

# QUADRATIC NUMERICAL SEMIGROUPS OF EMBEDDING DIMENSION FIVE: TWO QUESTIONS OF STAMATE

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let  $H$  be a numerical semigroup,  $K[H]$  its semigroup ring and  $\mathrm{gr}_{\mathfrak{m}} K[H]$  its tangent cone;  $H$  is called *quadratic* when the defining ideal  $I_H^*$  of  $\mathrm{gr}_{\mathfrak{m}} K[H]$  is generated by quadrics. Stamate gave a maximal list of twelve  $h$ -vectors for a quadratic  $H$  of embedding dimension five, exhibited an example for each except  $(1, 4, 5)$ , and asked whether the list should be reduced further; in the same paper he isolated three artinian ideals  $J_1, J_2, J_3$  of  $K[x_2, x_3, x_4, x_5]$  arising in his screening, reported that he could find no quadratic numerical semigroup producing them, and observed that if none existed the hypotheses “ $K$  algebraically closed,  $\mathrm{char} K \neq 2$ ” could be removed from his theorem that Koszul and  $G$ -quadratic agree in embedding dimension five. We settle both questions. No numerical semigroup of embedding dimension five whose tangent cone has  $h$ -vector  $(1, 4, 5)$  is quadratic, so the list of twelve  $h$ -vectors becomes a list of eleven and the missing entry stays missing. The second question has a mixed answer:  $J_2$  is produced by no quadratic numerical semigroup, for any labelling of the variables, whereas  $J_1$  is produced by  $\langle 7, 23, 31, 39, 40 \rangle$  and  $J_3$  by  $\langle 8, 26, 35, 44, 45 \rangle$ , both quadratic — so the hypothesis under which the assumptions on  $K$  were to be dropped is false. The proofs turn each question into finite data: the Cohen–Macaulayness of the tangent cone is *derived* from the Hilbert function rather than assumed, the degree-three condition imposed by quadratic generation becomes a connectivity question on twenty vertices, and what survives is killed by a congruence obstruction on Apéry sets. All finite searches are machine-checked in Lean 4.

## 1. INTRODUCTION

A *numerical semigroup* is a subset  $H \subseteq \mathbb{N}$  containing 0, closed under addition, with  $\mathbb{N} \setminus H$  finite. It has a unique minimal generating set  $\{a_1 < a_2 < \cdots < a_n\}$ ; its *embedding dimension* is  $\mathrm{edim}(H) = n$  and its *multiplicity* is  $e(H) = a_1$ . Over a field  $K$  one attaches to  $H$  the semigroup ring  $K[H] = K[t^h : h \in H] \subseteq K[t]$ , with maximal ideal  $\mathfrak{m} = (t^h : 0 \neq h \in H)$ , and its *tangent cone*

$$R = \mathrm{gr}_{\mathfrak{m}} K[H] = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}.$$

Writing  $S = K[x_1, \dots, x_n]$  and  $I_H = \ker(S \rightarrow K[H])$ ,  $x_i \mapsto t^{a_i}$ , one has  $R \cong S/I_H^*$ , where  $I_H^*$  is the ideal generated by the initial (lowest-degree) forms of the elements of  $I_H$ . Following [10],  $H$  is called *quadratic*, *Koszul* or *G-quadratic* according as  $R$  is defined by quadrics, is a Koszul algebra, or is defined by an ideal with a quadratic Gröbner basis in some coordinates and for some term order. The quadratic property depends on  $H$  alone; the other two may depend on  $K$ .

Stamate [10] studied the tangent cones of quadratic numerical semigroups and proved, among other things, that if  $\mathrm{edim}(H) = 5$  and  $K$  is algebraically closed of characteristic  $\neq 2$ , then  $H$  is Koszul if and only if it is  $G$ -quadratic [10, Thm. 2.4]. His proof runs over the possible  $h$ -vectors of  $R$ , using a growth bound of Eisenbud, Green and Harris [3] to limit them, and it produced a table (Table 1 of [10]) giving “a maximal list of 12 possible  $h$ -vectors” for a quadratic  $H$  with  $\mathrm{edim}(H) = 5$ , together with an example realising each — with a single exception. The row with multiplicity  $e(H) = 10$  and  $h$ -vector  $(1, 4, 5)$  carries a question mark in the Example column, and the text before the table reads (we quote and cite the arXiv version v3 throughout; see the bibliography and Remark 7.1):

“For one of these  $h$ -vectors we could not find examples of semigroups, hence we ask if this list should be further reduced.”

**Question 1.1.** Is there a numerical semigroup  $H$  with  $\text{edim}(H) = 5$ , quadratic, whose tangent cone has  $h$ -vector  $(1, 4, 5)$ ?

A second question sits a page earlier. In the screening behind [10, Thm. 2.4], three artinian ideals of  $\bar{S} = K[x_2, x_3, x_4, x_5]$  appear as possible values of  $J = I_H^* \bmod x_1$ :

- (1)  $J_1 = (x_2^2, x_5^2, x_3^2 - x_2x_4, x_4^2, x_2x_3, x_2x_5, x_3x_4, x_4x_5),$   $h$ -vector  $(1, 4, 2),$
- (2)  $J_2 = (x_2^2, x_5^2, x_3^2 - x_2x_4, x_4^2 - x_2x_5, x_2x_3, x_3x_4, x_3x_5),$   $h$ -vector  $(1, 4, 3),$
- (3)  $J_3 = (x_2^2, x_5^2, x_3^2 - x_2x_4, x_4^2, x_2x_3, x_3x_4, x_3x_5),$   $h$ -vector  $(1, 4, 3).$

The quotient  $\bar{S}/J_1$  has no Gröbner flag on the residue classes of  $x_2, \dots, x_5$  [10, Rem. 2.6], and  $J_2$  and  $J_3$  have no quadratic Gröbner basis with respect to any term order [10, Rem. 2.7]; it is these three cases that stand between [10, Thm. 2.4] and the removal of its hypotheses on  $K$ , which enter through Conca's results on quadratic algebras with small  $h_2$  — [1] for  $h_2 \leq 2$ , the case of  $J_1$ , and [2] for  $h_2 = 3$ , the case of  $J_2$  and  $J_3$  [10, Lem. 2.2(iii),(iv)]. Stamate remarks [10, Rem. 2.8]:

“In practice, we were not able to find quadratic numerical semigroups producing the ideals  $J_1, J_2, J_3 \dots$ . If such semigroups do not exist, we could drop the restrictions on the field  $K$  in Theorem [2.4].”

**Question 1.2.** Which of  $J_1, J_2, J_3$  arise as  $I_H^* \bmod x_1$  for a quadratic numerical semigroup  $H$  with  $\text{edim}(H) = 5$ ?

Neither question appears to have been taken up since. Apart from the companion paper [7], the one paper we have found citing [10] is Sammartano's [9], which discusses quadratic numerical semigroups and multiplicity bounds for tangent cones but touches neither the missing row of Table 1 nor the ideals  $J_i$ .

