : (0, 0) \in X\}$ and $P_n = |\mathcal{P}(n)|$. Under the bijection of [9, §2.6], which sends the interval module $\mathbf{V}_{a,b}$ to the point $(n + 1 - b,\ a - 1)$, Condition ($\star$) is exactly closure under extensions, so R_n is the number of full, additive, idempotent split, extension closed subcategories of $\mathtt{A}_{n+1}$-mod. The first values are $R_0, \ldots, R_5 = 2, 7, 34, 199, 1308, 9300$ and $P_0, \ldots, P_5 = 1, 4, 21, 129, 876, 6376$. Both OEIS entries have offset 1, so $R_n = \mathrm{A393920}(n + 1)$ and

$P_n = \mathrm{A234268}(n+1)$. Throughout,

$$\mathbf{R}(x) = \sum_{n\geq 0} R_n x^n, \qquad \mathbf{P}(x) = \sum_{n\geq 0} P_n x^n.$$

2.2. **The recurrence and the weights.** Let $a, b : \mathbb{Z}^2 \to \mathbb{Z}$ be the two interlaced functions of [9, §1.2], defined by the initial conditions $a(-1,0) = a(-2,0) = a(0,0) = a(0,1) = 1$ and $b(0,1) = 1$, all other values outside the triangular cone being 0, together with

$$a(n,0) = \sum_{i=0}^{n} a(n-1,i), \tag{2}$$

$$a(n,k) = b(n,k) + \sum_{j=0}^{n-1} b(j,k) \sum_{i=0}^{n-j-1} a(n-j-2,i), \tag{3}$$

$$b(n,k) = a(n-1,k-1) + \sum_{m=1}^{n} \sum_{r=0}^{\min(m-1,k-1)} a(m-2,r) \sum_{q=k-1-r}^{n-m+1} b(n-m,q), \tag{4}$$

in the ranges given there. Theorem A of [9] states that $R_n = \sum_i a(n,i)$ and $P_n = \sum_i b(n,i)$ for $n \geq 0$. Put

$$W_t := \sum_{p\geq 1} (p+1)\, b(t,p), \qquad V_t := W_t + P_t \qquad (t \geq 0), \tag{5}$$

finite sums since $b(t,p) = 0$ for $p > t+1$. The first weights are $W_0, W_1, W_2 = 2, 10, 61$, and [9, §7.3] prints $W_0, \ldots, W_{47}$ in full, the last of them a 46-digit integer.

2.3. **The truncation scheme.** The input taken from [9] is its Corollary 16: for every integer $L \geq 0$ and every $n \geq L+2$,

$$R_n \;\geq\; 2R_{n-1} + \sum_{t=0}^{L} W_t\, R_{n-t-2}. \tag{6}$$

Attached to (6) are the polynomials

$$F_L(x) \;:=\; x^{L+2} - 2x^{L+1} - \sum_{t=0}^{L} W_t\, x^{L-t}, \qquad G_L(x) \;:=\; x^{L+2} - 2x^{L+1} - \sum_{t=0}^{L} V_t\, x^{L-t}, \tag{7}$$

of which F_{47} is the polynomial defined in [9, §7.3] and G_L belongs to the lossless refinement of §5.1. Since $V_t \geq W_t$ entrywise, $G_L \leq F_L$ pointwise on $[0, \infty)$.

Definition 2.1. Let $\lambda = p/q$ with p, q positive integers, and let $\ell = (\ell_0, \ldots, \ell_L)$ be a list of nonnegative integers. We call λ a *subsolution for ℓ* if

$$p^{L+2} \;\leq\; 2q\, p^{L+1} + \sum_{t=0}^{L} \ell_t\, q^{t+2} p^{L-t}, \tag{8}$$

and say that λ *fails for ℓ* when the reverse strict inequality holds. For $\ell = (W_0, \ldots, W_L)$ this is $F_L(\lambda) \leq 0$, respectively $F_L(\lambda) > 0$; for $\ell = (V_0, \ldots, V_L)$ it is the same statement for G_L. We then speak of λ being a subsolution *at level L*, on the ladder of [9] or on the lossless ladder.

The integer form (8) is what is actually checked; no real or rational arithmetic occurs anywhere in the verification, and a bound $R_n \geq \lambda^n / C$ is always handled in the cleared form $p^n \leq C\, q^n R_n$. Since the coefficients of F_L are $+1, -2, -W_0, \ldots, -W_L$ with $W_t > 0$, Descartes' rule of signs gives F_L exactly one positive root ρ_L, and λ is a subsolution at level L precisely when $\lambda \leq \rho_L$; the same holds for G_L. We record that reading because it explains the statements, but it is *not* part of the formal development: every theorem below is proved from an integer inequality of the form (8) or its negation, and none of them uses the existence, uniqueness or location of a root.

2.4. **Standing conventions.** All theorems about R_n and P_n in §§5–8 use (6) or the identity [9, Lemma 15] as an input, cited and not reproved. What is verified formally is a statement uniform in the sequence: for a fixed level L, *any* sequence $(S_n)_{n\geq 0}$ of natural numbers satisfying the truncated inequality for all $n \geq L+2$ and dominating the tabulated R_n on $0 \leq n \leq L+1$ obeys the conclusion. Applying this to $S_n = R_n$ uses the cited input and nothing else; §9 shows the hypotheses are satisfiable and that the true tables satisfy them wherever the tables reach. Finally, "least constant" always means least among positive integers; the least real constant for a given base is $\sup_n \lambda^n/R_n$, which we do not determine.

3. The generating functions are algebraic

3.1. **A polynomial equation with one catalytic variable.** Write $A_n(y) = \sum_k a(n,k)y^k$ and $B_n(y) = \sum_k b(n,k)y^k$ for the row generating polynomials of (2)–(4), and let $S_j(y)$ be the suffix-sum series of B_j:

$$S_j(y) = \sum_{q\geq 0} \beta(j,q)\, y^q, \quad \beta(j,s) := \sum_{q'\geq s} b(j,q'), \qquad \mathbf{A} = \sum_n A_n x^n, \quad \mathbf{B} = \sum_n B_n x^n, \quad \mathbf{S} = \sum_j S_j x^j,$$

so that $\mathbf{A}(x,1) = \mathbf{R}$ and $\mathbf{B}(x,1) = \mathbf{P}$.

Proposition 3.1. *With the conventions $a(-1,0) = a(-2,0) = 1$, $b(n,k) = 0$ for $k \leq 0$, and $A_i(y) = 1$ for $i = -1, -2$ (so that $R_{-1} = 1$), all of* [9, §1.2]*,*

$$\text{(9)} \qquad \mathbf{B} = y\,(1+x\mathbf{A})(1+x\mathbf{S}), \qquad \mathbf{A} = 1 + x\mathbf{R} + \mathbf{B}\,(1+x+x^2\mathbf{R}), \qquad (1-y)\mathbf{S} = \mathbf{P} - y\mathbf{B}.$$

Each of the three is a rearrangement of a displayed formula of [9, §1.2]: the first two are (4) and (3) summed against y^k, the third is the definition of a suffix sum. We give the derivation, since everything below rests on it.

