THE 82-QUEEN COVER OF Q_{163}: MINIMALITY, AND AN OBSTRUCTION AT Q_{167}

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Abstract. The queen's graph Q_n has the squares of the $n \times n$ board as vertices, two squares adjacent when they share a row, a column or a diagonal, and $\gamma(Q_n)$ is its domination number. A recent preprint of Kong prints 82 squares of the 163×163 board and asserts, on the evidence of a short script, that they dominate it; with the Finozhenok–Weakley bound $\gamma(Q_n) \geq \lceil n/2 \rceil$ this gives $\gamma(Q_{163}) = 82$, an exact value beyond the published list, and the type A structure of the set improves the constant in Östergård–Weakley's amplification theorem to 17/33. We first re-establish every printed property of that certificate by exhaustive verified computation over all $163^2 = 26\,569$ squares, in place of the script. We then prove two statements the source does not make. The certificate is a *minimal* dominating set: each of its 82 queens owns a square that no other queen of the set covers, so no 81 of them dominate. And — the main result — it does not extend: with the 82 squares kept in place, no dominating set of Q_{167} of size at most 84 contains them, and none of Q_{165} of size at most 83 does. The first is a nonexistence statement over all $27\,889^2$ ordered pairs of added squares, reduced to $499 \times 27\,889$ by an anchor argument whose completeness is proved rather than assumed. Since $\gamma(Q_{167}) \geq 84$ by Finozhenok–Weakley, a new exact value at 167 needs a new construction and not an extension of this one. Everything is machine-checked in Lean 4; the two published theorems we rely on — the Finozhenok–Weakley bound and Östergård–Weakley's amplification theorem — are cited rather than reproved, and the Finozhenok–Weakley bound appears as an explicit hypothesis of the statements that use it.

1. Introduction

The *queen's graph* Q_n has as vertices the n^2 squares of the $n \times n$ chessboard, two squares being adjacent when they lie in a common row, a common column or a common diagonal. Its domination number $\gamma(Q_n)$ is the least number of queens that together occupy or attack every square of the board. Computing $\gamma(Q_n)$ is a classical problem, and exact values are scarce: the standard reference table, Östergård–Weakley [2, p. 17], lists the orders for which $\gamma(Q_n) = \lceil n/2 \rceil$ and *stops at* $n = 131$, and the OEIS entry A075458 [6] for the sequence has 25 terms, its comment recording only that $\gamma(Q_n) \in \{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ has been proved for $n \leq 132$ apart from $n = 3, 11$. Complete searches by SAT solving have so far reached $n \leq 19$ [5]. Later work has not lengthened the table of exact values. Proposition 21 of [8] collects the even orders with $\gamma(Q_n) = n/2$ and reaches $n = 130$; [9] improves *lower* bounds and exhibits no dominating sets. We are aware of no exact value $\gamma(Q_n) = \lceil n/2 \rceil$ for any $n > 131$ outside [1]: a sweep of the table of [2], of [6], and of the arXiv listings for queen domination, repeated on 29 August 2026, turned up none.

Two general results frame the subject. On the one hand Finozhenok and Weakley [3, Corollary 4] proved

$$\gamma(Q_n) \ \geq \ \lceil n/2 \rceil \qquad \text{for every positive integer } n \notin \{3, 11\}, \tag{1}$$

so an exact value $\gamma(Q_n) = \lceil n/2 \rceil$ is established as soon as a set of that size is exhibited and checked. On the other hand Östergård–Weakley [2, Theorem 5] showed that one such finite set *amplifies*: a cover of one board Q_n of a suitable type yields, for all k, an upper bound $\gamma(Q_k) \leq \frac{d+3}{n+2}k + O(1)$, where d is the size of the cover. Their own 66-queen cover of Q_{131} gives $\gamma(Q_k) \leq \frac{69}{133}k + O(1)$ [2, Corollary 6], and a different amplification, due to Burger and Mynhardt, gives $\gamma(Q_k) \leq \frac{101}{195}k + O(1)$ [4, Corollary 12]. Both constants exceed 1/2, which (1) shows is the floor; whether 1/2 is the truth is not settled by any of them.

On 23 August 2026 Kong [1] posted 82 explicit squares of the 163×163 board, together with the assertion — supported there by a standard-library Python script, printing `uncovered: 0` — that they dominate Q_{163}. With (1) this gives $\gamma(Q_{163}) = 82$, an exact value 32 beyond the published list, and since the 82 squares form a type A 1-cover in the sense of [2] they feed [2, Theorem 5] and give

$$\gamma(Q_k) \;\leq\; \tfrac{17}{33}\,k + O(1),$$

below both published coefficients. A construction of this kind invites an immediate question that [1] does not address: $\lceil n/2 \rceil$ grows by one each time n grows by two, so can the 82 queens simply be kept and the extra queens the bound allows be spent, producing further exact values at 165, 167, 169?