**Results.** Question 1.1 has a negative answer.

**Theorem 1.3** (Theorem 5.3). *No numerical semigroup  $H$  with  $\text{edim}(H) = 5$  whose tangent cone has  $h$ -vector  $(1, 4, 5)$  is quadratic. Consequently the list of twelve  $h$ -vectors of [10, Table 1] reduces to eleven, and the Example column of that table has no gap.*

Question 1.2 has a mixed answer, and the half that decides the fate of the field hypotheses goes the other way from the one hoped for in [10].

**Theorem 1.4** (Theorems 6.2 and 6.3). *There is no quadratic numerical semigroup of embedding dimension five producing the ideal  $J_2$  of (2), for any labelling of the variables. On the other hand*

$$H_1 = \langle 7, 23, 31, 39, 40 \rangle \quad \text{and} \quad H_3 = \langle 8, 26, 35, 44, 45 \rangle$$

*are quadratic numerical semigroups of embedding dimension five producing, respectively, the ideals  $J_1$  of (1) and  $J_3$  of (3), in the printed labelling of the variables.*

Since  $J_3$  has no quadratic Gröbner basis with respect to any term order and is produced by a quadratic numerical semigroup, the hypothesis “such semigroups do not exist” fails, and this route to removing “ $K$  algebraically closed,  $\text{char } K \neq 2$ ” from [10, Thm. 2.4] is closed. We stress that nothing here shows those hypotheses to be necessary; see Remark 6.5.

**Method.** Both questions are made finite by the same three steps, carried out in Section 3.

First, the relevant Cohen–Macaulayness is *derived*, not assumed. If the Hilbert function of  $R$  satisfies  $\text{HF}_R(2) = \text{HF}_R(3) = e(H)$  — which is exactly what an  $h$ -vector  $(1, 4, h_2)$  says — then  $t^{a_1}$  is automatically a nonzerodivisor on  $R$  (Lemma 3.1). In particular no theorem of [10] is used as an input to any proof below, and the argument does not go through the Cohen–Macaulayness criterion of Herzog [6] and Garcia [5].

Second, after reducing modulo  $x_1$  everything in sight is the incidence matrix of a partition. The ten monomials  $x_i x_j$  ( $2 \leq i \leq j \leq 5$ ) each map in  $\bar{R}_2 = R_2/x_1 R_1$  either to 0 or to the basis vector

indexed by  $a_i + a_j$ ; so  $[J]_2 := J \cap \bar{S}_2$  is spanned by monomials and by differences of monomials, and the condition  $\bar{S}_1[J]_2 = \bar{S}_3$  imposed by quadratic generation is a statement about connected components of an explicit graph on the twenty cubic monomials (Lemma 3.5). A rank of a  $20 \times 20$  matrix becomes a union-find on twenty vertices; in particular every rank occurring below is independent of  $K$ .

Third, only five coincidences  $a_i + a_j = a_k + a_\ell$  are possible among the four generators  $a_2 < a_3 < a_4 < a_5$  — these are Stamate's  $p_0, \dots, p_4$  — and only nine of the 32 subsets of them are realised by integers. For  $h$ -vector  $(1, 4, 5)$  this leaves  $9 \times 2^{10}$  configurations, of which exactly one survives the degree-three test (Proposition 4.1); the survivor forces

$$J = (x_2^2, x_3^2, x_4^2, x_5^2, x_2x_5 - x_3x_4), \quad a_2 + a_5 = a_3 + a_4,$$

and hence forces the Apéry set of  $H$  with respect to  $e(H) = 10$  to be

$$\text{Ap}(H, 10) = \{0, a_2, a_3, a_4, a_5, a_2+a_3, a_2+a_4, a_2+a_5, a_3+a_5, a_4+a_5\}.$$

Those ten integers must be pairwise incongruent modulo 10, and they never are: at most nine residues are attainable (Propositions 5.1 and 5.2). The same three steps applied to  $J_2$  give an eight-element Apéry set that must be a complete residue system modulo 8 and never is (Proposition 6.1).

Section 7 records an independent recomputation of the data of [10, Table 1], which is the calibration for everything else, and Section 8 describes the formal verification.

## 2. PRELIMINARIES

Throughout,  $K$  is an infinite field,  $H = \langle a_1 < a_2 < \dots < a_n \rangle$  is a numerical semigroup given by its minimal generating set,  $e = e(H) = a_1$ , and  $R = \text{gr}_{\mathfrak{m}} K[H] = S/I_H^*$  with  $S = K[x_1, \dots, x_n]$ . General background on numerical semigroups is in [8], on tangent cones and initial ideals in [4].

**Definition 2.1.** The *order* of  $h \in H$  is  $\text{ord}(h) = \max\{k : h = a_{i_1} + \dots + a_{i_k} \text{ for some } i_1, \dots, i_k\}$ , with  $\text{ord}(0) = 0$ . The *Apéry set* of  $H$  with respect to  $e$  is  $\text{Ap}(H, e) = \{h \in H : h - e \notin H\}$ ; it contains exactly one element of each residue class modulo  $e$ , namely the smallest element of  $H$  in that class.

We use the following classical description of  $R$  without further comment; it is the reason every computation below is a computation with  $\text{ord}$ .

**Lemma 2.2.** For each  $i \geq 0$  the residue classes of the monomials  $t^h$  with  $h \in H$ ,  $\text{ord}(h) = i$  form a  $K$ -basis of  $R_i = \mathfrak{m}^i / \mathfrak{m}^{i+1}$ . Under this identification the  $K$ -algebra map  $S \rightarrow R$  sends a monomial  $x^u$  of degree  $d$  to the basis vector indexed by  $\langle a, u \rangle = \sum_i u_i a_i$  if  $\text{ord}\langle a, u \rangle = d$ , and to 0 if  $\text{ord}\langle a, u \rangle > d$ .

Consequently  $\text{HF}_R(i) = \#\{h \in H : \text{ord}(h) = i\}$ . As  $R$  is a one-dimensional standard graded  $K$ -algebra of multiplicity  $e(H)$ , its Hilbert function equals  $e$  for  $i \gg 0$  and its  $h$ -vector is the first difference  $h_i = \text{HF}_R(i) - \text{HF}_R(i-1)$ , with  $\sum_i h_i = e$ ; see [10, Introduction]. We say that  $H$  has  $h$ -vector  $\mathbf{h}$  when  $R$  does.

**Lemma 2.3.** For  $h \in H$  one has  $\text{ord}(h + e) \geq \text{ord}(h) + 1$ . Hence two elements of  $H$  in the same residue class modulo  $e$  have distinct orders, and  $\text{HF}_R(i) \leq e$  for every  $i$ .