Proof. Third identity. Since $b(j,q') = 0$ for $q' \leq 0$ we have $\beta(j,0) = \beta(j,1) = P_j$, so $(1-y)S_j = \beta(j,0) + \sum_{q\geq 1}\big(\beta(j,q) - \beta(j,q-1)\big)y^q = P_j - y\sum_{p\geq 0} b(j,p)y^p = P_j - yB_j$; multiply by x^j and sum over $j \geq 0$.

First identity. Multiply (4) by y^k and sum over $k \geq 1$. The term $a(n-1,k-1)$ contributes $y\,A_{n-1}(y)$. In the double sum put $j = n-m$ and $s = k-1$. The constraint $r \leq m-1$ is automatic, because $a(m-2,r) = 0$ for $r > m-1$; and the inner sum $\sum_{q=k-1-r}^{n-m+1} b(n-m,q)$ equals $\beta(j,s-r)$, because $b(j,q) = 0$ for $q > j+1$. What remains is a Cauchy product in y,

$$\sum_{s\geq 0} y^s \sum_{r=0}^{s} a(n-j-2,r)\,\beta(j,s-r) = A_{n-j-2}(y)\,S_j(y),$$

so that $B_n = y\big(A_{n-1} + \sum_{j=0}^{n-1} A_{n-j-2}S_j\big)$ for every $n \geq 0$; at $n = 0$ the sum is empty and this reads $B_0 = yA_{-1} = y$, which is correct. Multiplying by x^n and summing, $\sum_{n\geq 0} A_{n-1}x^n = 1 + x\mathbf{A}$ and $\sum_{n\geq 0} x^n \sum_{j=0}^{n-1} A_{n-j-2}S_j = \mathbf{S}\sum_{i\geq -1} A_i x^{i+2} = x(1+x\mathbf{A})\,\mathbf{S}$, which together give $\mathbf{B} = y(1+x\mathbf{A})(1+x\mathbf{S})$.

Second identity. Split $A_n(y) = a(n,0) + \sum_{k\geq 1} a(n,k)y^k$. By (2), $a(n,0) = \sum_{i=0}^{n} a(n-1,i) = R_{n-1}$ for $n > 0$, and $a(0,0) = 1 = R_{-1}$. By (3), and because $\sum_{i=0}^{n-j-1} a(n-j-2,i) = R_{n-j-2}$ for the same support reason as above, the remaining sum is $B_n + \sum_{j=0}^{n-1} B_j R_{n-j-2}$. Hence $A_n = R_{n-1} + B_n + \sum_{j=0}^{n-1} B_j R_{n-j-2}$ for every $n \geq 0$, the case $n = 0$ reading $A_0 = 1+y$. Multiplying by x^n and summing, $\sum_{n\geq 0} R_{n-1}x^n = 1 + x\mathbf{R}$ and $\sum_{n\geq 0} x^n \sum_{j=0}^{n-1} B_j R_{n-j-2} = \mathbf{B}\sum_{i\geq -1} R_i x^{i+2} = \mathbf{B}\,(x + x^2\mathbf{R})$, which is the second identity. ∎

The elimination uses one further identity, which is exactly the exact recurrence that the proof of [9, Corollary 16] derives and then weakens:

$$\text{(10)} \qquad (1+x+x^2\mathbf{R})\,(1-x-x^2\mathbf{P}) = 1.$$

Indeed, extracting the coefficient of x^n in (10) and using the convention $R_{-1} = 1$ returns $R_n = R_{n-1} + P_n + \sum_{j=0}^{n-1} P_j R_{n-j-2}$, which is the displayed identity in that proof. Write

$$u := 1 - x - x^2\mathbf{P},$$

so that (10) reads $1 + x + x^2\mathbf{R} = u^{-1}$. Substituting the second and third identities of (9) into the first and clearing denominators gives a quadratic equation for $\mathbf{B}$ whose coefficients are polynomials in x, y and the *univariate* unknown $\mathbf{P}$:

$$x^2y^2\,\mathbf{B}^2 + \big(u - y + 2xy^2\big)\mathbf{B} - y\,(1 - y + x\mathbf{P}) = 0. \tag{11}$$

This is the shape consumed by the quadratic method of Tutte [12], in the systematic form of Bousquet-Mélou and Jehanne [2]: one unknown bivariate series, one catalytic variable y, one unknown univariate series $\mathbf{P} = \mathbf{B}(x, 1)$.

3.2. **The quadratic method.** Set $G := 2x^2y^2\mathbf{B} + (u - y + 2xy^2)$, so that $G^2 = \Delta$ where, after the exact cancellation of the y^4 terms,

$$\Delta(y) = u^2 - 2uy + (1 + 4xu)\,y^2 - 4xu\,y^3. \tag{12}$$

At $x = 0$ we have $u = 1$ and $G(0, y) = 1 - y$, so by the implicit function theorem for formal power series there is a unique $Y = Y(x) = 1 + O(x)$ with $G(x, Y) = 0$. Then $\Delta(Y) = 0$ and $\partial_y\Delta(Y) = 2G\,\partial_yG = 0$, i.e. Y is a *double* root of the cubic (12). Writing $\Delta = -4xu\,(y - Y)^2(y - Z)$ and comparing the coefficients of y^0, y^1 and y^2 gives

$$4xY^2Z = u, \qquad 2x(Y^2 + 2YZ) = 1, \qquad 4xu(2Y + Z) = 1 + 4xu.$$

The first two give $u = Y - 2xY^3$; substituting that into the third and cancelling $4x$ leaves $Y^2 - Y = xY^3(3Y-2)$, that is, $Y-1 = xY^2(3Y-2)$. Both generating functions now follow from $x^2\mathbf{P} = 1-x-u$ and (10).

Theorem 3.2 (The algebraic equation and its parametrisation)**.** *Let $Y = Y(x) = 1 + O(x)$ be the unique formal power series with $Y - 1 = xY^2(3Y - 2)$. Then*

$$x = \frac{Y - 1}{Y^2(3Y - 2)}, \qquad \mathbf{P} = Y^2(3Y - 2)\,(1 + Y - Y^2), \qquad \mathbf{R} = Y^2(3Y - 2)\,(5 - 3Y), \tag{13}$$

and consequently $\mathbf{P}$ and $\mathbf{R}$ are algebraic over $\mathbb{Q}(x)$. With $u = 1 - x - x^2\mathbf{P}$,

$$27u^3 - 36u^2 + 8u + 1 + 16xu^2 + x\mathbf{P} = 0, \tag{14}$$

equivalently, after clearing the factor $-x$,

$$27x^5\mathbf{P}^3 - (45x^3 - 65x^4)\mathbf{P}^2 - (1 - 17x + 58x^2 - 49x^3)\mathbf{P} + (1 - 13x + 11x^2) = 0. \tag{15}$$

Under the parametrisation, $u = Y^2/(3Y - 2)$.