Results. This paper does three things.

Verification (Sections 3–5). Every property of the certificate printed in [1] is re-established by exhaustive computation over the whole board, checked by machine rather than by a script: that the 82 squares are distinct and occupy each odd row and each odd column exactly once, that not one of the 26 569 squares escapes them, the two closed forms for the multisets of occupied diagonals, the parameters $(e, f, u) = (16, 15, 24)$ — recomputed from the definitions of [2] rather than copied — and the type A condition, which holds *with equality*, $e + f + 2u = 79 = (163 - 5)/2$. The domination number is defined from first principles and $\gamma(Q_{163}) \leq 82$ is proved; $\gamma(Q_{163}) = 82$ is stated with (1) carried as an explicit hypothesis, so that what is proved and what is cited stay separated inside the statement.

Minimality (Section 6). Each of the 82 queens has a private square: a square of the board covered by that queen and by no other queen of the set. Consequently no proper subset of the certificate dominates Q_{163} (Corollary 6.2). This does not use (1), which would give the far stronger — but only cited — statement that no 81 squares whatsoever dominate.

Non-extension (Section 8). The certificate cannot be pushed to the next boards inside the budget that (1) allows. On Q_{165} it leaves 8 squares uncovered and no single added queen covers all eight (Theorem 8.1). On Q_{167} it leaves 340 squares uncovered and no *pair* of added queens covers them all (Theorem 8.2); with $\gamma(Q_{167}) \geq 84$ from (1), this says that no minimum dominating set of Q_{167} of the size the lower bound permits contains the certificate. The cheapest route from Kong's construction to a further exact value is therefore closed, and Section 9 records what a new construction would have to look like, together with the measured cost of the searches that failed to find one.

2. Preliminaries

2.1. **Boards, covering, domination.** It is convenient to use the centred coordinates of [1]. For $H \geq 0$ put

$$B_H = \{-H, -H+1, \ldots, H\}^2 \subset \mathbb{Z}^2, \qquad |B_H| = (2H+1)^2,$$

and identify the board of Q_n, $n = 2H + 1$, with B_H. A queen at $q = (q_1, q_2)$ *covers* the square $p = (p_1, p_2)$ when

$$q_1 = p_1 \quad \text{or} \quad q_2 = p_2 \quad \text{or} \quad q_2 - q_1 = p_2 - p_1 \quad \text{or} \quad q_1 + q_2 = p_1 + p_2, \tag{2}$$

that is, when the two squares share a row, a column, a difference diagonal or a sum diagonal; in particular a queen covers its own square. A set $D \subseteq B_H$ *dominates* Q_n if every $p \in B_H$ is covered by some $q \in D$, and

$$\gamma(Q_n) = \min\{|D| : D \subseteq B_H,\ D \text{ dominates } Q_n\}.$$

For a finite $D \subseteq \mathbb{Z}^2$ write $\Delta(D)$ for the multiset $\{q_2 - q_1 : q \in D\}$ of its difference diagonals and $\Sigma(D)$ for the multiset $\{q_1 + q_2 : q \in D\}$ of its sum diagonals; the two *long* diagonals are those indexed by 0.

2.2. **Orthodoxy, covers, and the parameters** (e, f, u)**.** The following notions are Östergård–Weakley's [2, §2], transcribed into the coordinates above. Fix $p \in \{0, 1\}$. A set $D \subseteq B_H$ is *p-orthodox* if for every $v \in \{-H, \dots, H\}$ with $v \equiv p \pmod 2$ there is a square of D in row v and a square of D in column v; it is a *p-cover* if, in addition, every square $(x, y) \in B_H$ with $x \not\equiv p$ and $y \not\equiv p \pmod 2$ shares a diagonal with some square of D, i.e. satisfies $y - x \in \Delta(D)$ or $x + y \in \Sigma(D)$. For $H = 81$, so $n = 163$, a 1-orthodox set meets each of the 82 odd rows and each of the 82 odd columns, and the squares constrained by the second condition are the $81^2 = 6561$ squares both of whose coordinates are even.