*Proof.* A decomposition of  $h$  into  $\text{ord}(h)$  generators, extended by  $a_1 = e$ , decomposes  $h + e$  into  $\text{ord}(h) + 1$  generators. If  $h < h'$  lie in  $H$  and  $h' \equiv h \pmod{e}$  then  $h' = h + ke$  with  $k \geq 1$ , so  $\text{ord}(h') \geq \text{ord}(h) + k > \text{ord}(h)$ . Each of the  $e$  residue classes therefore contributes at most one element of each order.  $\square$

Finally,  $a_2, \dots, a_n$  lie in pairwise distinct nonzero residue classes modulo  $e$ : if  $a_i \equiv a_j \pmod{e}$  with  $2 \leq i < j \leq n$  then  $a_j \in \langle a_1, a_i \rangle$ , and if  $a_j \equiv 0$  then  $a_j \in \langle a_1 \rangle$ , either way contradicting minimality.

## 3. FROM SEMIGROUPS TO FINITE DATA

This section contains the dictionary that both questions are read through. Fix  $H$  with  $\text{edim}(H) = n$  and  $e = e(H)$ , and put  $\bar{S} = K[x_2, \dots, x_n]$ .

**Lemma 3.1.** *Suppose  $\text{HF}_R(2) = \text{HF}_R(3) = e$ . Then:*

- (i)  $\text{HF}_R(i) = e$  for every  $i \geq 2$ ; the class of  $t^{a_1}$  is a nonzerodivisor on  $R$ , so  $R$  is Cohen–Macaulay; and the  $h$ -vector of  $R$  is  $(1, n-1, e-n)$ .
- (ii) Every element of  $\text{Ap}(H, e)$  has order at most 2, and  $\text{Ap}(H, e) = \{0\} \cup \{a_2, \dots, a_n\} \cup W$  where  $W$  is the set of its  $e-n$  elements of order 2. Moreover  $x_1 R_1$  has basis the classes of  $t^{2e}$  and of  $t^{a_j+e}$ ,  $2 \leq j \leq n$ , and  $\bar{R}_2 := R_2/x_1 R_1$  has basis the classes of  $t^w$ ,  $w \in W$ .

*Proof.* By Lemma 2.3 each residue class modulo  $e$  carries at most one element of each order;  $\text{HF}_R(2) = e$  therefore says that each class carries exactly one element of order 2, and likewise for order 3.

For a residue class  $c$  write  $w_c = \min(H \cap c)$ , so that  $H \cap c = \{w_c + ke : k \geq 0\}$ , with strictly increasing orders by Lemma 2.3. In the class of 0 we have  $\text{ord}(\ell e) = \ell$  for every  $\ell \geq 0$  (a decomposition into  $k$  generators gives  $ke \leq \ell e$ , so  $k \leq \ell$ ), so its order-2 element is  $2e$ . In the class of  $a_j$  ( $2 \leq j \leq n$ ) the minimum is  $a_j$ , of order 1, because  $a_j - e \notin H$ ; the order-2 element is  $a_j + ke$  for some  $k \geq 1$ , and  $k \geq 2$  would force  $2 \leq \text{ord}(a_j + e) < 2$ , so it is  $a_j + e$ . In each of the remaining  $e-n$  classes the minimum  $w_c$  is neither 0 nor a generator — the classes of  $a_1 = e, a_2, \dots, a_n$  are the  $n$  just treated — so  $\text{ord}(w_c) \geq 2$ , and  $\text{ord}(w_c) \geq 3$  would leave the class without an element of order 2; hence  $\text{ord}(w_c) = 2$ . This proves (ii) and, at the same time, that multiplication by  $x_1$  maps the basis of  $R_1$  into the basis of  $R_2$  bijectively onto its image (namely  $[t^{\ell e}] \mapsto [t^{(\ell+1)e}]$  and  $[t^{a_j}] \mapsto [t^{a_j+e}]$ ), so  $R_1 \rightarrow R_2$  is injective.

The same argument applied to the order-2 element of each class, using  $\text{HF}_R(3) = e$ , shows that  $x_1$  carries the basis of  $R_2$  into that of  $R_3$ , so  $R_2 \rightarrow R_3$  is injective; as  $\dim R_2 = \dim R_3 = e$  it is bijective, so  $R_3 = x_1 R_2$ . Then  $R_4 = R_1 R_3 = R_1 x_1 R_2 = x_1 R_1 R_2 = x_1 R_3$ , and inductively  $R_{i+1} = x_1 R_i$  for all  $i \geq 2$ . Hence  $\text{HF}_R$  is non-increasing on  $i \geq 2$ ; being  $\leq e$  by Lemma 2.3, equal to  $e$  at  $i = 2$  and equal to  $e$  for  $i \gg 0$ , it is identically  $e$  on  $i \geq 2$ . Multiplication by  $x_1$  is then a surjection between spaces of equal finite dimension in every degree  $\geq 2$ , hence bijective there, and it is injective in degrees 0 and 1. So  $t^{a_1}$  is a nonzerodivisor. The  $h$ -vector is  $(1, n-1, e-n)$  because  $\text{HF}_R = (1, n, e, e, \dots)$ .  $\square$

Lemma 3.1 applies verbatim whenever the  $h$ -vector of  $R$  is  $(1, n-1, h_2)$  with no further entries, since then  $\text{HF}_R = (1, n, n+h_2, n+h_2, \dots)$  and  $e = \sum_i h_i = n+h_2$ .

From now on  $n = 5$ . Write  $P = \{(i, j) : 2 \leq i \leq j \leq 5\}$  for the ten index pairs and  $m_{ij} = x_i x_j$  for the corresponding monomials of  $\bar{S}_2$ ; put  $\sigma(i, j) = a_i + a_j$ .

**Definition 3.2.** Assume the hypothesis of Lemma 3.1. A pair  $(i, j) \in P$  is *null* if  $\sigma(i, j) \notin \text{Ap}(H, e)$ , and *live* otherwise;  $N \subseteq P$  denotes the set of null pairs. Two live pairs are *equivalent*, written  $(i, j) \sim (k, \ell)$ , when  $\sigma(i, j) = \sigma(k, \ell)$ . The pair  $(N, \sim)$  is the *configuration* of  $H$ .

**Lemma 3.3.** *Assume the hypothesis of Lemma 3.1 with  $n = 5$ , and set  $J = (I_H^* + (x_1))/(x_1) \subseteq \bar{S}$ , and  $[J]_d = J \cap \bar{S}_d$  for its degree- $d$  component (the brackets keep this apart from Stamate’s ideals  $J_1, J_2, J_3$ ). Then:*

- (i)  $m_{ij}$  maps to 0 in  $\bar{R}_2$  if and only if  $(i, j)$  is null; two live monomials have the same nonzero image if and only if they are equivalent; and the images of the live monomials are exactly the  $e-5$  basis vectors of  $\bar{R}_2$ . In particular  $\sim$  has exactly  $h_2 = e-5$  classes.
- (ii)  $[J]_2$  is spanned by  $\{m_{ij} : (i, j) \in N\}$  together with  $\{m_{ij} - m_{k\ell} : (i, j) \sim (k, \ell)\}$ , and  $\dim_K [J]_2 = 10 - h_2$ .
- (iii)  $[J]_d = \bar{S}_d$  for every  $d \geq 3$ .
- (iv) If  $H$  is quadratic then  $\bar{S}_1[J]_2 = \bar{S}_3$ . Conversely, if  $\bar{S}_1[J]_2 = \bar{S}_3$  then  $H$  is quadratic.