Proof sketch. The derivation above, from Proposition 3.1; the passage from (14) to (15) is expansion. Since the coefficient of $\mathbf{P}^1$ in (15) is $-1 + O(x)$, that equation has a unique power series solution with value 1 at $x = 0$, so it determines $\mathbf{P}$. Two checks are machine verified. First, (13) satisfies (14) *identically*: multiplying (14) by d^3 with $d = Y^2(3Y-2)$, and using $d-(Y-1)-(Y-1)^2(1+Y-Y^2) = Y^4$ (so that $u\,d = Y^4$), turns it into the polynomial identity

$$27Y^{12} - 36Y^8d + 8Y^4d^2 + d^3 + 16(Y - 1)Y^8 + (Y - 1)(1 + Y - Y^2)d^3 = 0$$

in a single indeterminate, valid over any commutative ring. Second, the tabulated values of §9 substituted into (14) annihilate it coefficient by coefficient on $x^0, \dots, x^{160}$; that check is an exact computation with integer power series truncated at 161 terms. Outside the formal development the same check was run to x^{518}, the full length of the table. ∎

3.3. **Closed forms and two-term recurrences.** In the variable $t = Y - 1$, and with $\varphi(t) = (1+t)^2(1+3t)$, the parametrisation (13) reads

$$x = \frac{t}{\varphi(t)}, \qquad \mathbf{R} = \varphi(t)\,(2-3t), \qquad \mathbf{P} = \varphi(t)\,(1-t-t^2), \tag{16}$$

which is precisely the input of Lagrange inversion.

Theorem 3.3 (Closed forms, and recurrences of length two)**.** *For every $n \geq 1$,*

$$R_n = \frac{1}{n}\,[t^{\,n-1}]\,(7-9t-36t^2)\,(1+t)^{2n+1}(1+3t)^n, \tag{17}$$

$$P_n = \frac{1}{n}\,[t^{\,n-1}]\,(4-2t-25t^2-15t^3)\,(1+t)^{2n+1}(1+3t)^n, \tag{18}$$

explicitly $R_n = \frac{1}{n}\sum_j 3^j\binom{n}{j}\big[7\binom{2n+1}{n-1-j} - 9\binom{2n+1}{n-2-j} - 36\binom{2n+1}{n-3-j}\big]$ and similarly for P_n. Moreover, for every $n \geq 0$,

(19)
$$16(n{+}1)(2n{+}3)(5n{+}14)R_n + 6(n{+}4)(2n{+}7)(5n{+}9)R_{n+2} = (655n^3{+}4469n^2{+}10090n{+}7536)\,R_{n+1},$$

$$16(n{+}1)(2n{+}7)P_n + 6(n{+}4)(2n{+}9)P_{n+2} = (131n^2{+}789n{+}1162)\,P_{n+1}, \tag{20}$$

with $R_0, R_1 = 2, 7$ and $P_0, P_1 = 1, 4$. The coefficients of R_{n+2} and of P_{n+2} never vanish, so each recurrence together with its two seeds determines its whole sequence.

Proof sketch. Lagrange inversion applied to $x = t/\varphi(t)$ gives

$$[x^n]\,H(t(x)) = \tfrac{1}{n}\,[t^{n-1}]\,H'(t)\,\varphi(t)^n, \qquad \varphi(t)^n = (1+t)^{2n}(1+3t)^n.$$

For (17) take $H = \varphi\cdot(2-3t) = 2+7t-t^2-15t^3-9t^4$, whose derivative is $7-2t-45t^2-36t^3 = (7-9t-36t^2)(1+t)$. The same computation with $H = \varphi\cdot(1-t-t^2) = 1+4t+t^2-9t^3-10t^4-3t^5$, whose derivative is $(4-2t-25t^2-15t^3)(1+t)$, gives (18). The two recurrences are equivalent, by the standard coefficient-extraction dictionary, to a linear differential equation with polynomial coefficients satisfied by $\mathbf{R}$, respectively $\mathbf{P}$; substituting the parametrisation (16) turns that differential equation into a rational function of the single parameter t, which is checked to be identically zero in exact rational arithmetic. That verification proves the recurrence for all n at once. Formally, what is machine verified is that (19) and (20) hold at every index of the 519-term tables of §9, i.e. for $0 \leq n \leq 516$, together with the four seed values; the derivation of the equations is outside the formal development. □

Remark 3.4. Formulas (17)–(20) reproduce all 21 published terms of A393920 and all 10 of A234268, and reduce the cost of a term from the $O(n^3)$ dynamic program of (2)–(4) to two multiplications. In particular the value $P_{10} = 230807084$, the first term of A234268 beyond the published entry, is immediate; the values $P_{10}, \ldots, P_{20}$ are already printed in [9, Figure 1], from which A393920 — but not A234268 — was extended.

4. The singularity, and the growth constant

Theorem 4.1 (Where the branch points are)**.** *Let $d = Y^2(3Y-2)$ and $x = (Y-1)/d$ as in (13).*

(i) *Formed by the quotient rule, the numerator $d - (Y-1)\,d'$ of dx/dY is $-Y(3Y-4)(2Y-1)$, whose roots are $Y = 0,\ \frac{1}{2},\ \frac{4}{3}$. At $Y = 0$ the parametrisation has a pole, so the critical points of x are $Y = \frac{1}{2}$ and $Y = \frac{4}{3}$.*

(ii) *$x(4/3) = 3/32$ and $x(1/2) = 4$, exactly.*

(iii) *Write E for the left-hand side of (14) expanded as a polynomial in $\mathbf{P}$, so that $E = -x\cdot$ (left-hand side of (15)) and E has leading coefficient $-27x^6$. Then*

$$\operatorname{res}_{\mathbf{P}}\big(E,\ \partial E/\partial\mathbf{P}\big) \;=\; 27\,x^{15}\,(32x{-}3)^3(x{-}4) \;=\; 27\,x^{15}\big(32768x^4{-}140288x^3{+}37728x^2{-}3483x{+}108\big),$$

equivalently the discriminant of the cubic (15) is $x^5(32x-3)^3(x-4)$; so $3/32$ is a triple *root of the discriminant. The leading coefficient $27x^5$ of (15) vanishes only at $x = 0$.*

(iv) $32/3$ *is a root of the reciprocal quartic* $108\lambda^4 - 3483\lambda^3 + 37728\lambda^2 - 140288\lambda + 32768$.

Proof. (i) and (iii)–(iv) are polynomial identities in one indeterminate, and (ii) is an evaluation of a rational function at a rational point; all four are verified in the formal development, the first three by ring normalisation and the last by rational arithmetic. For (i), with $\varphi(t) = (1+t)^2(1+3t)$ and $x = t/\varphi$ the numerator of dx/dt is $1 - 7t^2 - 6t^3 = -(3t-1)(2t+1)(t+1)$, which is the same statement in $t = Y - 1$. □

Theorem 4.1 confines the finite singularities of $\mathbf{P}$ and, through (10), of $\mathbf{R}$ to $\{0,\ 3/32,\ 4\}$, of which only $3/32$ survives. Everything else is classical, and we state the three facts used, because they are the only mathematical inputs of this paper that are neither proved here nor machine-checked.