Next, for a set D containing a square of each long diagonal, $e(D)$ is the largest integer e such that $2i \in \Delta(D)$ for every i with $|i| \le e$; $f(D)$ is defined in the same way from $\Sigma(D)$; and $u(D)$ is the largest integer u such that for every i with $1 \le i \le u$ all four of

$$\pm\bigl(2e(D) + 4i\bigr) \in \Delta(D), \qquad \pm\bigl(2f(D) + 4i\bigr) \in \Sigma(D)$$

hold. Östergård–Weakley's Theorem 3 is a *characterisation* of p-covers in these terms, not merely a sufficient condition, and we quote it in full.

Theorem 2.1 (Östergård–Weakley [2, Theorem 3])**.** *Let n be an odd positive integer and let $p \in \{0, 1\}$. Let D be a p-orthodox set for Q_n that contains at least one square from each of the long diagonals, and let $e = e(D)$, $f = f(D)$ and $u = u(D)$. Then D is a p-cover of Q_n if and only if either*

$$e + f \equiv p \pmod 2 \quad \text{and} \quad e + f + 2u \;\ge\; \tfrac{n-5}{2}, \tag{2A}$$

or

$$e + f \equiv 1 - p \pmod 2 \quad \text{and} \quad e + f \;\ge\; \tfrac{n-3}{2}. \tag{2B}$$

A p-cover is of *type A* or *type B* according to which of (2A), (2B) it satisfies. Finally we record the amplification theorem in the form in which it is used.

Theorem 2.2 (Östergård–Weakley [2, Theorem 5], first branch)**.** *Let n be an odd positive integer, let $p \in \{0, 1\}$, and let D be a type A p-cover of Q_n that contains d squares, including a square of each long diagonal. If $p = 0$ and $n \equiv 1 \pmod 4$, or $p = 1$ and $n \equiv -1 \pmod 4$, then for all k,*

$$\gamma(Q_k) \;\le\; \frac{d+3}{n+2}\, k + O(1).$$

Both theorems are stated in [2] — which we read in full, and whose numbering we use — and are proved there by reference to [7]. Theorems 2.1 and 2.2 are cited, not proved here, and neither is (1).

3. The certificate and its verified properties

Definition 3.1. $D_0 \subset B_{81}$ is the set of 82 squares printed in [1, §2]. Its x-coordinates are the 82 odd integers of $[-81, 81]$, each once, so D_0 is the graph of a map $x \mapsto y$ on those integers, and we present it that way:

x	-81	-79	-77	-75	-73	-71	-69	-67	-65	-63	-61	-59	-57	-55
y	-41	-19	31	25	-9	49	55	-39	-29	-51	51	-67	15	-23
x	-53	-51	-49	-47	-45	-43	-41	-39	-37	-35	-33	-31	-29	-27
y	27	77	-69	57	-25	-43	75	-71	39	-45	-57	61	-33	69
x	-25	-23	-21	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1
y	-77	65	-7	11	67	53	-11	-5	13	19	-61	-17	9	-1
x	3	5	7	9	11	13	15	17	19	21	23	25	27	29
y	21	-63	-81	-21	-15	7	59	73	-3	-75	5	-59	79	37
x	31	33	35	37	39	41	43	45	47	49	51	53	55	57
y	-49	81	23	41	63	-79	-73	-31	47	-55	35	-47	71	-35
x	59	61	63	65	67	69	71	73	75	77	79	81		
y	-13	1	-65	17	3	29	-53	45	-37	33	43	-27		

All the statements of this section and the next three are decided by evaluating a Boolean function on the finite data above; each proof sketch says which finite object was swept.

Proposition 3.2. $|B_{81}| = 26\,569 = 163^2$*, and the covering relation* (2) *is not identically true: the queen at* $(-81, -41)$ *does not cover the centre* $(0, 0)$*.*

Proof sketch. The board is enumerated and counted, and (2) is evaluated at one pair. The first assertion is the control that makes the exhaustive sweeps below meaningful — they run over a list of this length and not over an accidentally empty one. □

Proposition 3.3. D_0 *consists of* 82 *pairwise distinct squares of* B_{81}*. Its multiset of* x*-coordinates and its multiset of* y*-coordinates are each exactly the* 82 *odd integers of* $[-81, 81]$*, each occurring once. In particular* D_0 *is* 1*-orthodox on* Q_{163}*, and every coordinate of every square of* D_0 *is odd.*

Proof sketch. Length, duplicate-freeness and membership in B_{81} are computed directly; the two axis claims are verified as identities between sorted lists, so that they carry the multiplicity-one assertion and not merely the equality of underlying sets. This is equation (eq:axis-set) of [1]. □

Theorem 3.4. D_0 *dominates* Q_{163}*: none of the* 26 569 *squares of* B_{81} *escapes the* 82 *queens. Consequently* $\gamma(Q_{163}) \leq 82$*.*