*Proof.* (i) By Lemma 2.2,  $m_{ij}$  maps in  $R_2$  to 0 if  $\text{ord} \sigma(i, j) > 2$  and to the basis vector  $[t^{\sigma(i, j)}]$  otherwise. Suppose  $\sigma(i, j) \in \text{Ap}(H, e)$ . It is a sum of two generators, so its order is at least 2, and

at most 2 by Lemma 3.1(ii); it is neither 0 nor a minimal generator, so it lies in  $W$  and its class in  $\bar{R}_2$  is a basis vector. Suppose instead  $\sigma(i, j) \notin \text{Ap}(H, e)$ . If  $\text{ord } \sigma(i, j) > 2$  its image in  $R_2$  is already 0. If  $\text{ord } \sigma(i, j) = 2$ , write  $\sigma(i, j) = h + e$  with  $h \in H$ ; then  $\text{ord}(h) \leq 1$  by Lemma 2.3, and  $h \neq 0$  because  $\sigma(i, j) \geq 2a_2 > e$ , so  $h$  is a generator and  $[t^{\sigma(i, j)}] \in x_1 R_1$  by Lemma 3.1(ii). Either way the image in  $\bar{R}_2$  is 0. Finally  $\bar{S}_2 \rightarrow \bar{R}_2$  is surjective because  $\bar{R} = \bar{S}/J$  is standard graded, so every basis vector of  $\bar{R}_2$  is the image of a live monomial; distinct basis vectors are indexed by distinct values  $\sigma$ .

(ii) is (i) together with  $[J]_2 = \ker(\bar{S}_2 \rightarrow \bar{R}_2)$ : a kernel of a map sending each basis monomial to 0 or to a basis vector is spanned by the monomials of the first kind and the differences of monomials with a common image.

(iii)  $\bar{R}_3 = R_3/x_1 R_2 = 0$  by Lemma 3.1(i), so  $[J]_3 = \bar{S}_3$ , and then  $[J]_d = \bar{S}_d$  for  $d \geq 3$ .

(iv) If  $I_H^*$  is generated by quadrics then  $(I_H^*)_3 = S_1(I_H^*)_2$ . Applying the map  $S \rightarrow \bar{S}$ ,  $x_1 \mapsto 0$ , whose image of  $(I_H^*)_d$  is  $[J]_d$ , gives  $\bar{S}_3 = [J]_3 = \bar{S}_1[J]_2$ . Note that this direction uses only  $\bar{R}_3 = 0$  and the description of  $\bar{R}_2$ , not the full strength of Lemma 3.1(i). Conversely, since  $x_1$  is a nonzerodivisor on  $R$ , tensoring a minimal graded free resolution of  $R$  over  $S$  with  $S/(x_1)$  yields a minimal graded free resolution of  $\bar{R}$  over  $\bar{S}$ ; so  $I_H^*$  and  $J$  have minimal generators in the same degrees. By (iii),  $J$  needs no generators in degree  $\geq 4$ , and  $\bar{S}_1[J]_2 = \bar{S}_3 = [J]_3$  says it needs none in degree 3 either. Hence  $J$ , and with it  $I_H^*$ , is generated in degree 2.  $\square$

The remaining ingredient turns  $\bar{S}_1[J]_2 = \bar{S}_3$  into a connectivity statement. Let  $T_3$  denote the twenty monomials of degree 3 in  $\bar{S}$ .

**Definition 3.4.** Given a configuration  $(N, \sim)$ , let  $\Gamma = \Gamma(N, \sim)$  be the graph on the vertex set  $T_3$  in which a vertex  $u$  is *marked* when  $u = x_c m_{ij}$  for some  $c \in \{2, 3, 4, 5\}$  and some  $(i, j) \in N$ , and in which  $u, u'$  are joined by an edge when  $u = x_c m_{ij}$  and  $u' = x_c m_{k\ell}$  for a common  $c$  and equivalent live pairs  $(i, j) \sim (k, \ell)$ . Write  $c(\Gamma)$  for the number of connected components of  $\Gamma$  containing no marked vertex.

**Lemma 3.5.**  $\dim_K \bar{S}_1[J]_2 = 20 - c(\Gamma)$ . In particular  $\bar{S}_1[J]_2 = \bar{S}_3$  if and only if every connected component of  $\Gamma$  contains a marked vertex, and this condition does not depend on  $K$ .

*Proof.* By Lemma 3.3(ii),  $\bar{S}_1[J]_2$  is spanned by the marked vertices of  $\Gamma$  together with the differences  $u - u'$  along edges. Restricted to one connected component  $C$ , the differences span the hyperplane of  $\bigoplus_{u \in C} Ku$  of vectors with coordinate sum 0, of dimension  $|C| - 1$ ; if  $C$  carries a marked vertex the span is all of  $\bigoplus_{u \in C} Ku$ . Summing over components gives  $20 - c(\Gamma)$ . The count involves no scalars, whence the independence of  $K$ .  $\square$

**Lemma 3.6.** Let  $a_2 < a_3 < a_4 < a_5$  be positive integers. The only coincidences  $\sigma(i, j) = \sigma(k, \ell)$  with  $(i, j) \neq (k, \ell)$  that can occur are

$$\begin{aligned} p_0 : a_2 + a_5 &= a_3 + a_4, & p_1 : 2a_3 &= a_2 + a_4, & p_2 : 2a_3 &= a_2 + a_5, \\ p_3 : 2a_4 &= a_2 + a_5, & p_4 : 2a_4 &= a_3 + a_5, \end{aligned}$$

and the set of those which hold is one of the nine sets

$$\emptyset, \{p_0\}, \{p_1\}, \{p_2\}, \{p_3\}, \{p_4\}, \{p_1, p_3\}, \{p_2, p_4\}, \{p_0, p_1, p_4\}.$$

In each of the nine, the pairs identified by  $\sim$  are disjoint transpositions of  $P$ .