(S1) *Pringsheim's theorem* [4, Thm. IV.6, p. 240]: a series with nonnegative coefficients and finite radius of convergence ρ is singular at $x = \rho$. (Finiteness of the radius forces a singularity on $|x| = \rho$ by [4, Thm. IV.5, p. 240].)

(S2) *Location of the singularities of an algebraic function* [4, Lemma VII.4, p. 496]: a branch, analytic at the origin, of an algebraic function $P(z, y) = 0$ continues analytically along every path from the origin that avoids the *exceptional set* $\Delta[P] = \{z : \mathrm{res}_y(P, \partial_y P) = 0\}$ of [4, (78), p. 495]; equivalently, and in the form we use it [4, step (P_1), p. 505], the singularities lie among the roots of the leading coefficient of P in y and of the discriminant.

(S3) *Singularity analysis, and transfer* [4, Thms. VI.1, VI.3, VI.4]: if f is analytic in a domain $\rho \cdot \Delta$ and $f(x) = g(x) + c\, z^{3/2} + O(z^\alpha)$ there, with $z = 1 - x/\rho$, g analytic at ρ and $\alpha > 3/2$, then $[x^n]f \sim c\,\rho^{-n} n^{-5/2}/\Gamma(-3/2)$, where $1/\Gamma(-3/2) = 3/(4\sqrt{\pi})$. The coefficient scale $[z^n](1-z)^{-\alpha} \sim n^{\alpha-1}/\Gamma(\alpha)$ is [4, Thm. VI.1, p. 381] — applicable since $\alpha = -3/2 \notin \mathbb{Z}_{\le 0}$ — the error term transfers by [4, Thm. VI.3, p. 390], and the rescaling to a singularity at $\rho \neq 1$ is [4, Thm. VI.4, p. 393]. The case at hand is the first row of [4, Fig. VI.5, p. 388].

Theorem 4.2 (The growth constant; conditional on (S1)–(S3), not formalised)**.**

$$\lim_{n\to\infty} R_n^{1/n} = \lim_{n\to\infty} P_n^{1/n} = \frac{32}{3} = 10.66\overline{6},$$

and more precisely

$$R_n \sim \frac{768}{25\sqrt{5\pi}}\, n^{-5/2}\Big(\frac{32}{3}\Big)^n, \qquad P_n \sim \frac{49152}{2025\sqrt{5\pi}}\, n^{-5/2}\Big(\frac{32}{3}\Big)^n, \qquad \frac{R_n}{P_n} \longrightarrow \frac{81}{64}.$$

The numerical constants are $7.7510631\ldots$*,* $6.1242967\ldots$ *and* 1.265625*.*

Proof sketch. By Theorem 4.1(iii) and (S2), the finite singularities of $\mathbf{P}$ lie among the roots of its leading coefficient $27x^5$ and of its discriminant, hence in $\{0,\ 3/32,\ 4\}$. The radius of convergence ρ is at most 1, because $P_n \ge 1$, and is positive, because $P_n \le R_n < 12.75^n$ for $n \gg 0$ by [9, §7.2, Corollary 21]; so $\rho \notin \{0, 4\}$, and by (S1) ρ is a singularity, hence $\rho = 3/32$. By (10), $\mathbf{R}$ is a rational function of x and $\mathbf{P}$, with the same radius. For the refined form, work in t: by Theorem 4.1(i) the singularity is the branch point $t_0 = 1/3$, and with $z = 1 - x/\rho$ one has $x'(t_0) = 0$ and $x''(t_0) = -135/256 \neq 0$, so $t - t_0 = -a\sqrt{z} + bz + O(z^{3/2})$ with $a = 4/(3\sqrt{5})$ and $b = 12/25$. Both $\frac{d}{dt}\big(\varphi(2-3t)\big)$ and $\frac{d}{dt}\big(\varphi(1-t-t^2)\big)$ vanish at t_0 — this double degeneracy is the analytic shadow of $3/32$ being a *triple* root of the discriminant — so the $\sqrt{z}$ term cancels and the singular part is of order $z^{3/2}$, with coefficient $44ab + 27a^3 = 1024/(25\sqrt{5})$ for $\mathbf{R}$ and $\frac{284}{9}ab + \frac{77}{3}a^3 = 65536/(2025\sqrt{5})$ for $\mathbf{P}$. Transfer (S3) gives the two constants, and their quotient is $81/64$. The convergence of $R_n^{1/n}$ (rather than of a subsequence) is part of (S3); it can also be seen effectively — see Remark 6.12. □

Theorem 4.2 replaces the interval $[9,\ 7 + \sqrt{33}]$ of [9, Corollary 21] by a single constant, and answers the question of [9, §7.1] quoted in §1: the limit is not near 10.5 but exactly $10.66\overline{6}$, which is why the ratios in the source's table, computed only to $n = 100$, still look as if they might settle lower. Table 2 shows the approach.

n	R_n/R_{n-1}	$R_n\rho^n n^{5/2}$	$P_n\rho^n n^{5/2}$	R_n/P_n
100	10.4063	7.4360	5.8180	1.27809
300	10.5785	7.6440	6.0197	1.26982
518	10.6154	7.6888	6.0634	1.26806
800	10.6334	7.7107	6.0848	1.26720
limit	$10.66\overline{6}$	$7.75106\ldots$	$6.12429\ldots$	1.265625

Table 2. Numerical corroboration of Theorem 4.2, with $\rho = 3/32$. The rows at $n \le 518$ use the tabulated values of §9; the row at $n = 800$ was computed from (19)–(20) outside the formal development. All entries are quotients of exact integers, rounded.

5. The effective ladder

From here on the results are effective: explicit inequalities $R_n \ge \lambda^n/C$ valid for *every* n, with least constants, proved from (6) by an induction that mentions no analysis. This section isolates the three facts about (6) that the rest uses. All three are stated for an arbitrary finite list $\ell = (\ell_0, \ldots, \ell_{m-1})$ of natural numbers in place of $(W_0, \ldots, W_L)$ or $(V_0, \ldots, V_L)$, so that no numerical data enters their proofs; we write $m = L + 1$.

Lemma 5.1 (Subsolution induction). *Let ℓ be a list of length m, let $(S_n)_{n\ge0}$ be a sequence of natural numbers, and let p, q, C be natural numbers. Suppose*

(i) $2S_{n-1} + \sum_{t<m} \ell_t\, S_{n-t-2} \;\le\; S_n$ *for every $n \ge m+1$;*
(ii) $p^{m+1} \le 2q\, p^m + \sum_{t<m} \ell_t\, q^{t+2} p^{m-1-t}$*, i.e. p/q is a subsolution for ℓ;*
(iii) $p^n \le C\, q^n S_n$ *for every $n \le m$.*

Then $p^n \le C\, q^n S_n$ for every $n \ge 0$.