Proof sketch. Exhaustive: for each of the 26 569 squares $p \in B_{81}$, the list of 82 queens is scanned for one covering p, and the list of squares surviving that scan is checked to be empty. This is Proposition 1 of [1], whose only support there is the output of a script. The passage from the Boolean sweep to the statement "D_0 dominates" is proved, not assumed: it uses that every square satisfying the coordinate bounds occurs in the enumerated board, and that (2) evaluated to `true` is equivalent to the covering relation. The bound on γ then follows from the definition, using Proposition 3.3 for the cardinality. □

Corollary 3.5. *If* $\gamma(Q_{163}) \geq 82$*, then* $\gamma(Q_{163}) = 82$*. The hypothesis holds by* (1)*, which gives* $\gamma(Q_{163}) \geq \lceil 163/2 \rceil = 82$*.*

Proof. Antisymmetry, from Theorem 3.4. The hypothesis is stated rather than discharged because (1) is a theorem about all n which is cited here and not reproved; the formal statement carries it as an explicit assumption for the same reason. □

Proposition 3.6 (Controls)**.** *Deleting from* D_0 *the queen at* $(-81, -41)$ *leaves exactly* 212 *squares of* B_{81} *uncovered. The same* 82 *queens leave exactly* 8 *squares of the larger board* B_{82} *uncovered, so they do not dominate* Q_{165}*. Translating every queen by* $+2$ *in the* y *direction, and wrapping the unique queen with* $y = 81$ *to* $y = -81$*, leaves exactly* 49 *squares of* B_{81} *uncovered.*

Proof sketch. Three further sweeps of the same kind, over 26 569, 27 225 and 26 569 squares respectively; the statements pin exact counts rather than inequalities, so a perturbation of the certificate cannot satisfy them accidentally. The three are the too-small, the too-large and the perturbation control: they show that the cover is exactly tuned to 82 queens on Q_{163} and is not a generic pattern. □

Remark 3.7. The weakest single deletion — taken over all 82 choices of the deleted queen — still leaves 172 squares uncovered. This measurement was made outside the formal development and is quoted here only for scale.

4. Diagonal structure and the type A condition

Proposition 4.1. *As multisets,*

$$\Delta(D_0) = \{2j : -16 \le j \le 16\} \uplus \{\pm(32+4j) : 1 \le j \le 24\} \uplus \{0\}, \tag{3}$$

$$\Sigma(D_0) = \{2j : -15 \le j \le 15\} \uplus \{\pm(30+4j) : 1 \le j \le 24\} \uplus \{-80, 38, 42\}. \tag{4}$$

Both sides have 82 *elements. In particular the long difference diagonal is occupied twice and the long sum diagonal once.*

Proof sketch. The two multisets are formed from the 82 squares, sorted, and compared entrywise with the sorted right-hand sides; the comparison is therefore of multisets and not of underlying sets. These are equations (eq:diff-multiset) and (eq:sum-multiset) of [1]. □

Proposition 4.2. $\big(e(D_0), f(D_0), u(D_0)\big) = (16, 15, 24)$, *and condition* (2A) *holds at* $p = 1$, $n = 163$ with equality*:*

$$e + f = 31 \equiv 1 \pmod 2, \qquad e + f + 2u = 16 + 15 + 48 = 79 = \tfrac{163-5}{2}.$$

Proof sketch. The three parameters are computed from the definitions in Section 2 by searching for the first failure in each defining family — not read off [1] — and the two arithmetic conditions are then evaluated. Concretely: $\pm 34 \notin \Delta(D_0)$ gives $e = 16$, $\pm 32 \notin \Sigma(D_0)$ gives $f = 15$, and $\pm 132 \notin \Delta(D_0)$ gives $u = 24$; all three are visible in (3)–(4). This is equation (eq:parameters) of [1], and the equality in the second condition means that no occupied diagonal anywhere can be spared. □

Proposition 4.3. *Every one of the* 6561 *squares of* B_{81} *with both coordinates even satisfies* $y - x \in \Delta(D_0)$ *or* $x + y \in \Sigma(D_0)$*. Hence* D_0 *is a* 1*-cover of* Q_{163}*.*

Proof sketch. Exhaustive over the 6561 even–even squares. Together with the 1-orthodoxy of Proposition 3.3 this is exactly the definition of a 1-cover, so the conclusion is reached from the definition of [2] directly and independently of Theorem 2.1 — which, by Proposition 4.2 and (2A), gives the same conclusion by a second route, and in addition says the cover is of type A. With Proposition 4.1 (both long diagonals occupied), $|D_0| = 82$ and $163 \equiv -1 \pmod 4$, this is precisely the input Theorem 2.2 consumes. □

5. The parity profile

The exhaustive sweep of Theorem 3.4 can be replaced, for sets of this shape, by two inequalities. For $\varepsilon \in \{0, 1\}$ let $a_\varepsilon(D)$ be the least $r \ge 0$ with $r \equiv \varepsilon \pmod 2$ such that $2r \notin \Delta(D)$ or $-2r \notin \Delta(D)$, and let $b_\varepsilon(D)$ be defined in the same way from $\Sigma(D)$.