*Proof.* Order 2-multisets of  $\{2, 3, 4, 5\}$  componentwise after sorting. If one of two multisets dominates the other, the corresponding sums are strictly comparable; so a coincidence forces the two multisets to be incomparable, and there are exactly five incomparable pairs, namely  $p_0, \dots, p_4$ . (This list is already in [10, §2].) For the second assertion, each of the 32 subsets is either one of the nine or is contradicted by the strict inequalities: for instance  $p_0 \wedge p_2$  gives  $a_3 = a_4$ ,  $p_1 \wedge p_2$  gives  $a_4 = a_5$ ,  $p_3 \wedge p_4$  gives  $a_2 = a_3$ , and any two of  $p_0, p_1, p_4$  imply the third, so no subset containing exactly two of them

is the exact coincidence set. Each of the 23 exclusions is a linear consequence of  $a_2 < a_3 < a_4 < a_5$ . The nine sets do occur, as

$$\begin{aligned} \emptyset : (1, 2, 4, 8), \quad \{p_0\} : (1, 3, 4, 6), \quad \{p_1\} : (1, 2, 3, 7), \quad \{p_2\} : (1, 5, 6, 9), \\ \{p_3\} : (1, 2, 4, 7), \quad \{p_4\} : (1, 2, 4, 6), \quad \{p_1, p_3\} : (1, 2, 3, 5), \quad \{p_2, p_4\} : (1, 3, 4, 5), \end{aligned}$$

and  $\{p_0, p_1, p_4\} : (3, 4, 5, 6)$ , the case of four generators in arithmetic progression. The final assertion is read off the nine lists.  $\square$

*Remark 3.7.* Stamate observes in [10, §2] that  $p_1, p_2$  cannot both hold, that  $p_3, p_4$  cannot both hold, and that any two of  $p_0, p_1, p_4$  imply the third. Lemma 3.6 is the completion of that list to all 32 subsets, which is what makes the classification below a finite check.

#### 4. THE CLASSIFICATION OF DEGREE-THREE CONFIGURATIONS

We now classify the configurations that a quadratic  $H$  with  $h$ -vector  $(1, 4, 5)$  could carry. Here  $e = 10$  and  $h_2 = 5$ , so by Lemma 3.3(i) the live pairs fall into exactly five classes. Two pairs with  $\sigma(i, j) = \sigma(k, \ell)$  have the same image in  $\bar{R}_2$  and so are null together: nullity is constant on the fibres of  $\sigma$ . Finally, by Lemma 3.3(iv) and Lemma 3.5, quadratic generation says that every component of  $\Gamma$  carries a marked vertex. These three conditions refer only to the combinatorial data  $(N, \sim)$ , and by Lemma 3.6 there are  $9 \times 2^{10}$  candidate data in all.

**Proposition 4.1.** *Among the nine coincidence patterns of Lemma 3.6 and all  $2^{10}$  subsets  $N \subseteq P$  — that is, among 9216 configurations — exactly one satisfies all three of*

- (a) *nullity is constant on pairs with equal  $\sigma$ ;*
- (b) *the live pairs fall into exactly five classes;*
- (c) *every connected component of  $\Gamma$  carries a marked vertex,*

*namely the pattern in which  $p_0$  is the only coincidence, together with  $N = \{(2, 2), (3, 3), (4, 4), (5, 5)\}$ . Equivalently, the only possible value of  $J$  is*

$$J = (x_2^2, x_3^2, x_4^2, x_5^2, x_2x_5 - x_3x_4), \quad \text{with } a_2 + a_5 = a_3 + a_4.$$

*Moreover this configuration does satisfy (a)–(c), so the classification is not vacuous.*

*Proof.* The nine patterns of Lemma 3.6 give nine relations  $\sim$  on  $P$ ; for each and for each of the  $2^{10}$  subsets  $N$  the three conditions are decidable by inspection, the third by computing the connected components of a graph on twenty vertices. Running the  $9 \times 1024$  cases leaves the single configuration named. The computation is exhaustive and has been machine-checked; see Section 8.  $\square$

*Remark 4.2.* Two special cases explain the shape of the survivor. The vertex  $x_2^3$  of  $\Gamma$  is  $x_2 \cdot m_{22}$  and nothing else, and no  $p_i$  involves  $(2, 2)$ , so  $x_2^3$  is isolated in  $\Gamma$  and must be marked, i.e.  $x_2^2$  must be null; symmetrically  $x_5^2$  must be null. That  $x_3^2$  and  $x_4^2$  are null as well, and that  $p_0$  is the surviving coincidence, is what the enumeration contributes.

#### 5. THE RESIDUE OBSTRUCTION AND THE MAIN THEOREM

The surviving configuration determines the Apéry set of  $H$  completely. Indeed, by Lemma 3.1(ii) and Lemma 3.3(i),  $\text{Ap}(H, 10)$  consists of 0, the four generators  $a_2, \dots, a_5$ , and the five values  $\sigma$  of the five live classes. For  $N = \{(2, 2), (3, 3), (4, 4), (5, 5)\}$  and  $p_0$  the only coincidence, the live classes are  $\{(2, 3)\}, \{(2, 4)\}, \{(2, 5), (3, 4)\}, \{(3, 5)\}, \{(4, 5)\}$ , so

$$(4) \text{ Ap}(H, 10) = \{0, a_2, a_3, a_4, a_5, a_2+a_3, a_2+a_4, a_2+a_5, a_3+a_5, a_4+a_5\}, \quad a_2+a_5 = a_3+a_4.$$

Since an Apéry set with respect to  $e$  meets each residue class modulo  $e$  exactly once, those ten integers must be pairwise incongruent modulo 10. They are not.

**Proposition 5.1.** *For no  $b, c, d \in \mathbb{Z}/10$  are the ten elements*

$$0, \quad b, \quad c, \quad d, \quad c+d-b, \quad b+c, \quad b+d, \quad c+d, \quad 2c+d-b, \quad c+2d-b$$

*of  $\mathbb{Z}/10$  pairwise distinct. The bound is sharp: nine of them are distinct for  $(b, c, d) = (9, 8, 6)$ .*

*Proof.* A check of all  $10^3 = 1000$  triples  $(b, c, d)$ ; it has been machine-checked (Section 8). For sharpness,  $(b, c, d) = (9, 8, 6)$  gives the ten values  $0, 9, 8, 6, 5, 7, 5, 4, 3, 1$ , of which nine are distinct.  $\square$

Substituting  $a_5 = a_3 + a_4 - a_2$  into (4) turns the ten integers there into the ten elements of Proposition 5.1 with  $b, c, d$  the residues of  $a_2, a_3, a_4$ , whence:

**Proposition 5.2.** *Let  $a_2, a_3, a_4, a_5$  be natural numbers with  $a_2 + a_5 = a_3 + a_4$ . Then the ten integers*

$$0, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_2+a_3, \quad a_2+a_4, \quad a_2+a_5, \quad a_3+a_5, \quad a_4+a_5$$

*are not pairwise incongruent modulo 10. In particular they are not the Apéry set  $\text{Ap}(H, 10)$  of a numerical semigroup of multiplicity 10.*

Proposition 5.2 is a finite statement about  $(\mathbb{Z}/10)^3$  and is verified as such, but it also has a short direct proof, which we give because it explains the obstruction.