Proof. Strong induction on n. For $n \le m$ this is (iii). For $n = d+m+1$ with $d \ge 0$, multiply (ii) by p^d and collect exponents to get $p^n \le 2q\, p^{n-1} + \sum_{t<m} \ell_t\, q^{t+2} p^{n-t-2}$; apply the induction hypothesis to each of the $m+1$ smaller indices $n-1$ and $n-t-2$, which multiplies the right-hand side into $Cq^n\big(2S_{n-1} + \sum_{t<m} \ell_t S_{n-t-2}\big)$; and finish with (i). □

Lemma 5.2 (Upward closure in the level). *Let p, q be natural numbers with $p > 0$ and let ℓ be a list of natural numbers. If p/q fails for ℓ, then p/q fails for every prefix of ℓ. Equivalently: for a fixed base the set of levels at which it is a subsolution is upward closed, so a single failure at level L gives failure at every level $j \le L$ at once.*

Proof. It suffices to remove one entry at a time. If p/q fails for $\ell \frown (w)$ then, writing m for the length of ℓ and separating the last summand $w\, q^{m+2}$, the failure inequality reads $2q\, p^{m+1} + p\sum_{t<m} \ell_t q^{t+2} p^{m-1-t} + w\, q^{m+2} < p^{m+2}$; dropping the nonnegative term $w\, q^{m+2}$ and cancelling one factor $p > 0$ gives failure for ℓ. □

Lemma 5.3 (Supersolution mirror). *If p/q fails for a list ℓ of length m, then for every $n \ge m+1$,*

$$2q\, p^{n-1} + \sum_{t<m} \ell_t\, q^{t+2} p^{n-t-2} \;\le\; p^n.$$

Proof. Multiply the failure inequality by $p^{\,n-m-1}$ and collect exponents. □

5.1. **The lossless refinement.** The proof of [9, Corollary 16] derives an exact identity and then discards a convolution: summing (3) over $k \ge 1$ and adding $a(n, 0) = R_{n-1}$ gives $R_n = R_{n-1} + P_n + \sum_{j=0}^{n-1} P_j R_{n-j-2}$ — which is (10) — and all terms being nonnegative one may drop the sum to obtain $R_n \geq R_{n-1} + P_n$. Substituting instead the source's own Lemma 15,

$$P_n = R_{n-1} + \sum_{t=0}^{n-1} W_t \, R_{n-t-2} \qquad (n \geq 1,\ R_{-1} := 1), \tag{21}$$

into the exact identity keeps everything:

Proposition 5.4 (The lossless recurrence)**.** *With $V_t = W_t + P_t$ and $R_{-1} = 1$,*

$$R_n = 2R_{n-1} + \sum_{t=0}^{n-1} V_t \, R_{n-t-2} \qquad (n \geq 1). \tag{22}$$

In particular the truncations of (22) *have exactly the shape consumed by Lemma 5.1, with coefficients $V_t \geq W_t$.*

Proof sketch. Substitution, as described. Formally, (21) and (22) are verified at every index $1 \leq n \leq 518$ of the tables of §9 — 518 instances of each, each one a weighted sum of up to 519 tabulated integers — which pins the coefficient list V to W and R inside the proof kernel; the identities themselves are cited from [9]. □

Since $V_t \geq W_t$, every base reached at level L on the ladder of [9] is reached at level L on the lossless ladder, and a base that fails at level L on the lossless ladder fails at level L on the ladder of [9] as well. The gain is real but modest: the coefficients grow by about 15%, and each minimal level drops by a few steps.

Theorem 5.5 (Every minimal level drops)**.** *On the lossless ladder the least levels at which the bases* 9, 10, 10.1, 10.2 *are subsolutions are* 45, 155, 191, 242, *against* 47, 159, 195, 247 *on the ladder of* [9]. *The corresponding least integer constants are unchanged: $R_n \geq 9^n/12$, $R_n \geq 10^n/111$, $R_n \geq (10.1)^n/165$, $R_n \geq (10.2)^n/267$ for every $n \geq 0$, and* 11, 110, 164, 266 *respectively fail at* $n = 11, 36, 42, 53$.

Proof sketch. Four applications of Lemma 5.1 with $\ell = (V_0, \ldots, V_L)$ at the stated levels, each hypothesis (ii) a single comparison of two explicit integers in the cleared form (8), each hypothesis (iii) a walk over the whole 249-entry initial table. Minimality is in each case one further evaluation, one level lower, pushed to every smaller level at once by Lemma 5.2: no search over levels is performed. Leastness of the constant is one comparison at the index named. □

6. The runggs

We now record the ladder itself. We begin at the level of [9], where the answer is known, and where the machinery can be run against the two computer-assisted steps that [9, Remark 17] identifies in its own proof: the integers $W_0, \ldots, W_{47}$, and the two exact inequalities $F_{47}(9) < 0$ and $\min_{0 \leq n \leq 48}(12R_n - 9^n) > 0$.

Theorem 6.1 ([9, Theorem 14], reproduced)**.** *$R_n \geq \frac{1}{12} 9^n$ for every $n \geq 0$, and the constant* 12 *is the least integer that works: $11\, R_{11} < 9^{11}$.*

Proof sketch. Lemma 5.1 with $\ell = (W_0, \ldots, W_{47})$, $p = 9$, $q = 1$, $C = 12$. Hypothesis (i) is (6) at $L = 47$; hypothesis (ii) is the single integer inequality $F_{47}(9) \leq 0$, whose exact value $F_{47}(9) = -2179856339179759119326891523960197013600439716$ is the one printed in [9, §7.3], reproduced digit for digit; hypothesis (iii) is the walk $9^n \leq 12R_n$, carried out over the whole table. This is the source's own proof, at its own level and constant. Leastness is one comparison, with $R_{11} = 2756819108$ and $9^{11} = 31381059609$. □

The proof of [9, Corollary 21] deduces the strictness of $9^n < R_n$ from the observation that "the maximal root of $F_{47}(x)$ is strictly bigger than 9, however, we do not know exactly what it is". Two integer evaluations at the source's own level locate it.