Lemma 5.1. *Let* H *be odd and let* $D \subseteq B_H$ *be* 1*-orthodox with every coordinate of every square of* D *odd. Then* D *dominates* Q_{2H+1} *if and only if*

$$a_1(D) + b_1(D) > H - 1 \qquad \textit{and} \qquad a_0(D) + b_0(D) > H - 1.$$

Proof. A square with an odd coordinate lies in an occupied row or an occupied column, by 1-orthodoxy, hence is covered; and a square with both coordinates even lies in no occupied row or column, since all queens have odd coordinates, hence is covered if and only if $y - x \in \Delta(D)$ or $x + y \in \Sigma(D)$. Writing such a square as $x = s - d$, $y = s + d$, the conditions "x, y even" and "$|x|, |y| \le H - 1$" become "$d \equiv s \pmod 2$" and "$|d| + |s| \le H - 1$". So an uncovered square exists exactly when there are d, s of a common parity ε with $2d \notin \Delta(D)$, $2s \notin \Sigma(D)$ and $|d| + |s| \le H - 1$; the least such $|d|$ is $a_\varepsilon(D)$ and the least such $|s|$ is $b_\varepsilon(D)$. □

Lemma 5.1 is the specialisation, to odd-coordinate 1-orthodox sets, of the criterion of Theorem 2.1. The link runs through $e(D)+1 = \min\big(a_1(D), a_0(D)\big)$ and $f(D)+1 = \min\big(b_1(D), b_0(D)\big)$, which are immediate from the definitions; for the certificate these give $e = 16$ and $f = 15$, and the two displayed inequalities read $82 > 80$ twice, which is the profile form of the equality $e+f+2u = 79 = (163-5)/2$ of Proposition 4.2. The lemma is stated here because it is the form in which a search for further covers can be posed (Section 9); it is elementary, and it is proved by hand rather than machine-checked. What is machine-checked is the profile of the certificate:

Proposition 5.2. $\big(a_1, b_1, a_0, b_0\big)(D_0) = (17, 65, 66, 16)$, *and*

$$a_1(D_0) + b_1(D_0) = 82, \qquad a_0(D_0) + b_0(D_0) = 82,$$

against the threshold $H - 1 = 80$ *of Lemma 5.1.*

Proof sketch. The four quantities are computed from $\Delta(D_0)$ and $\Sigma(D_0)$ by scanning r upwards in each parity class; the values agree with the exhaustive sweep of Theorem 3.4 through Lemma 5.1. □

Both sums are as small as they can be. Indeed $a_1 + b_1$ is a sum of two odd numbers and $a_0 + b_0$ a sum of two even numbers, so both are even, and an even number exceeding 80 is at least 82: the certificate is tight in both parity classes at once, with no slack in either. This is the profile form of the equality $e + f + 2u = 79$ of Proposition 4.2.

6. Minimality

Theorem 6.1. *Every queen of D_0 has a private square: for each $q \in D_0$ there is a square $p \in B_{81}$ covered by q and by no other square of D_0.*

Proof sketch. One pass over the 26 569 squares assigns to each square, when exactly one queen of D_0 covers it, the index of that queen; this produces a table of at most one witness per queen, and the table is then checked to be complete — all 82 entries present — and correct: for each index k the recorded square lies in B_{81}, is covered by the k-th queen, and is covered by exactly one queen of D_0 in total. The verification of the table costs a further 82×82 evaluations of (2). The statement is not made in [1] and does not follow from the check performed there. □

Corollary 6.2. *No proper subset of D_0 dominates Q_{163}; in particular no* 81 *of the* 82 *queens dominate.*

Proof. A subset $D \subsetneq D_0$ omits some $q \in D_0$, whose private square from Theorem 6.1 is then covered by no member of D. □

Corollary 6.2 does not use (1). That bound gives more — with it, no 81 squares of the board whatsoever dominate Q_{163} — but only by citation; the argument above is self-contained.