*Direct proof of Proposition 5.2.* Suppose the ten integers form a complete residue system modulo 10. Their sum is  $4a_2 + 3a_3 + 3a_4 + 4a_5 = 4(a_2 + a_5) + 3(a_3 + a_4) = 7(a_2 + a_5)$ , while the sum of a complete residue system modulo 10 is  $0 + 1 + \dots + 9 = 45 \equiv 5$ . As 7 is a unit modulo 10 with inverse 3, we get

$$(5) \quad a_2 + a_5 \equiv 3 \cdot 5 = 15 \equiv 5 \pmod{10}.$$

In particular  $a_2 + a_5 \equiv a_3 + a_4 \equiv 0 \pmod{5}$ . Put  $x \equiv a_2$  and  $y \equiv a_3$  modulo 5; then  $a_5 \equiv -x$  and  $a_4 \equiv -y$ , and the ten values become, modulo 5,

$$0, \quad \pm x, \quad \pm y, \quad \pm(x+y), \quad \pm(x-y), \quad 0.$$

Each class modulo 5 must be hit exactly twice. The class of 0 is already hit twice, so  $x, y, x+y, x-y \neq 0$ ; in particular  $x \neq 0$ , so every nonzero class is  $ux$  for a unique  $u \in \{\pm 1, \pm 2\}$ , and  $y \equiv \pm x$  is excluded, leaving  $y \equiv \pm 2x \pmod{5}$ .

Now compare parities. If  $u \equiv v \pmod{5}$  but  $u \not\equiv v \pmod{10}$ , then  $u - v \equiv 5 \pmod{10}$  is odd, so  $u$  and  $v$  have opposite parities: the two members of each class modulo 5 have opposite parity. Applied to the class of 0, which contains 0 and  $a_2 + a_5$ , this is (5) again.

Suppose  $y \equiv 2x$ . Then  $a_3 + a_5 \equiv y - x \equiv x$ , so the class of  $x$  contains  $a_2$  and  $a_3 + a_5$ ; these have opposite parity, and with  $a_2 + a_5$  odd this forces  $a_3$  to be even. Likewise  $a_2 + a_4 \equiv x - y \equiv -x$ , so the class of  $-x$  contains  $a_5$  and  $a_2 + a_4$ ; the same reasoning makes  $a_4$  even, and then  $a_3 = a_2 + a_5 - a_4$  is odd. Contradiction.

Suppose  $y \equiv -2x$ . Then  $a_4 + a_5 \equiv -(x+y) \equiv x$ , so the class of  $x$  contains  $a_2$  and  $a_4 + a_5$ , forcing  $a_4$  even and hence  $a_3$  odd; and  $a_2 + a_3 \equiv x + y \equiv -x$ , so the class of  $-x$  contains  $a_5$  and  $a_2 + a_3$ , forcing  $a_3$  even. Contradiction.  $\square$

**Theorem 5.3.** *There is no numerical semigroup  $H$  with  $\text{edim}(H) = 5$  which is quadratic and whose tangent cone has  $h$ -vector  $(1, 4, 5)$ . Consequently [10, Table 1] lists eleven, not twelve,  $h$ -vectors, and Question 1.1 has a negative answer.*

*Proof.* Suppose such an  $H = \langle a_1 < \dots < a_5 \rangle$  exists. Its  $h$ -vector sums to  $e(H)$ , so  $e = a_1 = 10$  and  $\text{HF}_R = (1, 5, 10, 10, \dots)$ ; Lemma 3.1 applies. By Lemma 3.3(i) the configuration of  $H$  has exactly five live classes, nullity is constant on pairs with equal  $\sigma$ , and by Lemma 3.3(iv) and Lemma 3.5 every component of  $\Gamma$  is marked. By Lemma 3.6 the coincidence pattern of  $(a_2, a_3, a_4, a_5)$  is one of nine, so Proposition 4.1 applies and gives  $a_2 + a_5 = a_3 + a_4$  together with  $N = \{(2, 2), (3, 3), (4, 4), (5, 5)\}$ . Hence  $\text{Ap}(H, 10)$  is the set (4), whose elements are pairwise incongruent modulo 10 because  $\text{Ap}(H, 10)$  meets each class once. This contradicts Proposition 5.2.  $\square$

Theorem 5.3 was also confirmed on explicit data by a route that uses neither the classification of Section 4 nor the arithmetic of Proposition 5.2. For a numerical semigroup  $H$  and  $d \geq 2$  let  $\mu_d(H)$  denote the number of minimal generators of  $I_H^*$  of degree  $d$ ; as in Lemma 3.5 it equals the number of unmarked components of the degree- $d$  analogue of  $\Gamma$ , computed in all five variables, minus  $\text{HF}_R(d)$ , and needs no hypothesis on  $R$ .

**Proposition 5.4.** *Each of the eight numerical semigroups*

$$\begin{aligned} &\langle 10, 11, 12, 13, 16 \rangle, \quad \langle 10, 11, 12, 13, 17 \rangle, \quad \langle 10, 11, 12, 14, 15 \rangle, \quad \langle 10, 11, 12, 14, 17 \rangle, \\ &\langle 10, 11, 12, 15, 17 \rangle, \quad \langle 10, 11, 12, 15, 18 \rangle, \quad \langle 10, 11, 13, 14, 15 \rangle, \quad \langle 10, 11, 13, 14, 16 \rangle \end{aligned}$$

— the eight lexicographically smallest with  $h$ -vector  $(1, 4, 5)$  — satisfies  $\mu_3(H) > 0$ , and so is not quadratic.

*Proof.* A direct computation of the  $h$ -vector and of  $\mu_3$  from the order function for each of the eight; machine-checked, see Section 8.  $\square$

*Remark 5.5.* A further, non-formalised check was run outside Lean. Sweeping all 487,635 tuples  $10 < a_2 < a_3 < a_4 < a_5 \leq 70$  produced 14,612 numerical semigroups  $\langle 10, a_2, a_3, a_4, a_5 \rangle$  with  $h$ -vector  $(1, 4, 5)$ ; every one of them has between 2 and 7 unmarked components in  $\Gamma$ , where quadratic generation would require none, and the configuration of Proposition 4.1 occurs for none of them. Separately, the ten integers of Proposition 5.2 were computed for 30,643,864 quadruples of integers with  $a_2 + a_5 = a_3 + a_4$ ; the largest number of distinct residues modulo 10 observed was nine, attained for instance at  $(a_2, a_3, a_4, a_5) = (11, 12, 14, 15)$ . Neither computation is part of any proof above.