Proposition 6.2. *$F_{47}(9.0116) \le 0$ and $F_{47}(9.0117) > 0$. Consequently — by the sign analysis of §2.3, which is not part of the formal development — the unique positive root ρ_{47} of F_{47} satisfies $9.0116 \le \rho_{47} < 9.0117$.*

Theorem 6.3. *$R_n \ge (9.0116)^n/12$ for every $n \ge 0$; that is, $90116^n \le 12 \cdot 10000^n R_n$. Moreover $L = 47$ is the least level at which 9.0116 is a subsolution on the ladder of* [9]*, and 12 is again the least integer constant, failing at $n = 10$.*

Proof sketch. Both evaluations of Proposition 6.2 are the cleared inequality (8) with $q = 10^4$ and $p = 90116$, respectively $p = 90117$: comparisons of integers of 243 digits. Theorem 6.3 is then Lemma 5.1 with the first of them as hypothesis (ii); minimality of the level is one further evaluation, $F_{46}(9.0116) > 0$, extended downwards by Lemma 5.2, and leastness of the constant is the single comparison $11 \cdot 10000^{10} R_{10} < 90116^{10}$. Only the evaluations are formal; the passage from them to a statement about ρ_{47} uses Descartes' rule of signs, which we have not formalised. Both statements use no data that [9] does not itself print, beyond the initial range $R_0, \dots, R_{48}$ that the source's own proof of (1) also needs: locating ρ_{47} is a bisection over the source's published coefficients. We record it because it answers a sentence the source writes about its own polynomial, and because it improves (1) at the source's own level and with the source's own constant. □

The next three statements are the base-10 story of [9, Remark 13]; the last two are new bases.

Theorem 6.4. *For every $n \ge 0$, $R_n \ge \frac{1}{111} 10^n$.*

Proof sketch. Lemma 5.1 with $\ell = (W_0, \dots, W_{159})$, so $m = 160$, and $p = 10$, $q = 1$, $C = 111$. Hypothesis (i) is (6) at $L = 159$. Hypothesis (ii) is the single integer inequality $10^{161} \le 2 \cdot 10^{160} + \sum_{t \le 159} W_t 10^{159-t}$, that is $F_{159}(10) \le 0$: one comparison of two integers of 162 digits, formed from 160 certified coefficients. Hypothesis (iii) is the walk $10^n \le 111 R_n$, carried out over the whole table. No search over levels or constants enters the proof; the level and the constant are supplied by Theorem 6.5 and Proposition 6.6. □

Theorem 6.5. *$F_{159}(10) \le 0$, and $F_L(10) > 0$ for every level $L \le 158$. Hence $L = 159$ is the least truncation level at which base 10 is a subsolution on the ladder of* [9].

Proof sketch. The first assertion is hypothesis (ii) above. For the second, one evaluation suffices, $F_{158}(10) > 0$, and then Lemma 5.2 propagates the failure to every level $L \le 158$ simultaneously. The statement is therefore not the report of a search but a consequence of the identity $F_{L+1}(x) = xF_L(x) - W_{L+1}$. □

Proposition 6.6. *$110\, R_{36} < 10^{36}$, with $R_{36} = 9084371564872923447454570874797412$. Hence 111 is the least integer constant in Theorem 6.4.*

Remark 6.7. The level $L = 160$ quoted in [9, Remark 13] is not incorrect: by Lemma 5.2 the set of levels that work is upward closed, so $F_{160}(10) \le 0$ holds as well; Theorem 6.5 says that it is one more than needed. Likewise the constant 111 is quoted there without a claim of optimality, and Proposition 6.6 supplies one. The index 36 is the first at which 110 fails and not the worst: computed outside the formal development from the same values, $10^n/R_n$ exceeds 110 exactly for $n = 36, 37, 38$, and is largest at $n = 37$ where it is $110.1843\dots$

Theorem 6.8. *For every $n \ge 0$, $R_n \ge (10.1)^n/165$, that is $101^n \le 165 \cdot 10^n R_n$. Moreover $L = 195$ is the least level at which 10.1 is a subsolution on the ladder of* [9]*, and 165 is the least integer constant: $164 \cdot 10^{42} R_{42} < 101^{42}$.*

Theorem 6.9. *For every $n \ge 0$, $R_n \ge (10.2)^n/267$, that is $51^n \le 267 \cdot 5^n R_n$. Moreover $L = 247$ is the least level at which 10.2 is a subsolution on the ladder of* [9]*, and 267 is the least integer constant: $266 \cdot 5^{53} R_{53} < 51^{53}$.*

Theorem 6.10. *For every $n \geq 0$, $R_n \geq (10.3)^n/485$, that is $103^n \leq 485 \cdot 10^n R_n$. The least level at which* 10.3 *is a subsolution is* 325 *on the lossless ladder and* 331 *on the ladder of* [9]*, and* 485 *is the least integer constant:* $484 \cdot 10^{68} R_{68} < 103^{68}$.

Theorem 6.11. *For every $n \geq 0$, $R_n \geq (10.4)^n/1071$, that is $52^n \leq 1071 \cdot 5^n R_n$. The least level at which* 10.4 *is a subsolution is* 475 *on the lossless ladder and* 483 *on the ladder of* [9]*, and* 1071 *is the least integer constant:* $1070 \cdot 5^{96} R_{96} < 52^{96}$.

Proof sketch for Theorems 6.8–6.11. All four are Lemma 5.1 with the coefficient list truncated at the stated level and (p, q, C) equal to $(101, 10, 165)$, $(51, 5, 267)$, $(103, 10, 485)$, $(52, 5, 1071)$. Hypothesis (ii) is in each case one comparison of two explicit integers in the cleared form (8): 395 digits a side at base 10.1 and level 195, 426 at base 10.2 and level 247, 659 at base 10.3 and level 325, and 819 at base 10.4 and level 475. Hypothesis (iii) is in each case a walk over the whole table rather than only over the range the induction needs. Minimality of the level is one further evaluation, one level lower, on the ladder concerned, extended downwards by Lemma 5.2; leastness of the constant is one comparison at the index named — the first index at which the next smaller constant fails. For Theorems 6.10 and 6.11 the bound is proved twice, once from each ladder, since the two truncations are different hypotheses. $\square$

Remark 6.12. Theorems 6.4–6.11 are, by design, unconditional and effective: each is an inequality valid for every n, proved from (6) by Lemma 5.1. They also give the lower half of Theorem 4.2 without any analysis. In the converse direction — and here, unlike above, Theorem 4.2 is used as an input, so what follows is conditional on (S1)–(S3) — that theorem makes V_t grow at rate 32/3, so $\sum_t V_t \lambda^{-t-2}$ diverges for every $\lambda < 32/3$, so some finite truncation of (22) makes λ a subsolution, and Lemma 5.1 then yields $R_n \geq \lambda^n/C$ for all n with an explicit C. Every base below the growth rate is therefore reachable at some finite level; the rungs of Table 1 exhibit six of them.

7. Where the scheme stops

Theorem 7.1 (The ceiling)**.** *None of the bases* 10.45*,* 32/3*,* 10.7*,* 11 *is a subsolution at any level $L \leq 518$, on either ladder.*

Proof sketch. Four evaluations, one per base, of the failure inequality at the deepest truncation computed, each a comparison of explicit integers formed from all 519 lossless coefficients; then Lemma 5.2, which propagates each failure to every smaller level at once. Since $V_t \geq W_t$ entrywise, failure on the lossless ladder implies failure on the ladder of [9] at the same level. As in Theorem 6.5, no search over levels is run. $\square$

Theorem 7.2. *$F_L(11) > 0$ for every level $L \leq 247$; and $111\, R_{12} < 11^{12}$, with $R_{12} = 24321036896$.*

Theorem 7.3. *Bases* 10.3 *and* 11 *fail at every level $L \leq 248$ on the lossless ladder as well; in particular, putting back the convolution that the proof of* [9, Corollary 16] *discards moves neither of these two ceilings.*

Remark 7.4. How little it moves them can be quantified, outside the formal development, by locating the root ρ_L numerically: the maximal base that is a subsolution rises from 9.0116815 to 9.0571845 at level 47, from 10.0008311 to 10.0136854 at level 159, and from 10.2016754 to 10.2097850 at level 248, when the convolution is restored.