7. The amplification coefficient

By Propositions 4.1, 4.2 and 4.3 the certificate is a 1-cover of Q_{163} of size $d = 82$ occupying both long diagonals and satisfying (2A) — that is, a type A 1-cover — and $163 \equiv -1 \pmod 4$, so Theorem 2.2 applies with $p = 1$ and yields the coefficient $(d+3)/(n+2) = 85/165$.

Proposition 7.1. $\dfrac{82+3}{163+2} = \dfrac{17}{33}$, *and*

$$\frac{69}{133} - \frac{17}{33} = \frac{16}{4389}, \qquad \frac{101}{195} - \frac{17}{33} = \frac{2}{715}, \qquad \frac{1}{2} < \frac{17}{33} < \frac{101}{195} < \frac{69}{133}.$$

Moreover the same formula applied to Östergård–Weakley's own type A 1*-cover of size* 66 *on* Q_{131} *returns their published constant:* $(66+3)/(131+2) = 69/133$*.*

Proof. Rational arithmetic. These four identities involve no computation over the board and are checked by the proof kernel alone, with no appeal to compiled evaluation. The last sentence is a control on our reading of Theorem 2.2: applied to the 2001 certificate it must reproduce the 2001 constant, and it does. □

So the certificate improves the type A coefficient 69/133 of [2] by 16/4389, and the coefficient 101/195 of [4] by 2/715, while remaining above 1/2 — it does not decide whether $\gamma(Q_k)/k \to 1/2$.

Remark 7.2 (Provenance of the two published constants). The constant 69/133 is [2, Corollary 6]. The constant 101/195 is [4, Corollary 12]: for all n large enough, $\gamma(Q_n) \le \frac{101}{195}n + O(1)$. It is obtained there by substituting $k = 32$ into [4, Corollary 11], which gives $\gamma(Q_n) \le \frac{3k+5}{6k+3}n + O(1)$ for all large n whenever Q_{4k+1} carries a dominating set of cardinality $2k+1$ of a certain type; Q_{129} carries one of size 65 by [2], and $(3 \cdot 32 + 5)/(6 \cdot 32 + 3) = 101/195$. Access is worth recording, because [4] is served by its publisher only behind a paywall: the statements above were read on 29 August 2026 in the Internet Archive's capture of the publisher's own full-text page, together with the publisher's still-public images of the displayed formulas, and the general form of Corollary 11 agrees with the zbMATH review of [4] (Zbl 1015.05066). The improvement is not universally reflected in the literature: a 2026 preprint on the three-dimensional problem [10] still quotes 69/133 as the two-dimensional record.

8. The certificate does not extend

By (1), $\gamma(Q_{165}) \ge 83$, $\gamma(Q_{167}) \ge 84$ and $\gamma(Q_{169}) \ge 85$. So the cheapest conceivable route from $\gamma(Q_{163}) = 82$ to a further exact value is to keep the 82 queens and add one, two or three more. Throughout this section D_0 is regarded as a set of squares of the larger board through the common centre — that is, the 82 queens keep the coordinates of Definition 3.1, and the 163×163 board sits concentrically inside the larger one.

Since D_0 dominates B_{81} (Theorem 3.4), every square it leaves uncovered on a larger board lies in the frame outside B_{81}; on B_{83} that frame has $167^2 - 163^2 = 1320$ squares.

Theorem 8.1. D_0 *leaves exactly* 8 *squares of* B_{82} *uncovered, and for every square* $a \in B_{82}$ *the set* $D_0 \cup \{a\}$ *fails to dominate* Q_{165}*. Consequently no dominating set of* Q_{165} *of size at most* 83 *contains* D_0*.*

Proof sketch. The residual set is computed by a sweep over the 27 225 squares of B_{82}; it is

$$\{(-82,-48),\ (82,48),\ (-48,-82),\ (48,82),\ (-82,50),\ (82,-50),\ (50,-82),\ (-50,82)\},$$

eight squares of the two added rows and columns. The search is then exhaustive over all 27 225 candidate squares a: for each, one of the eight residual squares is found that a does not cover. The final sentence follows because a dominating set D with $D_0 \subseteq D \subseteq B_{82}$ and $|D| \le 83$ is either D_0 itself, which is excluded by Proposition 3.6, or $D_0 \cup \{a\}$ for a single square a. The verified statement pins the cardinality of the residual set and the outcome of the search; the eight coordinates are displayed for the reader. □

The board Q_{165} is a different structural regime from Q_{163} and Q_{167}: $165 \equiv 1$ and $163 \equiv 167 \equiv -1 \pmod 4$, so a cover of the shape considered in Section 2 would there be a 0-cover, with queens on even rows and columns. The residual set of size 8 is small for that reason and not because Q_{165} is nearly covered in any useful sense.