## 6. THE IDEALS $J_1, J_2, J_3$

We turn to Question 1.2. Throughout this section  $H$  is a numerical semigroup with  $\text{edim}(H) = 5$  whose  $h$ -vector is  $(1, 4, h_2)$  with  $h_2 \in \{2, 3\}$ ; then  $e(H) = 5 + h_2$  and Lemma 3.1 applies, so  $t^{a_1}$  is a nonzerodivisor on  $R$  and  $J = I_H^* \bmod x_1$  is the object appearing in (1)–(3). We say that  $H$  produces an ideal  $J'$  when  $J = J'$  after some permutation of  $x_2, x_3, x_4, x_5$ : the candidates  $J_2, J_3$  are introduced in the proof of [10, Thm. 2.4] as the exceptions occurring “up to a permutation of the variables”, so this is the reading Stamate’s question asks for.

Reading the generators of  $J_2$  through Lemma 3.3(ii): the monomial generators  $x_2^2, x_5^2, x_2x_3, x_3x_4, x_3x_5$  say that these five pairs are null, and the two binomials say that  $\{(3, 3), (2, 4)\}$  and  $\{(4, 4), (2, 5)\}$  are live classes; the remaining live pair  $(4, 5)$  is a class of its own, giving  $h_2 = 3$  and  $e(H) = 8$  as required. Let  $p, q, r, s$  denote the four generators  $a_2, \dots, a_5$  in the roles they play in  $J_2$ , that is, in the roles of  $x_2, x_3, x_4, x_5$ ; then the two binomials read  $2q = p + r$  and  $2r = p + s$ , and by Lemma 3.1(ii),

$$(6) \quad \text{Ap}(H, 8) = \{0, p, q, r, s, 2q, 2r, r + s\}, \quad 2q = p + r, \quad 2r = p + s.$$

Phrasing the conclusion in terms of the roles rather than the size order is what makes one statement cover all 24 relabellings of the variables at once, which is the generality in which [10] lists  $J_2$ .

**Proposition 6.1.** *For no  $p, q \in \mathbb{Z}/8$  are the eight elements*

$$0, \quad p, \quad q, \quad 2q - p, \quad 4q - 3p, \quad 2q, \quad 4q - 2p, \quad 6q - 4p$$

*of  $\mathbb{Z}/8$  pairwise distinct. The bound is sharp: seven of them are distinct for  $(p, q) = (1, 2)$ .*

*Proof.* A check of all  $8^2 = 64$  pairs  $(p, q)$ ; machine-checked, see Section 8. The eight expressions are those of (6) after eliminating  $r = 2q - p$  and  $s = 4q - 3p$ .  $\square$

**Theorem 6.2.** *Let  $p, q, r, s$  be natural numbers with  $2q = p + r$  and  $2r = p + s$ . Then the eight integers  $0, p, q, r, s, 2q, 2r, r + s$  are not pairwise incongruent modulo 8, and so are not the Apéry set of a numerical semigroup of multiplicity 8. Consequently no quadratic numerical semigroup  $H$  with  $\text{edim}(H) = 5$  produces the ideal  $J_2$ , for any labelling of the variables.*

*Proof.* Reducing modulo 8 and eliminating  $r$  and  $s$  turns the eight integers into the eight elements of Proposition 6.1. If  $H$  produced  $J_2$ , then  $h_2 = 3$ ,  $e(H) = 8$ , Lemma 3.1 applies and (6) holds; but  $\text{Ap}(H, 8)$  meets each residue class modulo 8 exactly once, a contradiction.  $\square$

The other two ideals do occur.

**Theorem 6.3.** *The numerical semigroups*

$$H_1 = \langle 7, 23, 31, 39, 40 \rangle \quad \text{and} \quad H_3 = \langle 8, 26, 35, 44, 45 \rangle$$

*have embedding dimension five, are quadratic, have  $h$ -vectors  $(1, 4, 2)$  and  $(1, 4, 3)$ , and*

$$\text{Ap}(H_1, 7) = \{0, 23, 31, 39, 40, 62, 71\}, \quad \text{Ap}(H_3, 8) = \{0, 26, 35, 44, 45, 70, 71, 89\}.$$

*Consequently  $I_{H_1}^* \bmod x_1 = J_1$  and  $I_{H_3}^* \bmod x_1 = J_3$ , in the labelling of the variables printed in (1) and (3).*

*Proof.* The  $h$ -vectors, the Apéry sets and the vanishing of  $\mu_d$  for  $3 \leq d \leq 5$  are computations from the order function, machine-checked as described in Section 8. Given the  $h$ -vectors, Lemma 3.1 applies and Lemma 3.3(iv) turns  $\mu_3 = 0$  into quadraticity — the values  $\mu_4 = \mu_5 = 0$  are then automatic and were computed only as a check.

For  $H_1$ , with  $(a_2, a_3, a_4, a_5) = (23, 31, 39, 40)$ , the ten sums  $\sigma(i, j)$  are

$$46, 54, 62, 63, 62, 70, 71, 78, 79, 80,$$

of which exactly 62 (twice, as  $2a_3$  and as  $a_2 + a_4$ ) and  $71 = a_3 + a_5$  lie in  $\text{Ap}(H_1, 7)$ . So the null pairs are  $(2, 2)$ ,  $(2, 3)$ ,  $(2, 5)$ ,  $(3, 4)$ ,  $(4, 4)$ ,  $(4, 5)$ ,  $(5, 5)$ , and the live classes are  $\{(3, 3), (2, 4)\}$  and  $\{(3, 5)\}$ ; by Lemma 3.3(ii) this is exactly the generating set (1) of  $J_1$ .

For  $H_3$ , with  $(a_2, a_3, a_4, a_5) = (26, 35, 44, 45)$ , the ten sums are

$$52, 61, 70, 71, 70, 79, 80, 88, 89, 90,$$

of which 70 (twice, as  $2a_3$  and as  $a_2 + a_4$ ),  $71 = a_2 + a_5$  and  $89 = a_4 + a_5$  lie in  $\text{Ap}(H_3, 8)$ . So the null pairs are  $(2, 2)$ ,  $(2, 3)$ ,  $(3, 4)$ ,  $(3, 5)$ ,  $(4, 4)$ ,  $(5, 5)$  and the live classes are  $\{(3, 3), (2, 4)\}$ ,  $\{(2, 5)\}$ ,  $\{(4, 5)\}$ ; by Lemma 3.3(ii) this is exactly the generating set (3) of  $J_3$ .  $\square$

*Remark 6.4.* The two witnesses were found by an exhaustive sweep, run outside Lean, over the numerical semigroups of embedding dimension five with all generators at most 70: the 25,278 such semigroups with  $e(H) = 7$  and  $h$ -vector  $(1, 4, 2)$  realise 61 distinct configurations, and the 38,200 with  $e(H) = 8$  and  $h$ -vector  $(1, 4, 3)$  realise 140; four of the 61 are relabellings of  $J_1$  and four of the 140 are relabellings of  $J_3$ , and in each case the identity labelling occurs. None of the 140 is a relabelling of  $J_2$ , in agreement with Theorem 6.2. Both  $H_1$  and  $H_3$  need  $a_5 \approx 40$ , which may be why a search over small generators did not reach them. The sweep is not part of the proof of Theorem 6.3; only the two certified witnesses are.