The last three theorems say the same thing at increasing depth, and their point is what does *not* explain the ceiling. The convolution that [9] discards is not the obstruction: restoring it (Proposition 5.4) moves the maximal reachable base by at most 0.046 at the source's own level and by 0.008 at level 248, and moves the ceiling not at all. Nor is the depth of the truncation the obstruction, in the sense one might expect: by Remark 6.12 every base below 32/3 *is* reachable, so the ladder is complete in the limit — the levels simply grow quickly, from 325 at base 10.3 to 475 at base 10.4, and (extrapolating outside the formal development) to roughly 10^3 at base 10.5. What stops the ladder at 32/3 is that above 32/3 the conclusion is false.

Corollary 7.5 (conditional on Theorem 4.2)**.** *No inequality $R_n \geq \lambda^n/C$ with $\lambda \geq 32/3$ holds for all n; in particular there is no constant C with $R_n \geq 11^n/C$ for every n, and no truncation of* (6) *or of* (22) *at any level makes* 11 *a subsolution.*

This corrects an expectation that the finite data alone invited. With only Theorem 7.2 in hand — and with the upper bound $(7+\sqrt{33})^n \approx 12.745^n$ of [9, Theorem 20] leaving room, and the ratios R_n/R_{n-1} still rising through 10.56 at $n = 248$ — one would naturally read the failure at base 11 as a limitation of the method rather than as a fact about the sequence. It is the latter: $11 > 32/3$.

8. The convex topologies on a chain

The sequence P_n is the number of convex topologies on an $(n+1)$-element chain [3], equivalently the number of Tamari interval preposets [1, §6.4]. We have found no exponential bound for it in the literature. [3] gives an algorithm for the count, its Table 2 of the ten values $n \leq 10$, and, on p. 26, the upper bound $(n!)^2$, of which it says only that it "is not sharp"; it makes no asymptotic or growth statement. [1, §6.4] writes that no simple formulas appear to exist. And [9, §8.4], which establishes the identification, says only that the results of its Section 7 "provide information for the growth of A234268" and gives no number; its §7.2 proves $P_n \leq R_n \leq 2P_n$, so P_n inherits the growth rate of R_n with a factor 2 in the constant. The route below is different and gives better constants: the truncation of (21) itself converts an R-bound into a P-bound.

Lemma 8.1. *Let ℓ be a list of natural numbers of length m, let $p = r + q$ with $q, r > 0$, let C, C' be natural numbers with $C > 0$ and $C\,p \leq C'\,r$, and let $(R_n), (P_n)$ be sequences of natural numbers such that $R_{n-1} + \sum_{t<m} \ell_t R_{n-t-2} \leq P_n$ for $n \geq m+1$, that p/q is a subsolution for ℓ, that $p^n \leq C\,q^n R_n$ for every n, and that $p^n \leq C' q^n P_n$ for $n \leq m$. Then $p^n \leq C' q^n P_n$ for every n.*

Proof. Fix $n \geq m+1$. Multiplying the hypothesis by Cq^n, redistributing the powers of q, and applying $p^k \leq Cq^k R_k$ term by term gives $q\,p^{n-1} + \sum_{t<m} \ell_t q^{t+2} p^{n-t-2} \leq C\,q^n P_n$. The subsolution inequality, shifted by p^{n-m-1}, reads $p^n \leq 2q\,p^{n-1} + \sum_{t<m} \ell_t q^{t+2} p^{n-t-2}$, and $p^n = r\,p^{n-1} + q\,p^{n-1}$, so one copy of $q\,p^{n-1}$ pays for the leading term and $r\,p^{n-1} \leq C\,q^n P_n$. Multiplying by C' and using $Cp \leq C'r$ gives $C\,p^n \leq C\,C' q^n P_n$; cancel C. For $n \leq m$ the claim is the hypothesis. □

The gain over the route through $P_n \geq R_n/2$ is a factor $2(\lambda-1)/\lambda$, about 1.8 at these bases. At each of the five bases below, the constraint $Cp \leq C'r$ — that is, $C' \geq C\lambda/(\lambda-1)$ — is inactive, its right-hand side being 14, 124, 297, 538 and 1185; so the least constant that the tabulated P_n themselves permit is what survives.

Theorem 8.2 (The first explicit growth bounds for A234268)**.** *For every $n \geq 0$,*

$$P_n \geq \frac{9^n}{16}, \qquad P_n \geq \frac{10^n}{144}, \qquad P_n \geq \frac{(10.2)^n}{344},$$

and each constant is least among integers: 15, 143 *and* 343 *fail at $n = 10$,* 36 *and* 53 *respectively.*

Theorem 8.3. *For every $n \geq 0$, $P_n \geq (10.3)^n/623$ and $P_n \geq (10.4)^n/1369$, and both constants are least among integers:* 622 *fails at $n = 68$ and* 1368 *at $n = 95$.*

Proof sketch for both. Lemma 8.1 with $\ell = (W_0, \ldots, W_L)$ at the level of the corresponding R-bound — $L = 47, 159, 247, 331, 483$ for the five bases — the R-bound being Theorem 6.1, 6.4, 6.9, 6.10, 6.11 respectively, and with (p, q, r) the cleared base. Hypothesis (ii) of Lemma 5.1 is reused verbatim. The initial ranges are walks over the whole tabulated P; leastness of each constant is one comparison at the index named, against the tabulated P_n. □

9. The tables, and that nothing above is vacuous

Every numerical statement above is an inequality or an equality between explicit integers built from $R_0, \ldots, R_{518}$, $P_0, \ldots, P_{518}$, $W_0, \ldots, W_{518}$ and $V_t = W_t + P_t$. Those four lists of 519 integers, the largest of 528 digits, are carried in the formal development as literals and are tied to their definitions in four independent ways.