Theorem 8.2. *D_0 leaves exactly 340 squares of B_{83} uncovered, and for every pair of squares $a, b \in B_{83}$ the set $D_0 \cup \{a, b\}$ fails to dominate Q_{167}. Consequently no dominating set of Q_{167} of size at most 84 contains D_0: every dominating set of Q_{167} containing the certificate has at least 85 squares.*

Proof sketch. Let $U \subset B_{83}$ be the set of squares uncovered by D_0, computed by a sweep over all $167^2 = 27\,889$ squares; $|U| = 340$. Suppose $D_0 \cup \{a, b\}$ dominates Q_{167}. Fix the *anchor* $z = (-83, -82)$, the first element of U in the enumeration order of the board. Then z is covered by a or by b, and the hypothesis is symmetric in the pair, so we may assume a covers z. The set $A = \{c \in B_{83} : c \text{ covers } z\}$ of anchor candidates is computed and has exactly 499 elements. The search now runs over $A \times B_{83}$: for each $a \in A$ the residual $U_a = \{p \in U : a \text{ does not cover } p\}$ is formed, and every $b \in B_{83}$ is tested against it; in no case does a b cover all of U_a. That is $499 \times 27\,889 = 13\,916\,611$ ordered pairs, each requiring at most 340 evaluations of (2), and the sweep is exhaustive. The unrestricted search would have been over $27\,889^2 = 777\,796\,321$ ordered pairs.

The anchor restriction is the only reduction used, and it is a symmetry argument rather than a computation: it is discharged in the formal statement rather than assumed. The Boolean sweep is quantified over the restricted range $A \times B_{83}$, while the theorem is quantified over arbitrary $a, b \in B_{83}$; the passage between them is the two lines above, that z is uncovered by D_0 and that $\{a, b\}$ is unordered. The case $a = b$ is included, so the statement covers the addition of one queen or of none as well. For the last sentence, a dominating set D of Q_{167} with $D_0 \subseteq D \subseteq B_{83}$ and $|D| \leq 84$ is of the form $D_0 \cup \{a, b\}$. □

Corollary 8.3. *If $\gamma(Q_{167}) = 84$, then no minimum dominating set of Q_{167} contains D_0. In particular, the value $\gamma(Q_{167}) = \lceil 167/2 \rceil$, if true, cannot be certified by extending the certificate of* [1].

Remark 8.4 (Scope of Theorem 8.2). The theorem concerns the concentric placement of D_0 in B_{83} fixed at the start of this section. Inside the 167×167 board the 163×163 board has 25 translates, and the computation above covers one of them. Under the translate by (s, t), $|s|, |t| \leq 2$, the queens occupy the rows $\{v + s : v \text{ odd},\ |v| \leq 81\}$: for even s these are 82 of the 84 odd rows of B_{83}, so the placement is of the same parity type as the one treated here, while for odd s they are even rows and the type is different. The other 24 translates were not searched. Each would be a sweep of the same shape — the residual set of that translate, an anchor chosen inside it, and the candidates covering that anchor — and nothing stated in this paper depends on them.

Remark 8.5 (Q_{169} and Q_{171}). On B_{84} the certificate leaves 364 squares uncovered, and an exhaustive anchored search over triples of added squares — 129 221 249 nodes, 1.46 seconds in a C implementation — found none that covers them; on B_{85} the residual has 704 squares and the corresponding search over quadruples, estimated at 6.5×10^{10} nodes, was not run to completion. Both computations were carried out outside the formal development, their node counts and timings are working-note measurements, and neither is claimed as a theorem here.

9. What a new construction would have to look like

Theorem 8.2 closes the cheap route, so the natural target is a fresh certificate.

Question 9.1. For which $n \equiv 3 \pmod 4$ with $n > 163$ does Q_n carry a type A 1-cover of size $(n+1)/2$ occupying both long diagonals?

Such a cover would give both a new exact value $\gamma(Q_n) = (n+1)/2$, by (1), and by Theorem 2.2 an amplification coefficient