*Remark 6.5.* Combining Theorems 6.2 and 6.3: of the three ideals,  $J_1$  and  $J_3$  are produced by quadratic numerical semigroups and  $J_2$  is not. Since  $\bar{S}/J_3$  has no quadratic Gröbner basis with respect to any term order [10, Rem. 2.7], the condition under which [10] hoped to remove the hypotheses “ $K$  algebraically closed,  $\text{char } K \neq 2$ ” from [10, Thm. 2.4] — namely that no quadratic numerical semigroup produces  $J_1, J_2, J_3$  — is not satisfied. This closes that particular route. It does not show that the hypotheses are necessary. What remains is to decide whether  $\bar{S}/J_3$  is  $G$ -quadratic over an arbitrary field; note that [10, Rem. 2.7] exhibits a Koszul filtration for  $\bar{S}/J_3$ , with no restriction on  $K$ . That is a question about one explicit artinian algebra in four variables rather than about semigroups, and we leave it open.

## 7. THE DATA OF TABLE 1, RECOMPUTED

The computations above are only as trustworthy as the routine that produces  $h$ -vectors and  $\mu_d$  from a list of generators. That routine was calibrated on the published data of [10, Table 1] before any claim of this paper was made. Table 1 of [10] lists twelve  $h$ -vectors together with twelve example semigroups, one for each  $h$ -vector except  $(1, 4, 5)$ , which has none, and  $(1, 4, 4)$ , which has two.

*Remark 7.1.* Table 1 differs between the published version of [10] and the arXiv version v3 that we cite. In the published version *two* rows carry “?” in the Example column,  $(1, 4, 3, 1)$  as well as  $(1, 4, 5)$ , and the sentence quoted in Section 1 reads “For two of these  $h$ -vectors...”; v3 fills the  $(1, 4, 3, 1)$  row with  $\langle 9, 10, 11, 13, 17 \rangle$ , an example found by F. Strazzanti, and changes “two” to “one”. The published table also prints  $\langle 8, 12, 13, 18, 55 \rangle$  in the  $(1, 4, 2, 1)$  row, where  $55 = 3 \cdot 13 + 2 \cdot 8$  is not a minimal generator; v3 prints  $\langle 8, 12, 13, 18, 35 \rangle$ , which is the semigroup of [10, Ex. 1.6] and the one the construction [10, Prop. 1.7] produces. Proposition 7.2 is about the v3 table.

**Proposition 7.2.** *For each of the twelve numerical semigroups listed in [10, Table 1], namely*

$$\begin{array}{lll} \langle 5, 6, 7, 8, 9 \rangle, & \langle 6, 7, 8, 9, 10 \rangle, & \langle 7, 8, 9, 10, 11 \rangle, \\ \langle 8, 12, 13, 18, 35 \rangle, & \langle 8, 9, 10, 11, 12 \rangle, & \langle 9, 10, 11, 13, 17 \rangle, \\ \langle 9, 10, 11, 12, 15 \rangle, & \langle 9, 17, 20, 23, 25 \rangle, & \langle 10, 16, 19, 22, 25 \rangle, \\ \langle 11, 13, 14, 15, 19 \rangle, & \langle 12, 14, 16, 18, 27 \rangle, & \langle 16, 17, 18, 20, 24 \rangle, \end{array}$$

*the  $h$ -vector computed from the order function is the one printed there, namely*

$$\begin{array}{llllll} (1, 4), & (1, 4, 1), & (1, 4, 2), & (1, 4, 2, 1), & (1, 4, 3), & (1, 4, 3, 1), \\ (1, 4, 4), & (1, 4, 4), & (1, 4, 4, 1), & (1, 4, 5, 1), & (1, 4, 5, 2), & (1, 4, 6, 4, 1) \end{array}$$

*in the same order, and  $\mu_d = 0$  for  $3 \leq d \leq 5$  in every case. By contrast  $\mu_5((17, 18, 19, 20, 21)) = 4$ .*

*Proof.* A computation from the order function for each of the thirteen semigroups; machine-checked, see Section 8.  $\square$

The  $h$ -vectors are the data of [10] and the quadraticity of the twelve examples is asserted there; what is independent here is the route. The count  $\mu_d$  is computed in five variables directly from `ord`, with no Cohen–Macaulay hypothesis, which is why it also handles  $\langle 8, 12, 13, 18, 35 \rangle$ , whose tangent cone is not Cohen–Macaulay [10, Ex. 1.6]. The last semigroup is a negative control:  $\langle 17, 18, 19, 20, 21 \rangle$  has  $e(H) = 17 > 2^4$  and is therefore not quadratic by [10, Thm. 1.1(i)], a restatement of results of Herzog and Stamate [7], and the same routine finds four minimal generators of  $I_H^*$  in degree 5.

## 8. VERIFICATION

All the finite searches above have been formally verified in Lean 4 against *Mathlib*, and the development is free of `sorry`. The two arithmetic obstructions are proved by kernel evaluation and depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: this covers Proposition 5.1, a `decide` over the 1000 triples in  $(\mathbb{Z}/10)^3$ , together with the sharpness witness  $(b, c, d) = (9, 8, 6)$ ; Proposition 5.2; Proposition 6.1, a `decide` over the 64 pairs in  $(\mathbb{Z}/8)^2$ , together with the sharpness witness  $(p, q) = (1, 2)$ ; and Theorem 6.2; as well as the reduction of Lemma 3.6 to a 100-case check on index pairs followed by a 32-way case split closed by linear arithmetic. What is delegated to compiled evaluation, through Lean’s `native_decide`, is exactly the exhaustive searches over finite data: the  $9 \times 1024$  configurations of Proposition 4.1 together with its three controls (that the survivor passes, that the configuration of the  $(1, 4, 4)$  example  $\langle 9, 10, 11, 12, 15 \rangle$  of [10, Table 1] also passes, and that an explicit configuration fails), and the order-function computations behind Proposition 5.4, Theorem 6.3 and Proposition 7.2. Each such evaluation contributes a single named axiom recording precisely which computation the theorem rests on; Theorem 5.3 carries exactly one, the one belonging to Proposition 4.1. Two finite checks are left to the reader rather than formalised, both immediate by inspection: the nine quadruples exhibited in Lemma 3.6 realise the nine coincidence patterns, and the eight semigroups of Proposition 5.4 are the lexicographically smallest with

their  $h$ -vector. Every numerical value quoted was produced first by an independent implementation — in Python for the configuration enumeration and the residue searches, in C for the sweeps of Remark 5.5 — and the formal statement was written against it; the whole Lean development checks in about 233 seconds. Repository reference to be added at submission.

## REFERENCES

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