Proposition 9.1 (Published values)**.** *The tabulated $R_0, \ldots, R_{20}$ are exactly the* 21 *terms of A393920 as the entry stood, and the tabulated $P_0, \ldots, P_{20}$ are exactly the* 21 *values printed in* [9, Figure 1]*, of which the first* 10 *are the whole of A234268. The tabulated $W_0, \ldots, W_{47}$ are exactly the* 48 *integers printed in* [9, §7.3]*, including $W_{47} = 44225210278527555070483300450520284568293336 20$.*

Proposition 9.2 (The source's recurrence, re-run)**.** *Running* (2)–(4) *with the initial conditions of* [9, §1.2] *up to $n = 248$, and forming $\sum_i a(n,i)$, $\sum_i b(n,i)$ and* (5)*, reproduces the first* 249 *entries of all four tabulated lists exactly.*

Proposition 9.3 (The definition, by brute force)**.** *Enumerating all $2^{\binom{n+2}{2}}$ subsets of $\mathcal{Q}(n)$ and testing Condition* ($\star$) *directly gives, for $0 \leq n \leq 5$, $(R_n) = 2, 7, 34, 199, 1308, 9300$ and $(P_n) = 1, 4, 21, 129, 876, 6376$, in agreement with the tabulated values.*

Proposition 9.4 (The identities, and the extension)**.** *The tabulated lists satisfy* (21) *and* (22) *at every index $1 \leq n \leq 518$; their first* 249 *entries are the shorter tables of Propositions 9.2, entry for entry; and they satisfy* (19) *and* (20) *at every index $0 \leq n \leq 516$.*

Proof sketches. Propositions 9.1 and 9.4 are finite equalities and finite systems of equalities between integer lists, checked by the proof kernel. Proposition 9.2 runs the recurrence as an $O(n^3)$ dynamic program with suffix sums over the b-table; the reformulation matters, since (4) written literally is $O(n^4)$. Proposition 9.3 is exhaustive: it sweeps all 2^{15} subsets of the 15-point $\mathcal{Q}(4)$ and all 2^{21} subsets of the 21-point $\mathcal{Q}(5)$, testing Condition ($\star$) against each pair of distinct points that constrains it — 90 such pairs at $n = 4$ and 175 at $n = 5$, the condition being symmetric in the two points. At $n = 6$ the sweep is over 2^{28} subsets and we did not run it. Which of these are checked by the proof kernel and which by compiled evaluation is spelled out in §10. ☐

Proposition 9.3 is the check that matters most: it validates the tables against Condition ($\star$) itself, so a misreading of (2)–(4) could not propagate silently into every bound in this paper. Proposition 9.1 anchors the tables to the published record, Proposition 9.2 removes any external program from the chain, and Proposition 9.4 pins the long tables to the short ones and to the identities. Independently of all of it, four separate computations agree wherever they overlap: an $O(n^3)$ dynamic program for (2)–(5), run to $n = 500$; the closed forms (17)–(18), over the whole table; and a second $O(n^3)$ and an $O(n^4)$ implementation of the same recurrence, each run to $n = 320$.

Proposition 9.5 (Non-vacuity)**.** *The sequence $S_n = 2 \cdot 11^n$ satisfies every hypothesis of Lemma 5.1 at each level used above, on either ladder, and the pair $R_n = P_n = 2 \cdot 11^n$ satisfies every hypothesis of Lemma 8.1. Moreover the tabulated R_n satisfy* (6) *and its lossless form, and the tabulated P_n satisfy the truncation of* (21)*, at every index at which the tables reach far enough back to state them.*

Proof sketch. The recurrence side is Lemma 5.3 applied to the base-11 failures of Theorems 7.2 and 7.1, scaled by 2: a base at which a truncation *fails* is exactly a base whose powers form a supersolution. The initial side is the walk $R_n \leq 2 \cdot 11^n$, respectively $P_n \leq 2 \cdot 11^n$, over the whole table; the ratio is largest at $n = 0$, where it is exactly 2, respectively 1. The last assertion is a walk of several hundred instances of the truncated inequalities over the tabulated values. ☐

Together with the leastness statements, this closes the usual gaps: a constant that is too small is refuted at a named index, a base that is too large is refuted at every level computed, and the hypotheses are met by an explicit sequence, so no theorem above is true by vacuity.

10. Verification

Every lemma, proposition and theorem above except Theorem 4.2 and Corollary 7.5 has been formally verified in Lean 4 [7] (toolchain `v4.32.0`) against *Mathlib* [8]; the development contains no `sorry`, and a repository reference is to be added at submission. Five things are deliberately outside that guarantee, and each is flagged where it occurs: the truncated inequality (6) and the identity (21), which are inputs taken from [9, Corollary 16, Lemma 15] and carried as hypotheses; the derivation of Proposition 3.1 and the elimination leading to Theorem 3.2, of which the formal content is the two ring identities and the 161-coefficient series check described in its proof; the differential-equation certificate behind (19)–(20), an exact rational-function computation whose formal counterpart is the verification of both recurrences at all 517 indices of the tables; the three classical steps (S1)–(S3), which are what turn Theorem 4.1 into Theorem 4.2 and which no part of the formal development touches; and the sign analysis of §2.3, together with the remarks explicitly labelled as computed outside the formal development (Remarks 6.7, 6.12 and 7.4, the $n = 800$ row of Table 2, and the extrapolated level at base 10.5 in §7). The development is split so that the reasoning layer contains no numerical data and the data layer imports no library: Lemmas 5.1, 5.2, 5.3 and 8.1 are proved for an arbitrary coefficient list, arbitrary sequences and an arbitrary rational base, and depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. All arithmetic that carries a claim is checked by the proof kernel: the subsolution and failure evaluations (8) behind every rung and every ceiling statement; the initial-range walks, each run over the full table rather than only over the range the induction needs; every leastness comparison; the four polynomial identities and the two rational evaluations of Theorem 4.1, and the two ring identities of Theorem 3.2; the truncated series identity for (14) on $x^0, \dots, x^{160}$; the 518 instances each of (21) and (22) and the 517 instances each of (19)–(20); the list equalities of Propositions 9.1 and 9.4; and the witnesses of Proposition 9.5. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly two things, both controls rather than claims: the $O(n^3)$ dynamic program of Proposition 9.2, which is an array computation and not a kernel object, and the exhaustive subset sweeps of Proposition 9.3. No theorem of §§3–8 depends on either.

11. Questions left open

Formalising the analytic step, and an effective upper bound. The gap between Theorem 4.1 and Theorem 4.2 is three named classical results applied to explicit polynomials; nothing in the present situation is delicate, and the input is finite. Theorem 4.2 is also not effective: it gives no explicit C with $R_n \le C\,(32/3)^n$, whereas the lower half of the ladder is unconditional and explicit at every rung. The algebraic equation should supply the matching upper constant.

A bijective explanation of the double degeneracy. The subexponential factor is $n^{-5/2}$ rather than the $n^{-3/2}$ of a generic square-root singularity, because at the branch point $t_0 = 1/3$ not only $x'(t_0) = 0$ but also $\frac{d}{dt}\big(\varphi(2-3t)\big)$ and $\frac{d}{dt}\big(\varphi(1-t-t^2)\big)$ vanish — for R and for P simultaneously. An $n^{-5/2}$ law is what one sees for unrooted or symmetry-quotiented families. We do not know a combinatorial reason for it here.

The published entries. Both OEIS entries carried the keyword `more` and no formula when we checked them on 30 August 2026; Theorem 3.3 removes the obstacle in both cases, and [9, Figure 1] already prints eleven terms of A234268 beyond the entry.

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