$$\frac{(n+1)/2 + 3}{n+2} = \frac{n+7}{2(n+2)},$$

which is 87/169 at $n = 167$, strictly below 17/33. In the formulation of Lemma 5.1, with $n = 2m-1$, one is looking for a permutation of the m odd rows onto the m odd columns whose difference and sum diagonals satisfy the two inequalities. We report the cost of failing to find one, because a null result here is worth nothing unless the search is calibrated. Three attacks were run, all of them on the *known-solvable* instance $m = 82$, where the certificate above is a solution: a direct permutation encoding given to a CDCL solver (6724 variables, 544 886 clauses, no symmetry breaking) was undecided after 290 seconds and 2.4 million conflicts; min-conflicts local search with breakout weights on the coverage objective plateaued with 4 of the 160 required diagonals missing, over all 82 admissible profiles; and the dual formulation, in which the diagonals are fixed and distinct rows and columns are sought, plateaued at 9 conflicts. The searcher is not at fault: seeded with the known solution and perturbed by k random transpositions it recovers a solution reliably for $k \le 14$ and, with restarts, at $k = 18$, and fails at $k = 20$. The obstruction is the size of the basin, not the implementation, and therefore a null result at $m = 84$ would carry no information; none is claimed. Total cost of these searches: about 45 CPU-minutes. Every figure in this paragraph — variable and clause counts, timings, plateau values, transposition thresholds — is a measurement recorded in the working notes of the computation, made outside the formal development. They are quoted so that the null result can be judged, and none of them is a claim of this paper.

Two routes look more promising than blind search. The first is recursion: Östergård–Weakley [2] built their Q_{131} cover from their Q_{129} cover by adding 1 to every row and column index and inserting the single square $(-65, -65)$, which suggests that covers of this type come in families. The naive analogue fails here: raising u from 24 to 25 would need all four of the difference diagonals ± 132 and the sum diagonals ± 130, none of which D_0 occupies, and no square of B_{83} lies on one of the first pair and one of the second at once, so the two queens available at $n = 167$ cannot supply them. Theorem 8.2 proves the much stronger statement that no two squares whatsoever work. The second route is rigidity. Working out the counting behind Lemma 5.1 suggests that the admissible profiles of a size-m odd-coordinate 1-cover of Q_{2m-1} form two one-parameter families,

$$(a_1, b_1, a_0, b_0) = (\alpha,\ m-\alpha,\ m+1-\alpha,\ \alpha-1) \quad \text{or} \quad (\alpha,\ m-\alpha,\ m-1-\alpha,\ \alpha+1), \qquad \alpha \text{ odd},$$

and that the excess diagonals are then forced. The certificate sits at $\alpha = 17$, $m = 82$, in the first family, and the three excesses visible in (3)–(4) — the repeated 0 in Δ, the repeated 38 and 42 and the extra -80 in Σ — are exactly the ones the counting predicts. Rigidity of this kind is not new in outline: [2, Proposition 7] already constrains a type A p-cover of Q_n with $(n + 1)/2$ members to satisfy $e = f + 1$ when $p = 1$, and $e \in \{f, f + 2\}$ when $p = 0$, and the certificate obeys it ($16 = 15 + 1$). The profile statement above would be a sharper form of the same phenomenon, and it would narrow Question 9.1. We have not proved it. It was derived informally and checked against the certificate only; nothing in this paragraph is machine-checked, and it should be read as a proposal rather than as a claim.

10. Verification

Every theorem, proposition and corollary above has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [11, 12], with four exceptions: Theorems 2.1 and 2.2, which are quoted from [2]; Lemma 5.1, which is elementary and is proved by hand above; and Corollary 8.3, which is an immediate consequence of Theorem 8.2 and (1). The remarks, and the search figures of Section 9, lie outside the formal development and are flagged as such where they appear. The development contains no `sorry` and no hand-written axiom. Mathlib, at the pinned revision, carries no notion of graph domination and no chessboard graphs, so the board, the covering relation (2), dominating sets and γ are defined from scratch, as are p-orthodoxy and p-covers, transcribed from [2, §2]. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches: the 26 569-square sweep of Theorem 3.4, the three control sweeps of Proposition 3.6, the private-square table of Theorem 6.1, the multiset, parameter and profile computations of Sections 4–5, the 27 225-candidate search of Theorem 8.1 and the 13 916 611-pair search of Theorem 8.2. Everything else — in particular the passage from each Boolean computation to the quantified statement, including the discharge of the anchor restriction in Theorem 8.2 — is checked by the kernel alone and depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`; the four rational identities of

Proposition 7.1 use no compiled evaluation at all. Two published results are used and are cited rather than reproved, and the formal statements say so: the Finozhenok–Weakley bound (1) appears as an explicit hypothesis of Corollary 3.5 rather than as a fact, and Theorem 2.2 is used only through the arithmetic of Proposition 7.1. Theorem 2.1 is quoted for context and is used nowhere: the 1-cover property is established from the definition in Proposition 4.3, and condition (2A) is verified directly in Proposition 4.2. Every numerical value quoted in this paper was first produced by an independent implementation in Python and the formal statement was then written against it. Repository reference to be added at submission.

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